Extension theorems for reductive group schemes

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September 28, 2015

ABSTRACT. We prove several basic extension theorems for reductive group schemes via extending Lie algebras and via taking schematic closures. We also prove that for each scheme Y, the category in groupoids of adjoint group scheme over Y whose Lie algebras $\mathcal{O}_Y$-modules have perfect Killing forms is isomorphic via the differential functor to the category in groupoids of Lie algebras $\mathcal{O}_Y$-modules which have perfect Killing forms and which as $\mathcal{O}_Y$-modules are coherent and locally free.

Key words: reductive group schemes, purity, regular rings, and Lie algebras.

MSC 2010: Primary: 11G18, 14F30, 14G35, 14L15, 14L17, 14K10, and 17B45.

1. Introduction

A group scheme H over a scheme S is called *reductive* if the morphism $H \to S$ has the following two properties: (i) it is smooth and affine (and therefore of finite presentation) and (ii) its geometric fibres are reductive groups over spectra of fields and therefore are connected (cf. [DG, Vol. III, Exp. XIX, Subsects. 2.7, 2.1, and 2.9]). If moreover the *center* of H is trivial, then H is called an *adjoint* group scheme over S. Let $\mathcal{O}_S$ be the structure ring sheaf of S. Let $Lie(H)$ be the *Lie algebra* $\mathcal{O}_S$-module of H. As a $\mathcal{O}_S$-module, $Lie(H)$ is coherent and locally free.

The main goal of the paper is to prove Theorems 1.2 and 1.4 below (see Sections 3 and 4) and to apply them for obtaining a few new extension theorems for homomorphisms between reductive group schemes (see Section 5). We begin by introducing two groupoids on sets (i.e., two categories whose morphisms are all isomorphisms).

1.1. Two groupoids on sets. Let Y be an arbitrary scheme. Let Adj-perf$_Y$ be the category whose objects are adjoint group schemes over Y with the property that their Lie algebras $\mathcal{O}_Y$-modules have perfect *Killing forms* (i.e., the Killing forms induce naturally $\mathcal{O}_Y$-linear isomorphisms from them into their duals) and whose morphisms are isomorphisms of group schemes. Let Lie-perf$_Y$ be the category whose objects are Lie algebras $\mathcal{O}_Y$-modules which have perfect Killing forms and which as $\mathcal{O}_Y$-modules are coherent and locally free and whose morphisms are isomorphisms of Lie algebras $\mathcal{O}_Y$-modules.

1.2. Theorem. *Let $\mathcal{L}_Y$: Adj-perf$_Y$ $\to$ Lie-perf$_Y$ be the functor which associates to a morphism $f : G \tilde{\to} H$ of Adj-perf$_Y$ the morphism $df : Lie(G) \tilde{\to} Lie(H)$ of Lie-perf$_Y$ which is the differential of f. Then the functor $\mathcal{L}_Y$ is an equivalence of categories.*

We have a variant of this theorem for simply connected semisimple group schemes instead of adjoint group schemes, cf. Corollary 3.8. This theorem implies the classification of Lie algebras over fields of characteristic at least 3 that have non-degenerate Killing forms obtained previously by Curtis, Seligman, Mills, Block–Zassenhaus, and Brown in [C], [MS], [Mi], [BR], [S], and [Br] (see Remark 3.7 (a)). The functor $\mathcal{L}_Y$ is an equivalence of non-empty categories if and only if Y is a $\operatorname{Spec} Z[\frac{1}{2}]$-scheme, cf. Corollary 3.9. Directly from the Theorem 1.2 we get our first extension result:

1.3. Corollary. *We assume that $Y = \operatorname{Spec} A$ is an affine scheme. Let K be the ring of fractions of A. Let G_K be an adjoint group scheme over $\operatorname{Spec} K$ such that the symmetric bilinear Killing form on the Lie algebra $\operatorname{Lie}(G_K)$ of G_K is perfect (i.e., it induces naturally a K-linear isomorphism $\operatorname{Lie}(G_K) \xrightarrow{\sim} \operatorname{Hom}_K(\operatorname{Lie}(G_K), K)$). We assume that there exists a Lie algebra $\mathfrak{g}$ over A such that the following two properties hold:*

(i) *we have an identity $\operatorname{Lie}(G_K) = \mathfrak{g} \otimes_A K$ and the A-module $\mathfrak{g}$ is projective and finitely generated;*

(ii) *the symmetric bilinear Killing form on $\mathfrak{g}$ is perfect.*

Then there exists a unique adjoint group scheme G over Y which extends G_K and such that we have an identity $\operatorname{Lie}(G) = \mathfrak{g}$ that extends the identity of the property (i).

Let U be an open, Zariski dense subscheme of Y. We call the pair $(Y, Y \setminus U)$ *quasi-pure* if each finite étale cover of U extends uniquely to a finite étale cover of Y (to be compared with [G2, Exp. X, Def. 3.1]).

1.4. Theorem. *We assume that Y is a normal, noetherian scheme and the codimension of $Y \setminus U$ in Y is at least 2. Then the following two properties hold:*

(a) *Let G_U be an adjoint group scheme over U. We assume that the Lie algebra $\mathcal{O}_U$-module $\operatorname{Lie}(G_U)$ of G_U extends to a Lie algebra $\mathcal{O}_Y$-module that is a locally free $\mathcal{O}_Y$-module. Then G_U extends uniquely to an adjoint group scheme G over Y.*

(b) *Let H_U be a reductive group scheme over U. We assume that the pair $(Y, Y \setminus U)$ is quasi-pure and that the Lie algebra $\mathcal{O}_U$-module $\operatorname{Lie}(G_U)$ of the adjoint group scheme G_U of H_U extends to a Lie algebra $\mathcal{O}_Y$-module that is a locally free $\mathcal{O}_Y$-module. Then H_U extends uniquely to a reductive group scheme H over Y.*

The proof of the Theorem 1.2 we include combines the cohomology theory of Lie algebras with a simplified variant of [V1, Claim 2, p. 464] (see Subsections 3.3 and 3.5). The proof of Theorem 1.4 (a) is an application of [CTS, Cor. 6.12] (see Subsection 4.1). The classical purity theorem of Nagata and Zariski (see [G2, Exp. X, Thm. 3.4 (i)]) says that the pair $(Y, Y \setminus U)$ is quasi-pure, provided Y is *regular* and U contains all points of Y of codimension 1 in Y. In such a case, a slightly weaker form of Theorem 1.4 (b) was obtained in [CTS, Thm. 6.13]. In general, the hypotheses of the Theorem 1.4 are needed (see Remarks 4.3). See [MB] (resp. [FC], [V1], [V2], and [VZ]) for different analogues of Theorems 1.4 (a) and (b) for Jacobian (resp. for abelian) schemes. For instance, in [VZ, Cor. 1.5] it is proved that if Y is a regular, formally smooth scheme over the spectrum of a discrete valuation ring of mixed characteristic $(0, p)$ and index of ramification at most

$p-1$ and if U contains all points of Y that are of either characteristic 0 or codimension 1 in Y, then each abelian scheme over U extends uniquely to an abelian scheme over Y.

Section 2 presents notations and basic results. In Section 3 we prove Theorem 1.2. In Section 4 we prove the Theorem 1.4. Section 5 contains four results on extending homomorphisms between reductive group schemes. The first one is an application of Theorem 1.4 (b) (see Proposition 5.1) and the other three are refinements of results of [V1] (see Subsections 5.2 to 5.6; in particular, see Theorem 5.4).

To describe and motivate these four results, we will restrict in this introduction to the context in which $Y = \mathrm{Spec}\, A$ is an integral affine scheme of field of fractions K and we have a closed embedding homomorphism $\rho_K : G_K \hookrightarrow \boldsymbol{GL}_{d,K}$ from the generic fibre G_K of a reductive group scheme G over Y to the general linear group over $\mathrm{Spec}\, K$ of rank d. A fundamental problem of the representation theory is to identify all lattices M of K^d (i.e., all free A-submodules M of K^d rank d) which are G-modules and to identify all cases when the resulting homomorphism $\rho : G \to \boldsymbol{GL}_M$ is a closed embedding (or at least finite). If M is a G-module, then it is also a $\mathrm{Lie}(G)$-module but the converse most often does not hold.

Our first result (see Proposition 5.1) shows in particular that if A is normal, noetherian and if the desired homomorphism ρ exists in codimension 1, then ρ exists and it is finite and even a closed embedding under a natural required condition that pertains to characteristic 2.

Our second result (see Proposition 5.2) says in particular that ρ exists and is a closed embedding provided we have a family $(G_i)_{i=1}^n$ of commuting normal, closed subgroup schemes of G such that the restriction of ρ_K to each $G_{i,K}$ extends to a closed embedding homomorphism $G_i \hookrightarrow \boldsymbol{GL}_M$ between reductive group schemes over Y and moreover we have a direct sum decomposition $\mathrm{Lie}(G_K) = \oplus_{i=1}^n \mathrm{Lie}(G_{i,K})$. The role of this second result is to reduce the existence of ρ to simpler cases in which G is either a torus or a semisimple group scheme whose adjoint is absolutely simple.

Our third result (see Theorem 5.3.2) is a general criterion on the existence of ρ provided G is a split torus of rank 1. The interesting case is when G is a closed subgroup scheme of $\boldsymbol{SL}_{2,Y}$ and M is naturally a $\mathrm{Lie}(\boldsymbol{SL}_{2,Y})$-module.

Our fourth result (see Theorem 5.4) combines all the other three results in order to provide a criterion which guarantees that ρ exists and is a closed embedding provided M is a $\mathrm{Lie}(G)$-module and a few other conditions hold. Though these conditions are technical, currently it is the maximum we can get based on this paper, due to natural and serious limitations that come from the $\boldsymbol{SL}_{2,Y}$ case and thus from the fact that Theorem 5.3.2 can not be improved (see Example 5.3.3). However we have the following practical form of the Theorem 5.4 which is proved in Subsection 5.6.

1.5. Theorem. *Let $p \in \mathbb{N}^*$ be a prime. Let $Y = \mathrm{Spec}\, A$ be a local, integral scheme whose closed point has a residue field of characteristic p and which is either normal or strictly henselian. Let K be the field of fractions of A. Let M be a free A-module of finite rank. Let G_K be a reductive subgroup of $\boldsymbol{GL}_{M\otimes_A K}$ and let $Z^0(G_K)$ be the maximal torus of the center of G_K. Let $\mathfrak{h} := \mathrm{Lie}(G_K^{\mathrm{der}}) \cap \mathrm{Lie}(\boldsymbol{GL}_M)$, the intersection being taken inside $\mathrm{Lie}(\boldsymbol{GL}_{M\otimes_A K})$. Let $Z^0(G)$ and G be the schematic closures of $Z^0(G_K)$ and G_K (respectively) in $\boldsymbol{GL}_M$. We assume that the following three properties hold:*

(i) *$Z^0(G)$ is a closed subgroup scheme of $\boldsymbol{GL}_M$ that is a torus;*

(ii) *the Lie algebra* $\mathfrak{h}$ *is (as an* A*-module) a direct summand of the Lie algebra of* $\boldsymbol{GL}_M$ *and its Killing form is perfect;*

(iii) *there is a maximal torus of* G_K *generated by cocharacters that act on* $M \otimes_A K$ *via the trivial or the identity characters of* $\mathbb{G}_{m,K}$ *(i.e., via weights in the set* $\{0,1\}$*).*

Then G *is a reductive, closed subgroup scheme of* $\boldsymbol{GL}_M$*.*

Our main motivation for Theorems 1.2, 1.4, and 5.4 stems from the meaningful applications to crystalline cohomology one gets by combining them with either Faltings' results of [F, Sect. 4] (see [V1]; see also Subsections 5.4 to 5.6) or de Jong's extension theorem [dJ, Thm. 1.1] (see [V6]). The manuscript [V6] applies this paper to extend our prior work on integral canonical models of Shimura varieties of Hodge type in unramified mixed characteristic $(0,p)$ with $p \geq 5$ (see [V1]), to unramified mixed characteristics $(0,2)$ and $(0,3)$. In addition, this paper can be used to get relevant simplifications of certain parts of the mentioned prior work (see Remark 5.7 for details).

2. Preliminaries

Our notations are gathered in Subsection 2.1. In Subsections 2.2 to 2.5 we include four basic results that are of different nature and that are often used in Sections 3 to 5.

2.1. Notations and conventions. If K is a field, let $\bar{K}$ be an algebraic closure of it. Let H be a reductive group scheme over a scheme S. Let $Z(H)$, H^{der}, H^{ad}, and H^{ab} denote the center, the *derived group* scheme, the adjoint group scheme, and the *abelianization* of H (respectively). We have $H^{\mathrm{ab}} = H/H^{\mathrm{der}}$ and $H^{\mathrm{ad}} = H/Z(H)$. The center $Z(H)$ is a group scheme of *multiplicative type*, cf. [DG, Vol. III, Exp. XXII, Cor. 4.1.7]. Let $Z^0(H)$ be the maximal *torus* of $Z(H)$; the *quotient* group scheme $Z(H)/Z^0(H)$ is a finite, flat group scheme over S of multiplicative type. Let H^{sc} be the simply connected semisimple group scheme cover of the derived group scheme H^{der}.

See [DG, Vol. III, Exp. XXII, Cor. 4.3.2] for the quotient group scheme H/F of H by a flat, closed subgroup scheme F of $Z(H)$ which is of multiplicative type.

If X or X_S is an S-scheme, let X_{A_1} (resp. X_{S_1}) be its pull back via a morphism $\operatorname{Spec} A_1 \to S$ (resp. $S_1 \to S$).

If S is either affine or integral, let K_S be the ring of fractions of S. If S is a normal, noetherian, integral scheme, let $\mathcal{D}(S)$ be the set of local rings of S that are discrete valuation rings.

Let $\mathbb{G}_{m,S}$ be the rank 1 split torus over S; similarly, the group schemes $\mathbb{G}_{a,S}$, $\boldsymbol{GL}_{d,S}$ with $d \in \mathbb{N}^*$, etc., will be understood to be over S. Let $\mathit{Lie}(H)$ be the Lie algebra $\mathcal{O}_S$-module of H. If $S = \operatorname{Spec} A$ is affine, then let $\mathbb{G}_{m,A} := \mathbb{G}_{m,S}$, etc., and let $\mathrm{Lie}(F)$ be the Lie algebra over A of a closed subgroup scheme F of H. As A-modules, we identify $\mathrm{Lie}(F) = \mathrm{Ker}(F(A[x]/x^2) \to F(A))$, where the A-epimorphism $A[x]/(x^2) \twoheadrightarrow A$ takes x to 0. The Lie bracket on $\mathrm{Lie}(F)$ is defined by taking the (total) differential of the commutator morphism $[,] : F \times_S F \to F$ at identity sections. If $S = \operatorname{Spec} A$ is affine, then $\mathrm{Lie}(H) = \mathit{Lie}(H)(S)$ is the Lie algebra over A of global sections of $\mathit{Lie}(H)$ and it is a projective, finitely generated A-module.

If N is a projective, finitely generated A-module, let $N^* := \mathrm{Hom}_A(N, A)$, let $\boldsymbol{GL}_N$ be the reductive group scheme over $\mathrm{Spec}\, A$ of linear automorphisms of N, and let $\mathfrak{gl}_N := \mathrm{Lie}(\boldsymbol{GL}_N)$. Thus $\mathfrak{gl}_N$ is the Lie algebra associated to the A-algebra $\mathrm{End}_A(N)$. A bilinear form $b_N : N \times N \to A$ on N is called *perfect* if it induces an A-linear map $N \to N^*$ that is an isomorphism. If b_N is symmetric, then its *kernel* is the A-submodule

$$\mathrm{Ker}(b_N) := \{a \in N | b_N(a, b) = 0 \; \forall b \in N\}$$

of N. For a Lie algebra $\mathfrak{g}$ over A that is a projective, finitely generated A-module, let $\mathrm{ad} : \mathfrak{g} \to \mathfrak{gl}_{\mathfrak{g}}$ be the adjoint representation of $\mathfrak{g}$ and let $\mathcal{K}_{\mathfrak{g}} : \mathfrak{g} \times \mathfrak{g} \to A$ be the Killing form on $\mathfrak{g}$. For a, $b \in \mathfrak{g}$ we have $\mathrm{ad}(a)(b) = [a, b]$ and $\mathcal{K}_{\mathfrak{g}}(a, b)$ is the trace of the endomorphism $\mathrm{ad}(a) \circ \mathrm{ad}(b)$ of $\mathfrak{g}$. The kernel $\mathrm{Ker}(\mathcal{K}_{\mathfrak{g}})$ is an ideal of $\mathfrak{g}$.

We denote by k an arbitrary field. Let $n \in \mathbb{N}^*$. See [B2, Ch. VI, Sect. 4] and [H1, Ch. III, Sect. 11] for the classification of connected Dynkin diagrams. For

$$\flat \in \{A_n, B_n, C_n | n \in \mathbb{N}^*\} \cup \{D_n | n \geq 3\} \cup \{E_6, E_7, E_8, F_4, G_2\}$$

we say that H is of *isotypic* $\flat$ *Dynkin type* if the connected Dynkin diagram of each simple factor of an arbitrary geometric fibre of H^{ad}, is $\flat$; if H^{ad} is absolutely simple we drop the word isotypic. We recall that $A_1 = B_1 = C_1$, $B_2 = C_2$, and $A_3 = D_3$.

2.2. Proposition. *Let Y be a normal, noetherian, integral scheme. Let $K := K_Y$.*

(a) *If $Y = \mathrm{Spec}\, A$ is affine, then inside the field K we have $A = \cap_{V \in \mathcal{D}(Y)} V$.*

(b) *Let U be an open subscheme of Y such that $Y \setminus U$ has codimension in Y at least 2. Let W be an affine Y-scheme of finite type. Then the natural restriction map $\mathrm{Hom}_Y(Y, W) \to \mathrm{Hom}_Y(U, W)$ is a bijection. If moreover W is integral, normal and such that we have $\mathcal{D}(W) = \mathcal{D}(W_U)$, then W is determined (up to unique isomorphism) by W_U.*

(c) *Suppose that $Y = \mathrm{Spec}\, A$ is local, regular, and has dimension 2. Let y be the closed point of Y and let $U := Y \setminus \{y\}$. Then each locally free $\mathcal{O}_U$-module of finite rank, extends uniquely to a free $\mathcal{O}_Y$-module.*

Proof: See [M, (17.H), Thm. 38] for (a). To check (b), we can assume $Y = \mathrm{Spec}\, A$ is affine. We write $W = \mathrm{Spec}\, B$. The A-algebra of global functions of U is A, cf. (a). We have $\mathrm{Hom}_Y(U, W) = \mathrm{Hom}_A(B, A) = \mathrm{Hom}_Y(Y, W)$. If moreover B is a normal ring and we have $\mathcal{D}(W) = \mathcal{D}(W_U)$, then B is uniquely determined by $\mathcal{D}(W_U)$ (cf. (a)) and therefore by W_U. From this (b) follows. See [G2, Exp. X, Lemma 3.5] for (c). □

2.3. Proposition. *Let G be a reductive group scheme over a scheme Y. Then the functor on the category of Y-schemes that parametrizes maximal tori of G, is representable by a smooth, separated Y-scheme of finite type. Thus locally in the étale topology of Y, G has split, maximal tori.*

Proof: See [DG, Vol. II, Exp. XII, Cor. 1.10] for the first part. The second part follows easily from the first part (see also [DG, Vol. III, Exp. XIX, Prop. 6.1]). □

2.3.1. Lemma. *Let Y be a reduced scheme. Let G be a reductive group scheme over Y. Let $K := K_Y$. Let $f_K : G'_K \to G_K$ be a central isogeny of reductive group schemes over $\mathrm{Spec}\, K$. We assume that either G is split or Y is normal. We have:*

(a) *There exists (up to a canonical identification) at most one central isogeny $f : G' \to G$ that extends $f_K : G'_K \to G_K$. If Y is integral (i.e., K is a field), then there exists a unique central isogeny $f : G' \to G$ that extends $f_K : G'_K \to G_K$.*

(b) *If Y is normal and integral, then G' is the normalization of G in (the field of fractions of) G'_K.*

Proof: We first prove (a) in the case when G is split. Let T be a split, maximal torus of G. We first prove the existence part: thus K is a field. As f_K is a central isogeny, the inverse image T'_K of T_K in G'_K is a split torus. Thus G'_K is split. Let $\mathcal{R}' \to \mathcal{R}$ be the 1-morphism of root data in the sense of [DG, Vol. III, Exp. XXI, Def. 6.8.1] which is associated to the central isogeny $f_K : G'_K \to G_K$ that extends the isogeny $T'_K \to T_K$. Let $\tilde{f} : \tilde{G}' \to G$ be a central isogeny of split, reductive group schemes over Y which extends an isogeny of split tori $\tilde{T}' \to T$ and for which the 1-morphism of root data associated to it and to the isogeny $\tilde{T}' \to T$, is $\mathcal{R}' \to \mathcal{R}$ (cf. [DG, Vol. III, Exp. XXV, Thm. 1.1]). From loc. cit. we also get that there exists an isomorphism $i_K : \tilde{G}'_K \tilde{\to} G'_K$ such that we have $\tilde{f}_K = f_K \circ i_K$. Obviously, i_K is unique. Let G' be the unique group scheme over Y such that i_K extends (uniquely) to an isomorphism $i : \tilde{G}' \tilde{\to} G'$. Let $f := \tilde{f} \circ i^{-1} : G' \to G$; it is a central isogeny.

To check the uniqueness part, we consider two central isogenies $G' \to G$ and $G'_1 \to G$ that extend a central isogeny $f_K : G'_K \to G_K$ (thus $G'_K = G'_{1,K}$). Let G'_2 be the schematic closure of G'_K embedded diagonally into the product $G' \times_Y G'_1$. We are left to check that the two projections $\pi_1 : G'_2 \to G'$ and $\pi_2 : G'_2 \to G'_1$ are isomorphisms as in such a case the composite isomorphism $\pi_2 \circ \pi_1^{-1} : G' \tilde{\to} G'_1$ is an isomorphism that extends the identity $G'_K = G'_{1,K}$. This statement is local for the étale topology of Y and therefore we can assume based on Proposition 2.3 that the inverse images of T to G' and G'_1 are split tori. From this and [DG, Vol. III, Exp. XXIII, Thm. 4.1] we get that there exists a unique isomorphism $\theta : G_1 \tilde{\to} G'_1$ which extends the identity $G'_K = G_K$. This implies that G'_2 is the graph of θ and therefore the two projections π_1 and π_2 are isomorphisms. We conclude that (a) holds if G is split.

We now prove simultaneously (a) and (b) in the case when Y is normal. If a G' as in (a) exists, then it is a smooth scheme over the normal scheme Y and thus it is a normal scheme; from this and the fact that $f : G' \to G$ is a finite morphism, we get that G' is the normalization of G in G'_K and in particular it is unique.

Thus to end the proof of the Lemma, it suffices to show that the normalization G' of G in G'_K is a reductive group scheme equipped with a central isogeny $f : G' \to G$. This is a local statement for the étale topology of Y. As each connected, étale scheme over Y is a normal, integral scheme, based on Proposition 2.3 we can assume that G has a split, maximal torus T. Thus the fact that G' is a reductive group scheme equipped with a central isogeny $f : G' \to G$ follows from the previous three paragraphs. $\square$

2.3.2. Lemma. *Let $Y = \mathrm{Spec}\, A$ be an affine scheme. Let $K := K_Y$. Let T be a torus over Y equipped with a homomorphism $\rho : T \to G$, where G is a reductive group scheme over Y. Then the following three properties hold:*

(a) *the kernel $Ker(\rho)$ is a group scheme over Y of multiplicative type;*

(b) *the kernel $Ker(\rho)$ is trivial (resp. finite) if and only if the kernel $Ker(\rho_K)$ is trivial (resp. is finite);*

(c) *the quotient group scheme $T/Ker(\rho)$ is a torus and we have a closed embedding homomorphism $T/Ker(\rho) \hookrightarrow G$.*

Proof: The statements of the Lemma are local for the étale topology of Y. Thus we can assume that Y is local and (cf. Proposition 2.3) that T and G are split. As Y is connected, the split reductive group scheme G has constant Lie type. Thus G is the pull back to Y of a reductive group scheme $G_{\mathbb{Z}}$ over $\operatorname{Spec}\mathbb{Z}$, cf. [DG, Vol. III, Exp. XXV, Cor. 1.2]. As $G_{\mathbb{Z}}$ can be embedded into a general linear group scheme over $\operatorname{Spec}\mathbb{Z}$ (for instance, cf. [DG, Vol. I, Exp. VI_B, Rm. 11.11.1]), there exists a closed embedding homomorphism $G \hookrightarrow \boldsymbol{GL}_M$, with M a free A-module of rank $d \in \mathbb{N}^*$. By replacing ρ with its composite with this closed embedding homomorphism $G \hookrightarrow \boldsymbol{GL}_M$, we can assume that $G = \boldsymbol{GL}_M$ is a general linear group scheme over Y. The representation of T on M is a finite direct sum of representations of T of rank 1, cf. [J, Part I, Subsect. 2.11]. Thus ρ factors as the composite of a homomorphism $\rho_1 : T \to \mathbb{G}_{m,A}^m$ with a closed embedding homomorphism $\mathbb{G}_{d,A}^m \hookrightarrow \boldsymbol{GL}_M$. The kernel $\mathrm{Ker}(\rho_1)$ is a group scheme over Y of multiplicative type, cf. [DG, Vol. II, Exp. IX, Prop. 2.7 (i)]. As $\mathrm{Ker}(\rho) = \mathrm{Ker}(\rho_1)$, we get that (a) holds. As (a) holds, $\mathrm{Ker}(\rho)$ is flat over Y as well as the extension of a finite, flat group scheme T_1 by a torus T_0. But T_1 (resp. T_0) is a trivial group scheme if and only if $T_{1,K}$ (resp. $T_{0,K}$) is trivial. From this (b) follows. The quotient group scheme $T/\mathrm{Ker}(\rho)$ exists and is a closed subgroup scheme of $\mathbb{G}_{m,A}^m$ that is of multiplicative type, cf. [DG, Vol. II, Exp. IX, Prop. 2.7 (i) and Cor. 2.5]. As the fibres of $T/\mathrm{Ker}(\rho)$ are tori, we get that $T/\mathrm{Ker}(\rho)$ is a torus. Thus (c) holds. $\square$

2.4. Lemma. *Suppose that $k = \bar{k}$. Let H be a reductive group over $\operatorname{Spec} k$. Let $\mathfrak{n}$ be a non-zero ideal of $Lie(H)$ which is a simple left H-module. We assume that there exists a maximal torus T of H such that we have $Lie(T) \cap \mathfrak{n} = 0$. Then $char(k) = 2$ and H^{der} has a normal, subgroup F which is isomorphic to $\boldsymbol{SO}_{2n+1,k}$ for some $n \in \mathbb{N}^*$ and for which we have an inclusion $\mathfrak{n} \subseteq Lie(F)$.*

Proof: This is only a variant of [V4, Lemma 2.1]. $\square$

2.4.1. Remark. If $\mathfrak{n}$ is assumed to be a restricted Lie subalgebra of $\mathrm{Lie}(H)$ (for instance, this holds if $\mathfrak{n}$ is the Lie algebra of a subgroup of H), then there exists a purely inseparable isogeny $H \to H/\mathfrak{n}$ (see [Bo, Ch. V, Prop. 7.4]) and in this case Lemma 2.4 can be also deduced easily from [PY, Lemma 2.2] applied to such isogenies with H^{ad} absolutely simple. In this paper, Lemma 2.4 will be applied only in such situations in which $\mathfrak{n}$ is the Lie algebra of a subgroup of H.

2.5. Theorem. *Let $f : G_1 \to G_2$ be a homomorphism between group schemes over a scheme Y. We assume that G_1 is reductive, that G_2 is separated and of finite presentation, and that all fibres of f are closed embeddings. Then f is a closed embedding.*

Proof: As G_1 is of finite presentation over Y, the homomorphism f is locally of finite type. As the fibres of f are closed embeddings and thus monomorphisms, f itself is a monomorphism (cf. [DG, Vol. I, Exp. VI_B, Cor. 2.11]). Thus the Theorem follows from [DG, Vol. II, Exp. XVI, Cor. 1.5 a)]. □

2.5.1. Lemma. *Let G be an adjoint group scheme over an affine scheme $Y = \mathrm{Spec}\, A$. Let $Aut(G)$ be the group scheme over Y of automorphisms of G. Then the natural adjoint representation $Ad\colon Aut(G) \to \boldsymbol{GL}_{\mathrm{Lie}(G)}$ is a closed embedding.*

Proof: To prove the Lemma, we can work locally in the étale topology of Y and therefore (cf. Proposition 2.3) we can assume that G is split and that Y is connected. We have a short exact sequence $1 \to G \to \mathrm{Aut}(G) \to C \to 1$ that splits (cf. [DG, Vol. III, Exp. XXIV, Thm. 1.3]), where C is a finite, étale, constant group scheme over Y. Thus G is the identity component of $\mathrm{Aut}(G)$ and $\mathrm{Aut}(G)$ is a finite disjoint union of right translates of G via certain Y-valued points of $\mathrm{Aut}(G)$. If the fibres of Ad are closed embeddings, then the restriction of Ad to G is a closed embedding (cf. Theorem 2.5) and thus also the restriction of Ad to any right translate of G via a Y-valued point of $\mathrm{Aut}(G)$ is a closed embedding. The last two sentences imply that Ad is a closed embedding. Thus to end the proof, we are left to check that the fibres of Ad are closed embeddings. For this, we can assume that A is an algebraically closed field.

As G is adjoint and A is a field, the restriction of Ad to G is a closed embedding. Thus the representation Ad is a closed embedding if and only if each element $g \in \mathrm{Aut}(G)(A)$ that acts trivially on $\mathrm{Lie}(G)$, is trivial. We show that the assumption that there exists a non-trivial such element g leads to a contradiction. For this, we can assume that G is absolutely simple and that g is a non-trivial outer automorphism of G. Let T be a maximal torus of a Borel subgroup B of G and let n be the dimension of T.

For $t \in \mathrm{Lie}(T)$, let $C_G(t)$ be its centralizer in G. It is a subgroup of G that contains T. In this paragraph we check that, as G is adjoint, we can choose t such that we have $C_G(t)^0 = T$. We consider the root decomposition $\mathrm{Lie}(G) = \mathrm{Lie}(T) \bigoplus_{\alpha\in\Phi} \mathfrak{g}_\alpha$ with respect to T, where Φ is the root system of G and where each $\mathfrak{g}_\alpha$ is a one dimensional A-vector space normalized by T. Let Δ be the basis for Φ such that we have $\mathrm{Lie}(B) = \mathrm{Lie}(T) \bigoplus_{\alpha\in\Delta} \mathfrak{g}_\alpha$. As G is adjoint, Δ is a basis for the dual A-vector space $\mathrm{Lie}(T)^*$ (to be compared with [DG, Vol. III, Exp. XXI, Def. 6.2.6 and Exp. XXII, Def. 4.3.3]). Thus for each root $\alpha \in \Delta$, $\mathrm{Ker}(\alpha)$ is a A-vector subspace of $\mathrm{Lie}(T)$ of dimension $n-1$. As each $\alpha \in \Phi$ is conjugate under the Weyl group of Φ (equivalently of G) to an element of Δ (see [H1, Ch. III, Sect. 10, Thm.]), we get that for each $\alpha \in \Delta$ its kernel $\mathrm{Ker}(\alpha)$ is a A-vector subspace of $\mathrm{Lie}(T)$ of dimension $n-1$. We choose $t \in \mathrm{Lie}(T) \setminus (\cup_{\alpha\in\Phi}\mathrm{Ker}(\alpha))$. This implies that we have $\mathrm{Lie}(C_G(t)) = \mathrm{Lie}(T)$. From this and the fact that T is a subgroup of $C_G(t)$, we get that $C_G(t)$ is a smooth group of dimension n and therefore that $C_G(t)^0 = T$.

As g fixes t and $\mathrm{Lie}(B)$, g normalizes both $C_G(t)^0 = T$ and B. But it is well known that a non-trivial outer automorphism g of G that normalizes both T and B, can not fix $\mathrm{Lie}(B)$. Contradiction. Thus Ad is a closed embedding. □

We follow the ideas of [V1, Prop. 3.1.2.1 c) and Rm. 3.1.2.2 3)] in order to prove the next Proposition.

2.5.2. Proposition. *Let V be a discrete valuation ring whose residue field is k. Let $Y = \mathrm{Spec}\, V$ and let $K := K_Y$. Let $f : H_1 \to H_2$ be a homomorphism between flat, finite* *type, affine group schemes over Y such that H_1 is a reductive group scheme and the generic fibre $f_K : H_{1,K} \to H_{2,K}$ of f is a closed embedding. We have:*

(a) *The subgroup scheme $Ker(f_k : H_{1,k} \to H_{2,k})$ of $H_{1,k}$ has a trivial intersection with each torus $T_{1,k}$ of $H_{1,k}$. In particular, we have $Lie(Ker(f_k)) \cap Lie(T_{1,k}) = 0$.*

(b) *The homomorphism f is finite.*

(c) *If $char(k) = 2$, we assume that $H_{1,K}$ has no normal subgroup that is adjoint of isotypic B_n Dynkin type for some $n \in \mathbb{N}^*$. Then f is a closed embedding.*

Proof: Let $\rho : H_2 \hookrightarrow \boldsymbol{GL}_M$ be a closed embedding homomorphism, with M a free V-module of finite rank (cf. [DG, Vol. I, Exp. VI_B, Rm. 11.11.1]). To prove the Proposition we can assume that V is complete, that $k = \bar{k}$, and that $f_K : H_{1,K} \to H_{2,K}$ is an isomorphism. Let $H_{0,k} := \mathrm{Ker}(f_k)$. We now show that the group scheme $H_{0,k} \cap T_{1,k}$ is trivial by adapting arguments from [V1, Rm. 3.1.2.2 3) and proof of Lemma 3.1.6]. As V is strictly henselian, the maximal torus $T_{1,k}$ of $H_{1,k}$ is split and (cf. Proposition 2.3) it lifts to a maximal torus T_1 of H_1. The restriction of $\rho \circ f$ to T_1 has a trivial kernel (as its fibre over $\mathrm{Spec}\, K$ is trivial, cf. Lemma 2.3.2 (b)) and therefore it is a closed embedding (cf. Lemma 2.3.2 (c)). Thus the restriction of f to T_1 is a closed embedding homomorphism $T_1 \hookrightarrow H_2$. Therefore the intersection $H_{0,k} \cap T_{1,k}$ is a trivial group scheme. Thus (a) holds.

We check (b). The identity component of the reduced scheme of $\mathrm{Ker}(f_k)$ is a reductive group that has 0 rank (cf. (a)) and therefore it is a trivial group. Thus f is a quasi-finite, birational morphism. From Zariski Main Theorem (see [G1, Thm. (8.12.6)]) we get that H_1 is an open subscheme of the normalization H_2^{n} of H_2. Let H_3 be the smooth locus of H_2^{n} over $\mathrm{Spec}\, V$; it is an open subscheme of H_2^{n} that contains H_1. As H_3 is an open subscheme of the affine scheme H_2^{n}, it is a quasi-affine scheme.

As H_3 is smooth over $\mathrm{Spec}\, V$, the products $H_3 \times_{\mathrm{Spec}\, V} H_2^{\mathrm{n}}$ and $H_2^{\mathrm{n}} \times_{\mathrm{Spec}\, V} H_3$ are smooth over H_2^{n} and thus are normal schemes. The product $H_2^{\mathrm{n}} \times_{\mathrm{Spec}\, V} H_2^{\mathrm{n}}$ is a flat scheme over $\mathrm{Spec}\, V$ whose generic fibre is smooth over $\mathrm{Spec}\, K$. Its normalization $(H_2^{\mathrm{n}} \times_{\mathrm{Spec}\, V} H_2^{\mathrm{n}})^{\mathrm{n}}$ contains both $H_3 \times_{\mathrm{Spec}\, V} H_2^{\mathrm{n}}$ and $H_2^{\mathrm{n}} \times_{\mathrm{Spec}\, V} H_3$ as open subschemes and is equipped with a finite surjective morphism $(H_2^{\mathrm{n}} \times_{\mathrm{Spec}\, V} H_2^{\mathrm{n}})^{\mathrm{n}} \to H_2^{\mathrm{n}} \times_{\mathrm{Spec}\, V} H_2^{\mathrm{n}}$ whose generic fibre is an isomorphism. The product morphism $H_2 \times_{\mathrm{Spec}\, V} H_2 \to H_2$ induces a natural product type of morphism $\Theta : (H_2^{\mathrm{n}} \times_{\mathrm{Spec}\, V} H_2^{\mathrm{n}})^{\mathrm{n}} \to H_2^{\mathrm{n}}$. Its restrictions to $H_3 \times_{\mathrm{Spec}\, V} H_2^{\mathrm{n}}$ and $H_2^{\mathrm{n}} \times_{\mathrm{Spec}\, V} H_3$ induce product type of morphisms $H_3 \times_{\mathrm{Spec}\, V} H_2^{\mathrm{n}} \to H_2^{\mathrm{n}}$ and $H_2^{\mathrm{n}} \times_{\mathrm{Spec}\, V} H_3 \to H_2^{\mathrm{n}}$. This implies that for each valued point $z \in H_2^{\mathrm{n}}(V)$ it makes sense to speak about the left zH_3 and the right $H_3 z$ translations of H_3 through z; they are smooth open subschemes of H_2^{n} and thus of H_3. This implies that $H_3(V) = H_2^{\mathrm{n}}(V)$ and that Θ restricts to a product morphism $H_3 \times_{\mathrm{Spec}\, V} H_3 \to H_3$. The inverse automorphisms of the $\mathrm{Spec}\, V$-schemes H_1 and H_2 induce an inverse automorphism of the $\mathrm{Spec}\, V$-scheme H_2^{n} which restricts to an inverse automorphism of the $\mathrm{Spec}\, V$-scheme H_3. With respect to its product morphism, its inverse automorphism, and its identity section inherited from H_1, H_3 gets the structure of a (quasi-affine) group scheme over $\mathrm{Spec}\, V$ that is of finite type.

As V is complete, it is also excellent (cf. [M, Sect. 34]). Thus the morphism $H_2^{\mathrm{n}} \to H_2$ is finite. The homomorphism f is finite if and only if $H_1 = H_2^{\mathrm{n}}$ and thus if and only if the set $H_2^{\mathrm{n}}(k) \setminus H_1(k)$ is empty. We show that the assumption that $H_1 \neq H_2^{\mathrm{n}}$ leads to a contradiction. Let $x \in H_2^{\mathrm{n}}(k) \setminus H_1(k)$. From [V5, Lemma 4.1.5] applied to the completion of the local ring of x in H_2^{n}, we get that there exists a finite, flat discrete valuation ring extension V' of V for which we have a valued point $z' \in H_2^{\mathrm{n}}(V')$ that lifts x (we recall that loc. cit. is only a local version of the global result [G1, Cor. (17.16.2)]). The flat $\mathrm{Spec}\, V'$-scheme $H_{2,V'}^{\mathrm{n}}$ might not be normal but we have $H_1 \neq H_2^{\mathrm{n}}$ if and only if $H_{1,V'} \neq H_{2,V'}^{\mathrm{n}}$. Thus to reach a contradiction we can replace V by V' and therefore we can assume that x is such that there exists a valued point $z \in H_2^{\mathrm{n}}(V) = H_3(V)$ which lifts x. As $x \in H_2^{\mathrm{n}}(k) \setminus H_1(k)$, we have $z \in H_3(V) \setminus H_1(V)$. As H_1 is a subgroup scheme of H_3, all fibers of the homomorphism $H_1 \to H_3$ are closed. From this and Theorem 2.5 we get that H_1 is a closed subscheme of H_3. Thus, as H_3 is an integral scheme and as $H_{3,K} = H_{1,K}$, we get that $H_1 = H_3$. This contradicts the fact that $z \in H_3(V) \setminus H_1(V)$. Thus (b) holds.

We check (c). We show that the assumption that $\mathrm{Lie}(H_{0,k}) \neq 0$ leads to a contradiction. From Lemma 2.4 applied to $H_{1,k}$ and to any simple $H_{1,k}$-submodule of the left $H_{1,k}$-module $\mathrm{Lie}(H_{0,k})$, we get that $\mathrm{char}(k) = 2$ and that $H_{1,k}$ has a normal subgroup $H_{4,k}$ isomorphic to $\boldsymbol{SO}_{2n+1,k}$ for some $n \in \mathbb{N}^*$. As $H_{4,k}$ is adjoint, we have a product decomposition $H_{1,k} = H_{4,k} \times_{\mathrm{Spec}\, k} H_{5,k}$ of reductive groups. It lifts (cf. [DG, Vol. III, Exp. XXIV, Prop. 1.21]) to a product decomposition $H_1 = H_4 \times_{\mathrm{Spec}\, V} H_5$, where H_4 is isomorphic to $\boldsymbol{SO}_{2n+1,V}$ and where H_5 is a reductive group scheme over $\mathrm{Spec}\, V$. This contradicts the extra hypothesis of (c). Thus we have $\mathrm{Lie}(H_{0,k}) = 0$. Therefore $H_{0,k}$ is a finite, étale, normal subgroup of $H_{1,k}$. But $H_{1,k}$ is connected and thus its action on $H_{0,k}$ via inner conjugation is trivial. Therefore we have $H_{0,k} \leqslant Z(H_1)_k \leqslant T_{1,k}$. Thus $H_{0,k} = H_{0,k} \cap T_{1,k}$ is the trivial group, cf. (a). In other words, the homomorphism $f_k : H_{1,k} \to H_{2,k}$ is a closed embedding. Thus $f : H_1 \to H_2$ is a closed embedding homomorphism, cf. Theorem 2.5. $\square$

2.5.3. Remark. See [V3, Thm. 1.2 (b)] and [PY, Thm. 1.2] for two other proofs of Proposition 2.5.2 (c).

3. Lie algebras with perfect Killing forms

Let A be a commutative $\mathbb{Z}$-algebra. Let $\mathfrak{g}$ be a Lie algebra over A which as an A-module is projective and finitely generated. In this Section we will assume that the Killing form $\mathcal{K}_{\mathfrak{g}}$ on $\mathfrak{g}$ is perfect. Let $U_{\mathfrak{g}}$ be the enveloping algebra of $\mathfrak{g}$ i.e., the quotient of the tensor algebra $T_{\mathfrak{g}}$ of $\mathfrak{g}$ by the two-sided ideal of $T_{\mathfrak{g}}$ generated by the subset $\{x \otimes y - y \otimes x - [x,y] | x, y \in \mathfrak{g}\}$ of $T_{\mathfrak{g}}$. Let $Z(U_{\mathfrak{g}})$ be the center of $U_{\mathfrak{g}}$. The categories of left $\mathfrak{g}$-modules and of left $U_{\mathfrak{g}}$-modules are canonically identified. We view $\mathfrak{g}$ as a left $\mathfrak{g}$-module via the adjoint representation $\mathrm{ad} : \mathfrak{g} \to \mathfrak{gl}_{\mathfrak{g}}$; let $\mathrm{ad} : U_{\mathfrak{g}} \to \mathrm{End}(\mathfrak{g})$ be the A-homomorphism corresponding to the left $\mathfrak{g}$-module $\mathfrak{g}$. We refer to [CE, Ch. XIII] for the cohomology groups $H^i(\mathfrak{g}, \mathfrak{v})$ of a left $\mathfrak{g}$-module $\mathfrak{v}$ (here $i \in \mathbb{N}$). We denote also by $\mathcal{K}_{\mathfrak{g}} : \mathfrak{g} \otimes_A \mathfrak{g} \to A$ the A-linear map defined by $\mathcal{K}_{\mathfrak{g}} : \mathfrak{g} \times \mathfrak{g} \to A$. Thus we have $\mathcal{K}_{\mathfrak{g}} \in (\mathfrak{g} \otimes_A \mathfrak{g})^* = \mathfrak{g}^* \otimes_A \mathfrak{g}^*$. Let $\phi : \mathfrak{g} \xrightarrow{\sim} \mathfrak{g}^*$ be the A-linear isomorphism defined naturally by $\mathcal{K}_{\mathfrak{g}}$. It induces an A-linear isomorphism $\phi^{-1} \otimes \phi^{-1} : \mathfrak{g}^* \otimes_A \mathfrak{g}^* \xrightarrow{\sim} \mathfrak{g} \otimes_A \mathfrak{g}$. The image Ω of $\phi^{-1} \otimes \phi^{-1}(\mathcal{K}_{\mathfrak{g}}) \in \mathfrak{g} \otimes_A \mathfrak{g} \subseteq T_{\mathfrak{g}}$ in $U_{\mathfrak{g}}$ is called the Casimir element of the adjoint representation $\mathrm{ad} : \mathfrak{g} \to \mathfrak{gl}_{\mathfrak{g}}$.

3.1. Lemma. *For the Casimir element $\Omega \in U_{\mathfrak{g}}$ the following four properties hold:*

(a) *if the A-module $\mathfrak{g}$ is free and if $\{x_1,\ldots,x_m\}$ and $\{y_1,\ldots,y_m\}$ are two A-bases for $\mathfrak{g}$ such that for all $i,j\in\{1,\ldots,m\}$ we have $\mathcal{K}_{\mathfrak{g}}(x_i\otimes y_j)=\delta_{ij}$, then Ω is the image of the element $\sum_{i=1}^m x_i\otimes y_i$ of $T_{\mathfrak{g}}$ in $U_{\mathfrak{g}}$;*

(b) *we have $\Omega\in Z(U_{\mathfrak{g}})$;*

(c) *the Casimir element Ω is fixed by the group of Lie automorphisms of $\mathfrak{g}$ (i.e., if $\sigma : U_{\mathfrak{g}}\tilde{\to}U_{\mathfrak{g}}$ is the A-algebra automorphism induced by a Lie algebra automorphism $\sigma:\mathfrak{g}\tilde{\to}\mathfrak{g}$, then we have $\sigma(\Omega)=\Omega$);*

(d) *the Casimir element Ω acts identically on $\mathfrak{g}$ (i.e., $ad(\Omega)=1_{\mathfrak{g}}$).*

Proof: Parts (a) and (b) are proved in [B1, Ch. I, Sect. 3, Subsect. 7, Prop. 11]. Strictly speaking, loc. cit. is stated over a field but its proof applies over any commutative $\mathbb{Z}$-algebra. This is so as the essence of the proof of loc. cit. is [B1, Ch. I, Sect. 3, Subsect. 5, Example 2] which is worked out over any commutative $\mathbb{Z}$-algebra. In particular, [B1, Ch. I, Sect. 3, Subsect. 5, Example 3] can be easily stated over a commutative $\mathbb{Z}$-algebra (by involving a perfect invariant bilinear form over a commutative $\mathbb{Z}$-algebra instead of a non-degenerate invariant bilinear form over a field). We recall here that $\mathcal{K}_{\mathfrak{g}}$ is $\mathfrak{g}$-invariant i.e., for all $a,b,c\in\mathfrak{g}$ we have an identity $\mathcal{K}_{\mathfrak{g}}(\mathrm{ad}(a)(b),c)+\mathcal{K}_{\mathfrak{g}}(b,\mathrm{ad}(a)(c))=0$ (see [B1, Ch. I, Sect. 3, (13) and Prop. 8]) and this is the very essence of (b).

To check (c) and (d), we can assume that the A-module $\mathfrak{g}$ is free. Let $\{x_1,\ldots,x_m\}$ and $\{y_1,\ldots,y_m\}$ be two A-bases for $\mathfrak{g}$ as in (a). Thus Ω is the image of $\sum_{i=1}^m x_i\otimes y_i$ in $U_{\mathfrak{g}}$. Therefore $\sigma(\Omega)$ is the image of $\sum_{i=1}^m \sigma(x_i)\otimes\sigma(y_i)$ in $U_{\mathfrak{g}}$. As for $i,j\in\{1,\ldots,m\}$ we have $\mathcal{K}_{\mathfrak{g}}(\sigma(x_i),\sigma(y_j))=\delta_{ij}$, from (a) we get that the image of $\sum_{i=1}^m\sigma(x_i)\otimes\sigma(y_i)\in T_{\mathfrak{g}}$ in $U_{\mathfrak{g}}$ is Ω. Thus $\sigma(\Omega)=\Omega$.

We check (d). Let $z,w\in\mathfrak{g}$. We write $\mathrm{ad}(z)\circ\mathrm{ad}(w)(x_i)=\sum_{j=1}^m a_{ji}x_j$, with a_{ji}'s in A. Using the $\mathfrak{g}$-invariance of $\mathcal{K}_{\mathfrak{g}}$ we compute:

$$\mathcal{K}_{\mathfrak{g}}(\mathrm{ad}(\Omega)(z),w)=\mathcal{K}_{\mathfrak{g}}(\sum_{i=1}^m\mathrm{ad}(x_i)\circ\mathrm{ad}(y_i)(z),w)=-\sum_{i=1}^m\mathcal{K}_{\mathfrak{g}}(\mathrm{ad}(y_i)(z),\mathrm{ad}(x_i)(w))$$

$$=\sum_{i=1}^m\mathcal{K}_{\mathfrak{g}}(\mathrm{ad}(z)(y_i),\mathrm{ad}(x_i)(w))=-\sum_{i=1}^m\mathcal{K}_{\mathfrak{g}}(y_i,\mathrm{ad}(z)\circ\mathrm{ad}(x_i)(w))$$

$$=\sum_{i=1}^m\mathcal{K}_{\mathfrak{g}}(y_i,\mathrm{ad}(z)\circ\mathrm{ad}(w)(x_i))=\sum_{i,j=1}^m a_{ji}\delta_{ji}=\sum_{i=1}^m a_{ii}=\mathcal{K}_{\mathfrak{g}}(z,w)$$

(the last equality due to the very definition of $\mathcal{K}_{\mathfrak{g}}$). This implies that for each $z\in\mathfrak{g}$, we have $\mathrm{ad}(\Omega)(z)-z\in\mathrm{Ker}(\mathcal{K}_{\mathfrak{g}})=0$. Thus $\mathrm{ad}(\Omega)(z)=z$ i.e., (d) holds. $\square$

3.2. Fact. *Let $i\in\mathbb{N}$. Let $\mathfrak{v}$ be a left $\mathfrak{g}$-module on which Ω acts identically. Then the cohomology group $H^i(\mathfrak{g},\mathfrak{v})$ is trivial.*

Proof: We have an identity $H^i(\mathfrak{g},\mathfrak{v})=\mathrm{Ext}^i_{U_{\mathfrak{g}}}(A,\mathfrak{v})$ of $Z(U_{\mathfrak{g}})$-modules, cf. [CE, Ch. XIII, Sects. 2 and 8]. As $\Omega\in Z(U_{\mathfrak{g}})$ acts trivially on A and identically on $\mathfrak{v}$, the group $\Omega\mathrm{Ext}^i_{U_{\mathfrak{g}}}(A,\mathfrak{v})$ is on one hand trivial and on the other hand it is $\mathrm{Ext}^i_{U_{\mathfrak{g}}}(A,\mathfrak{v})$. Thus $\mathrm{Ext}^i_{U_{\mathfrak{g}}}(A,\mathfrak{v})=0$. Therefore $H^i(\mathfrak{g},\mathfrak{v})=0$. $\square$

3.3. Theorem. *We recall that the Killing form $\mathcal{K}_{\mathfrak{g}}$ on $\mathfrak{g}$ is perfect. Then the group scheme $Aut(\mathfrak{g})$ over $Spec\,A$ of Lie automorphisms of $\mathfrak{g}$ is smooth and locally of finite presentation.*

Proof: To check this, we can assume that the A-module $\mathfrak{g}$ is free. The group scheme $\mathrm{Aut}(\mathfrak{g})$ is a closed subgroup scheme of $\boldsymbol{GL}_{\mathfrak{g}}$ defined by a finitely generated ideal of the ring of functions of $\boldsymbol{GL}_{\mathfrak{g}}$. Thus $\mathrm{Aut}(\mathfrak{g})$ is of finite presentation. Thus to show that $\mathrm{Aut}(\mathfrak{g})$ is smooth over $\mathrm{Spec}\,A$, it suffices to show that for each affine morphism $\mathrm{Spec}\,B \to \mathrm{Spec}\,A$ and for each ideal $\mathfrak{j}$ of B such that $\mathfrak{j}^2 = 0$, the restriction map $\mathrm{Aut}(\mathfrak{g})(B) \to \mathrm{Aut}(\mathfrak{g})(B/\mathfrak{j})$ is onto (cf. [BLR, Ch. 2, Sect. 2.2, Prop. 6]). Not to introduce extra notations by repeatedly tensoring with B over A, we will assume that $B = A$. Thus $\mathfrak{j}$ is an ideal of A and we have to show that the restriction map $\mathrm{Aut}(\mathfrak{g})(A) \to \mathrm{Aut}(\mathfrak{g})(A/\mathfrak{j})$ is onto.

Let $\bar{\sigma} : \mathfrak{g}/\mathfrak{j}\mathfrak{g} \xrightarrow{\sim} \mathfrak{g}/\mathfrak{j}\mathfrak{g}$ be a Lie automorphism. Let $\sigma_0 : \mathfrak{g} \xrightarrow{\sim} \mathfrak{g}$ be an A-linear automorphism that lifts $\bar{\sigma}$. Let $\mathfrak{j}\mathfrak{g}_{\bar{\sigma}}$ be the left $\mathfrak{g}$-module which as an A-module is $\mathfrak{j}\mathfrak{g}$ and whose left $\mathfrak{g}$-module structure is defined as follows: if $x \in \mathfrak{g}$, then x acts on $\mathfrak{j}\mathfrak{g}_{\bar{\sigma}}$ in the same way as $\mathrm{ad}(\bar{\sigma}(x))$ (equivalently, as $\mathrm{ad}(\sigma_0(x))$) acts on the A-module $\mathfrak{j}\mathfrak{g} = \mathfrak{j}\mathfrak{g}_{\bar{\sigma}}$; this makes sense as $\mathfrak{j}^2 = 0$. Let $\theta : \mathfrak{g} \times \mathfrak{g} \to \mathfrak{j}\mathfrak{g}_{\bar{\sigma}}$ be the alternating map defined by the rule:

$$\theta(x, y) := [\sigma_0(x), \sigma_0(y)] - \sigma_0([x, y]) \quad \forall\, x, y \in \mathfrak{g}. \tag{1}$$

We check that θ is a 2-cocycle i.e., for all $x, y, z \in \mathfrak{g}$ we have an identity

$$d\theta(x, y, z) := x(\theta(y, z)) - y(\theta(x, z)) + z(\theta(x, y)) - \theta([x, y], z) + \theta([x, z], y) - \theta([y, z], x) = 0.$$

Substituting (1) in the definition of $d\theta$, we get that the expression $d\theta(x, y, z)$ is a sum of 12 terms which can be divided into three groups as follows. The first group contains the three terms $-\sigma_0([[x, y], z])$, $\sigma_0([[x, z], y])$, and $-\sigma_0([[y, z], x])$; their sum is 0 due to the Jacobi identity and the fact that σ_0 is an A-linear map. The second group contains the six terms $[\sigma_0(x), \sigma_0[y, z]]$, $-[\sigma_0(x), \sigma_0[y, z]]$, $[\sigma_0(y), \sigma_0[x, z]]$, $-[\sigma_0(y), \sigma_0[x, z]]$, $[\sigma_0(z), \sigma_0[x, y]]$, $-[\sigma_0(z), \sigma_0[x, y]]$; obviously their sum is 0. The third group contains the three terms $[\sigma_0(x), [\sigma_0(y), \sigma_0(z)]]$, $-[\sigma_0(y), [\sigma_0(x), \sigma_0(z)]]$, and $[\sigma_0(z), [\sigma_0(x), \sigma_0(y)]]$; their sum is 0 due to the Jacobi identity. Thus indeed $d\theta = 0$.

As Ω (i.e., $\mathrm{ad}(\Omega)$) acts identically on $\mathfrak{g}$ (cf. Lemma 3.1 (d)), it also acts identically on $\mathfrak{j}\mathfrak{g}$. But Ω modulo $\mathfrak{j}$ is fixed by the Lie automorphism $\bar{\sigma}$ of $\mathfrak{g}/\mathfrak{j}\mathfrak{g}$, cf. Lemma 3.1 (c). Thus Ω also acts identically on the left $\mathfrak{g}$-module $\mathfrak{j}\mathfrak{g}_{\bar{\sigma}}$. From this and the Fact 3.2 we get that $H^2(\mathfrak{g}, \mathfrak{j}\mathfrak{g}_{\bar{\sigma}}) = 0$. Thus θ is the coboundary of a 1-cochain $\delta : \mathfrak{g} \to \mathfrak{j}\mathfrak{g}_{\bar{\sigma}}$ i.e., we have

$$\theta(x, y) = x(\delta(y)) - y(\delta(x)) - \delta([x, y]) \quad \forall x,\, y \in \mathfrak{g}. \tag{2}$$

Let $\sigma : \mathfrak{g} \xrightarrow{\sim} \mathfrak{g}$ be the A-linear isomorphism defined by the rule $\sigma(x) := \sigma_0(x) - \delta(x)$; here $\delta(x)$ is an element of the A-module $\mathfrak{j}\mathfrak{g} = \mathfrak{j}\mathfrak{g}_{\bar{\sigma}}$. Due to formulas (1) and (2), we compute

$$\sigma([x, y]) = \sigma_0([x, y]) - \delta([x, y]) = [\sigma_0(x), \sigma_0(y)] - \theta(x, y) - \delta([x, y])$$

$$= [\sigma_0(x), \sigma_0(y)] - x(\delta(y)) + y(\delta(x)) = [\sigma_0(x), \sigma_0(y)] - \mathrm{ad}(\bar{\sigma}(x))(\delta(y)) + \mathrm{ad}(\bar{\sigma}(y))(\delta(x))$$

$$= [\sigma_0(x), \sigma_0(y)] - \mathrm{ad}(\sigma_0(x))(\delta(y)) + \mathrm{ad}(\sigma_0)(y)(\delta(x)) = [\sigma_0(x), \sigma_0(y)] - [\sigma_0(x), \delta(y)] + [\sigma_0(y), \delta(x)]$$

$$= [\sigma_0(x)-\delta(x), \sigma_0(y)-\delta(y)] - [\delta(x),\delta(y)] = [\sigma(x),\sigma(y)] - [\delta(x),\delta(y)] = [\sigma(x),\sigma(y)]$$

(the last identity as $\mathfrak{j}^2 = 0$). Thus σ is a Lie automorphism of $\mathfrak{g}$ that lifts the Lie automorphism $\bar{\sigma}$ of $\mathfrak{g}/\mathfrak{j}\mathfrak{g}$. Thus the restriction map $\mathrm{Aut}(\mathfrak{g})(A) \to \mathrm{Aut}(\mathfrak{g})(A/\mathfrak{j})$ is onto. $\square$

3.5. Proof of the Theorem 1.2. The functor $\mathcal{L}_Y$ is faithful, cf. Lemma 2.5.1. Thus to prove the Theorem 1.2 it suffices to show that $\mathcal{L}_Y$ is surjective on objects and that $\mathcal{L}_Y$ is fully faithful. To check this, as Adj-perf$_Y$ and Lie-perf$_Y$ are groupoids on sets and as $\mathcal{L}_Y$ is faithful, we can assume that $Y = \mathrm{Spec}\, A$ is affine. Thus to end the proof it suffices to check the following three properties:

(i) if $\mathfrak{g}$ is an object of Lie-perf$_Y$ (identified with a Lie algebra over A), then there exists a unique open subgroup scheme $\mathrm{Aut}(\mathfrak{g})^0$ of $\mathrm{Aut}(\mathfrak{g})$ which is an adjoint group scheme over Y and whose Lie algebra is the Lie subalgebra $\mathrm{ad}(\mathfrak{g})$ of $\mathfrak{gl}_{\mathfrak{g}}$ (therefore $\mathfrak{g} = \mathrm{ad}(\mathfrak{g})$ is the image through $\mathcal{L}_Y$ of the object $\mathrm{Aut}(\mathfrak{g})^0$ of Adj-perf$_Y$);

(ii) the group scheme $\mathrm{Aut}(\mathrm{Aut}(\mathfrak{g})^0)$ of automorphisms of $\mathrm{Aut}(\mathfrak{g})^0$ is $\mathrm{Aut}(\mathfrak{g})$ acting on $\mathrm{Aut}(\mathfrak{g})^0$ via inner conjugation (therefore $\mathrm{Aut}(\mathfrak{g})(A) = \mathrm{Aut}(\mathrm{Aut}(\mathfrak{g})^0)(A)$);

(iii) if G and H are two objects of Adj-perf$_Y$ such that $\mathrm{Lie}(G) = \mathrm{Lie}(H)$, then G and H are isomorphic.

To check the first two properties, we can assume that the A-module $\mathfrak{g}$ is free of rank $m \in \mathbb{N}^*$. Let k be the residue field of an arbitrary point $y \in Y$. It is well known that the Lie algebra $\mathrm{Lie}(\mathrm{Aut}(\mathfrak{g})_k)$ is the Lie algebra of derivations of $\mathfrak{g}_k := \mathfrak{g} \otimes_A k$. As this fact plays a key role in this paper, we include a proof of it. The tangent space of $\mathrm{Aut}(\mathfrak{g})_k$ at the identity element is identified with the set of automorphisms a of $\mathfrak{g}_k \otimes_k k[\varepsilon]/(\varepsilon^2)$ which modulo $\bar{\varepsilon} = \varepsilon + (\varepsilon^2)$ are the identity automorphism of $\mathfrak{g}_k$. We can write each such automorphism as $a = 1_{\mathfrak{g}_k \otimes_k k[\varepsilon]/(\varepsilon^2)} + D_a \otimes \bar{\varepsilon}$, where D_a is a k-linear endomorphism of $\mathfrak{g}_k$. The condition that a respects the Lie bracket (i.e., we have $a([u,v]\otimes 1) = [a(u\otimes 1), a(v \otimes 1)]$ for all $u, v \in \mathfrak{g}_k$) is equivalent to the condition that D_a is a derivation of $\mathfrak{g}_k$. The association $a \mapsto D_a$ identifies the tangent space of $\mathrm{Aut}(\mathfrak{g})_k$ at the identity element with the k-vector space of derivations of $\mathfrak{g}_k$. Under this identification, the Lie bracket of a with an automorphism b of $\mathfrak{g}_k \otimes_k k[\varepsilon_1]/(\varepsilon_1^2)$ which modulo $\bar{\varepsilon}_1 = \varepsilon_1 + (\varepsilon_1^2)$ is the identity automorphism of $\mathfrak{g}_k$, is the derivation of $\mathfrak{g}_k$ which corresponds to the automorphism $aba^{-1}b^{-1} = 1_{\mathfrak{g}_k \otimes k[\varepsilon\varepsilon_1]/(\varepsilon^2\varepsilon_1^2)} + [D_a, D_b]\bar{\varepsilon}\bar{\varepsilon}_1$ of $\mathfrak{g}_k \otimes_k k[\varepsilon\varepsilon_1]/(\varepsilon^2\varepsilon_1^2)$ and thus is the Lie bracket $[D_a, D_b]$ (ε_1 is used here instead of ε so that this last part makes sense). Therefore $\mathrm{Lie}(\mathrm{Aut}(\mathfrak{g})_k)$ is the Lie algebra of derivations of $\mathfrak{g}_k$.

As the Killing form $\mathcal{K}_{\mathfrak{g}_k}$ is perfect, as in [H1, Ch. II, Subsect. 5.3, Thm.] one argues that each derivation of $\mathfrak{g}_k$ is an inner derivation. Thus we have $\mathrm{Lie}(\mathrm{Aut}(\mathfrak{g})_k) = \mathrm{ad}(\mathfrak{g}) \otimes_A k$. As the group scheme $\mathrm{Aut}(\mathfrak{g})$ over Y is smooth and locally of finite presentation (cf. Theorem 3.3), from [DG, Vol. I, Exp. VI$_B$, Cor. 4.4] we get that there exists a unique open subgroup scheme $\mathrm{Aut}(\mathfrak{g})^0$ of $\mathrm{Aut}(\mathfrak{g})$ whose fibres are connected. The fibres of $\mathrm{Aut}(\mathfrak{g})^0$ are open-closed subgroups of the fibres of $\mathrm{Aut}(\mathfrak{g})$ and thus are affine.

Let N_k be a smooth, connected, unipotent, normal subgroup of $\mathrm{Aut}(\mathfrak{g})^0_k$. The Lie algebra $\mathrm{Lie}(N_k)$ is a nilpotent ideal of $\mathrm{Lie}(\mathrm{Aut}(\mathfrak{g})^0_k) = \mathrm{ad}(\mathfrak{g}) \otimes_A k$. Thus $\mathrm{Lie}(N_k) \subseteq \mathrm{Ker}(\mathcal{K}_{\mathrm{Lie}(\mathrm{Aut}(\mathfrak{g})^0_k)}) = \mathrm{Ker}(\mathcal{K}_{\mathrm{ad}(\mathfrak{g})\otimes_A k})$, cf. [B1, Ch. I, Sect. 4, Prop. 6 (b)]. As the Killing form $\mathcal{K}_{\mathrm{ad}(\mathfrak{g})\otimes_A k}$ is perfect, we get $\mathrm{Lie}(N_k) = 0$. Thus N_k is the trivial subgroup of

$\mathrm{Aut}(\mathfrak{g})_k^0$ and therefore the unipotent radical of $\mathrm{Aut}(\mathfrak{g})_k^0$ is trivial. Thus $\mathrm{Aut}(\mathfrak{g})_k^0$ is an affine, connected, smooth group over $\mathrm{Spec}\, k$ whose unipotent radical is trivial. Therefore $\mathrm{Aut}(\mathfrak{g})_k^0$ is a reductive group over $\mathrm{Spec}\, k$, cf. [Bo, Ch. IV, Subsect. 11.21]. As $\mathrm{Lie}(\mathrm{Aut}(\mathfrak{g})_k^0) = \mathrm{ad}(\mathfrak{g}) \otimes_A k$ has trivial center, the group $\mathrm{Aut}(\mathfrak{g})_k^0$ is semisimple. Thus the smooth group scheme $\mathrm{Aut}(\mathfrak{g})^0$ of finite presentation over Y has semisimple fibres. Therefore $\mathrm{Aut}(\mathfrak{g})^0$ is a semisimple group scheme over Y, cf. [DG, Vol. II, Exp. XVI, Thm. 5.2 (ii)]. As $Z(\mathrm{Aut}(\mathfrak{g})^0)_k$ acts trivially on $\mathrm{Lie}(\mathrm{Aut}(\mathfrak{g})_k^0) = \mathrm{ad}(\mathfrak{g}) \otimes_A k$ and as $Z(\mathrm{Aut}(\mathfrak{g})^0)_k$ is a subgroup of $\mathrm{Aut}(\mathfrak{g})_k$, the group $Z(\mathrm{Aut}(\mathfrak{g})^0)_k$ is trivial. This implies that the finite, flat group scheme $Z(\mathrm{Aut}(\mathfrak{g})^0)$ is trivial and thus $\mathrm{Aut}(\mathfrak{g})^0$ is an adjoint group scheme.

The Lie subalgebras $\mathrm{Lie}(\mathrm{Aut}(\mathfrak{g})^0)$ and $\mathrm{ad}(\mathfrak{g})$ of $\mathfrak{gl}_{\mathfrak{g}}$ are free A-submodules of the Lie subalgebra $\mathfrak{l}$ of $\mathfrak{gl}_{\mathfrak{g}}$ formed by derivations of $\mathfrak{g}$. As for each point y of Y we have $\mathrm{Lie}(\mathrm{Aut}(\mathfrak{g})_k^0) = \mathrm{ad}(\mathfrak{g}) \otimes_A k = \mathfrak{l} \otimes_A k$, $\mathfrak{l}$ is locally generated by either $\mathrm{Lie}(\mathrm{Aut}(\mathfrak{g})^0)$ or $\mathrm{ad}(\mathfrak{g})$. We easily get that we have identities $\mathrm{Lie}(\mathrm{Aut}(\mathfrak{g})^0) = \mathrm{ad}(\mathfrak{g}) = \mathfrak{l}$.

The group scheme $\mathrm{Aut}(\mathfrak{g})$ acts via inner conjugation on $\mathrm{Aut}(\mathfrak{g})^0$. As $\mathrm{Lie}(\mathrm{Aut}(\mathfrak{g})^0) = \mathrm{ad}(\mathfrak{g})$ and as $\mathrm{Aut}(\mathfrak{g})$ is a closed subgroup scheme of $\boldsymbol{GL}_{\mathfrak{g}}$, the inner conjugation homomorphism $\mathrm{Aut}(\mathfrak{g}) \to \mathrm{Aut}(\mathrm{Aut}(\mathfrak{g})^0)$ has trivial kernel. As $\mathrm{Aut}(\mathrm{Aut}(\mathfrak{g})^0)$ is a closed subgroup scheme of $\mathrm{Aut}(\mathrm{Lie}(\mathrm{Aut}(\mathfrak{g})^0)) = \mathrm{Aut}(\mathrm{ad}(\mathfrak{g}))$ (cf. Lemma 2.5.1), we can identify naturally $\mathrm{Aut}(\mathrm{Aut}(\mathfrak{g})^0)$ with a closed subgroup scheme of $\mathrm{Aut}(\mathfrak{g})$. From the last two sentences, we get that $\mathrm{Aut}(\mathrm{Aut}(\mathfrak{g})^0) = \mathrm{Aut}(\mathfrak{g})$. Thus both properties (i) and (ii) hold.

To check that the property (iii) holds, let $\mathfrak{g} = \mathrm{Lie}(G) = \mathrm{Lie}(H)$. It suffices to show that G and H are identified with $\mathrm{Aut}(\mathfrak{g})^0$. We will work only with G. The adjoint representation $G \to \boldsymbol{GL}_{\mathfrak{g}}$ factors as composite closed embedding homomorphisms $G \to \mathrm{Aut}(\mathfrak{g})^0 \to \mathrm{Aut}(\mathfrak{g}) \to \boldsymbol{GL}_{\mathfrak{g}}$ (cf. Lemma 2.5.1 and [DG, Vol. III, Exp. XXIV, Thm. 1.3]). We get a closed embedding homomorphism $G \to \mathrm{Aut}(\mathfrak{g})^0$ between adjoint group schemes that have the same Lie algebra $\mathfrak{g}$ (cf. also property (i)). By reasons of dimensions, the geometric fibers of the closed embedding homomorphism $G \to \mathrm{Aut}(\mathfrak{g})^0$ are isomorphisms and therefore $G \to \mathrm{Aut}(\mathfrak{g})^0$ is an isomorphism. Thus property (iii) holds as well. $\square$

The next Proposition details on the range of applicability of the Theorem 1.2.

3.6. Proposition. **(a)** *We recall that k is a field. Let H be a non-trivial semisimple group over $\mathrm{Spec}\, k$. Then the Killing form $\mathcal{K}_{\mathrm{Lie}(H)}$ is perfect if and only if the following two conditions hold:*

(i) *either $\mathrm{char}(k) = 0$ or $\mathrm{char}(k)$ is an odd prime p and H^{ad} has no simple factor of isotypic A_{pn-1}, $B_{pn+\frac{1-p}{2}}$, C_{pn-1}, or D_{pn+1} Dynkin type (here $n \in \mathbb{N}^*$);*

(ii) *if $\mathrm{char}(k) = 3$ (resp. if $\mathrm{char}(k) = 5$), then H^{ad} has no simple factor of isotypic E_6, E_7, E_8, F_4, G_2 (resp. of isotypic E_8) Dynkin type.*

(b) *If $\mathcal{K}_{\mathrm{Lie}(H)}$ is perfect, then the central isogenies $H^{\mathrm{sc}} \to H \to H^{\mathrm{ad}}$ are étale; thus, by identifying tangent spaces at identity elements, we have $\mathrm{Lie}(H^{\mathrm{sc}}) = \mathrm{Lie}(H) = \mathrm{Lie}(H^{\mathrm{ad}})$.*

Proof: We can assume that $k = \bar{k}$ and that $\mathrm{tr.deg.}(k) < \infty$. If $\mathrm{char}(k) = 0$, then $\mathrm{Lie}(H)$ is a semisimple Lie algebra over k and therefore the Proposition follows from [H1, Ch. II, Subsect. 5.1, Thm.]. Thus we can assume $\mathrm{char}(k)$ is a prime $p \in \mathbb{N}^*$. If the conditions (i) and (ii) hold, then p does not divide the order of the finite group scheme $Z(H^{\mathrm{sc}}) = \mathrm{Ker}(H^{\mathrm{sc}} \to H^{\mathrm{ad}})$ (see [B2, PLATES I to IX]) and therefore (a) implies (b).

Let $W(k)$ be the ring of p-typical Witt vectors with coefficients in k. Let $H_{W(k)}$ be a semisimple group scheme over $\operatorname{Spec} W(k)$ that lifts H, cf. [DG, Vol. III, Exp. XXIV, Prop. 1.21]. We have identities $\operatorname{Lie}(H^{\mathrm{sc}}_{W(k)})[\frac{1}{p}] = \operatorname{Lie}(H_{W(k)})[\frac{1}{p}] = \operatorname{Lie}(H^{\mathrm{ad}}_{W(k)})[\frac{1}{p}]$. This implies that:

(iii) $\mathcal{K}_{\operatorname{Lie}(H_{W(k)})}$ is the composite of the natural $W(k)$-linear map $\operatorname{Lie}(H_{W(k)}) \times \operatorname{Lie}(H_{W(k)}) \to \operatorname{Lie}(H^{\mathrm{ad}}_{W(k)}) \times \operatorname{Lie}(H^{\mathrm{ad}}_{W(k)})$ with $\mathcal{K}_{\operatorname{Lie}(H^{\mathrm{ad}}_{W(k)})}$;

(iv) $\mathcal{K}_{\operatorname{Lie}(H^{\mathrm{sc}}_{W(k)})}$ is the composite of the natural $W(k)$-linear map $\operatorname{Lie}(H^{\mathrm{sc}}_{W(k)}) \times \operatorname{Lie}(H^{\mathrm{sc}}_{W(k)}) \to \operatorname{Lie}(H_{W(k)}) \times \operatorname{Lie}(H_{W(k)})$ with $\mathcal{K}_{\operatorname{Lie}(H_{W(k)})}$.

We prove (a). We have $\operatorname{Ker}(\operatorname{Lie}(H) \to \operatorname{Lie}(H^{\mathrm{ad}})) \subseteq \operatorname{Ker}(\mathcal{K}_{\operatorname{Lie}(H)})$, cf. property (iii). If $\mathcal{K}_{\operatorname{Lie}(H)}$ is perfect, then $\operatorname{Ker}(\operatorname{Lie}(H) \to \operatorname{Lie}(H^{\mathrm{ad}})) = 0$ and therefore $\operatorname{Lie}(H) = \operatorname{Lie}(H^{\mathrm{ad}})$. Thus to prove (a) we can assume that $H = H^{\mathrm{ad}}$ is adjoint. Even more, to prove (a) we can also assume that the adjoint group H is simple; let $\flat$ be the Lie type of H. If $\flat$ is not of classical Lie type, then $\mathcal{K}_{\operatorname{Lie}(H)}$ is perfect if and only if either $p > 5$ or $p = 5$ and $\flat \neq E_8$ (cf. [H2, TABLE, p. 49]). Thus to prove (a), we can assume that $\flat$ is a classical Lie type. We fix a morphism $\operatorname{Spec} \mathbb{C} \to \operatorname{Spec} W(k)$.

Suppose that $\flat$ is either A_n or C_n. By the standard trace form on $\operatorname{Lie}(H^{\mathrm{sc}})$ (resp. on $\operatorname{Lie}(H^{\mathrm{sc}}_{W(k)})$ or on $\operatorname{Lie}(H^{\mathrm{sc}}_{\mathbb{C}})$) we mean the trace form $\mathcal{T}$ (resp. $\mathcal{T}_{W(k)}$ or $\mathcal{T}_{\mathbb{C}}$) associated to the faithful representation of H^{sc} (resp. $H^{\mathrm{sc}}_{W(k)}$ or $H^{\mathrm{sc}}_{\mathbb{C}}$) of rank $n+1$ if $\flat = A_n$ and of rank $2n$ if $\flat = C_n$. We have $\mathcal{K}_{\operatorname{Lie}(H^{\mathrm{sc}}_{\mathbb{C}})} = 2(n+1)\mathcal{T}_{\mathbb{C}}$, cf. [He, Ch. III, Sect. 8, (5) and (22)]. This identity implies that we also have $\mathcal{K}_{\operatorname{Lie}(H^{\mathrm{sc}}_{W(k)})} = 2(n+1)\mathcal{T}_{W(k)}$ and thus $\mathcal{K}_{\operatorname{Lie}(H^{\mathrm{sc}})} = 2(n+1)\mathcal{T}$. If p does not divide $2(n+1)$, then $\operatorname{Lie}(H^{\mathrm{sc}}) = \operatorname{Lie}(H)$ and it is well known that $\mathcal{T}$ is perfect; thus $\mathcal{K}_{\operatorname{Lie}(H^{\mathrm{sc}})} = \mathcal{K}_{\operatorname{Lie}(H)} = 2(n+1)\mathcal{T}$ is perfect. Suppose that p divides $2(n+1)$. This implies that $\mathcal{K}_{\operatorname{Lie}(H^{\mathrm{sc}})}$ is the trivial bilinear form on $\operatorname{Lie}(H^{\mathrm{sc}})$. From this and the property (iv) we get that the restriction of $\mathcal{K}_{\operatorname{Lie}(H)}$ to $\operatorname{Im}(\operatorname{Lie}(H^{\mathrm{sc}}) \to \operatorname{Lie}(H))$ is trivial. As $\dim_k(\operatorname{Lie}(H)/\operatorname{Im}(\operatorname{Lie}(H^{\mathrm{sc}}) \to \operatorname{Lie}(H))) = 1$ and as $\dim_k(\operatorname{Lie}(H)) \geq 3$, we easily get that $\mathcal{K}_{\operatorname{Lie}(H)}$ is degenerate.

Suppose that $\flat = B_n$ (resp. that $\flat = D_n$ with $n \geq 4$). If $p > 2$ we have $\operatorname{Lie}(H^{\mathrm{sc}}) = \operatorname{Lie}(H)$. Moreover, using [He, Ch. III, Sect. 8, (11) and (15)], as in the previous paragraph we argue that $\mathcal{K}_{\operatorname{Lie}(H)}$ is perfect if p does not divide $2(2n-1)$ (resp. if p does not divide $2(n-1)$) and is degenerate if p divides $2n-1$ (resp. if p divides $2(n-1)$).

We are left to show that $\mathcal{K}_{\operatorname{Lie}(H)}$ is degenerate if $p = 2$ and $\flat = B_n$. The group H is (isomorphic to) the $\boldsymbol{SO}$-group of the quadratic form $x_0^2 + x_1x_{n+1} + \cdots + x_nx_{2n}$ on $W := k^{2n+1}$. Let $\{e_{i,j} | i,j \in \{0,1,\ldots,n\}$ be the standard k-basis for $\mathfrak{gl}_W$. The direct sum $\mathfrak{n}_n := \oplus_{i=1}^{2n} ke_{0,i}$ is a nilpotent ideal of $\operatorname{Lie}(H)$, cf. [Bo, Ch. V, Subsect. 23.6]. Thus $\mathfrak{n}_n \subseteq \operatorname{Ker}(\mathcal{K}_{\operatorname{Lie}(H)})$, cf. [B1, Ch. I, Sect. 4, Prop. 6 (b)] applied to the adjoint representation of $\operatorname{Lie}(H)$. Therefore $\mathcal{K}_{\operatorname{Lie}(H)}$ is degenerate.

We conclude that $\mathcal{K}_{\operatorname{Lie}(H)}$ is perfect if and only if both conditions (i) and (ii) hold. Therefore (a) (and thus also (b)) holds. $\square$

3.7. Remarks. Let A and $\mathfrak{g}$ be as in the beginning of this Section.

(a) Let $p \in \mathbb{N}^*$ be a prime. Suppose that A is an algebraically closed field of characteristic p. Let G be an adjoint group over $\operatorname{Spec} A$ such that $\mathfrak{g} = \operatorname{Lie}(G)$, cf. Theorem

1.2. We have $p \neq 2$, cf. Proposition 3.6. Let $G_{\mathbb{Z}}$ be the unique (up to isomorphism) split, adjoint group scheme over $\operatorname{Spec} \mathbb{Z}$ such that G is the pull back of $G_{\mathbb{Z}}$ to $\operatorname{Spec} A$, cf. [DG, Vol. III, Exp. XXV, Cor. 1.3]. We have $\mathfrak{g} = \operatorname{Lie}(G_{\mathbb{Z}}) \otimes_{\mathbb{Z}} A$ i.e., $\mathfrak{g}$ has a canonical model $\operatorname{Lie}(G_{\mathbb{Z}})$ over $\mathbb{Z}$. For $p > 7$, this result was obtained in [C, Sect. 5, Thm.]. For $p > 3$, this result was obtained by Seligman, Mills, Block, and Zassenhaus (see [MS], [Mi], [BZ], and [S, II. 10]). For $p = 3$, this result was obtained in [Br, Thm. 4.1]. It seems to us that the fact that $p \neq 2$ (i.e., that all Killing forms of finite dimensional Lie algebras over fields of characteristic 2, are degenerate) is new.

(b) Let $B \twoheadrightarrow A$ be an epimorphism of commutative $\mathbb{Z}$-algebras whose kernel $\mathfrak{j}$ is a nilpotent ideal. Then $\mathfrak{g}$ has, up to isomorphisms, a unique lift to a Lie algebra over B which as a B-module is projective and finitely generated. One can prove this statement using cohomological methods as in the proof of Theorem 3.3. The statement also follows from the Theorem 1.2 and the fact that $\operatorname{Aut}(\mathfrak{g})^0$ has, up to isomorphisms, a unique lift to an adjoint group scheme over $\operatorname{Spec} B$ (this can be easily checked at the level of torsors of adjoint group schemes; see [DG, Vol. III, Exp. XXIV, Cors. 1.17 and 1.18]).

3.8. Corollary. *Let Sc-perf$_Y$ be the category whose objects are simply connected semisimple group schemes over Y with the property that their Lie algebras $\mathcal{O}_Y$-modules have perfect Killing forms and whose morphisms are isomorphisms of group schemes. Then the functor $\mathcal{L}_Y^{\mathrm{sc}}$: Sc-perf$_Y \to$ Lie-perf$_Y$ which associates to a morphism $f : G \xrightarrow{\sim} H$ of Sc-perf$_Y$ the morphism df : Lie$(G) \xrightarrow{\sim}$ Lie(H) of Lie-perf$_Y$ which is the differential of f, is an equivalence of categories.*

Proof: The functor $\mathcal{L}_Y^{\mathrm{sc}}$ is the composite of the canonical ('division by the centers') functor $\mathcal{Z}_Y$: Sc-perf$_Y \to$ Ad-perf$_Y$ with $\mathcal{L}_Y$; the functor $\mathcal{Z}_Y$ makes sense (cf. Lemma 3.6 (b)) and it is an equivalence of categories. Thus the Corollary follows from the Theorem 1.2. $\square$

3.9. Corollary. *The category Lie-perf$_Y$ has a non-zero object if and only if Y is a non-empty $\operatorname{Spec} \mathbb{Z}[\frac{1}{2}]$-scheme.*

Proof: The if part is implied by the fact that an $\mathfrak{sl}_2$ Lie algebra $\mathcal{O}_Y$-module has perfect Killing form. The only if part follows from the relation $p \neq 2$ of the Remark 3.7 (a). $\square$

4. Proof of the Theorem 1.4

In this Section we prove the Theorem 1.4. See Subsections 4.1 and 4.2 for the proofs of Theorems 1.4 (a) and (b) (respectively). In Remarks 4.3 we point out that the hypotheses of the Theorem 1.4 are indeed needed in general. We will use the notations listed in Section 1.

4.1. Proof of Theorem 1.4 (a). To prove the Theorem 1.4 we can assume Y is also integral. Let $K := K_Y$; it is a field. If H is a reductive group scheme over Y, then we have $\mathcal{D}(H) = \mathcal{D}(H_U)$ and thus the uniqueness parts of the Theorem 1.4 follow from Proposition 2.2 (b). Let $\mathfrak{l}$ be the Lie algebra $\mathcal{O}_Y$-module which extends $\operatorname{Lie}(G_U)$.

We prove Theorem 1.4 (a). Due to the uniqueness part, to prove Theorem 1.4 (a) we can assume $Y = \operatorname{Spec} A$ is also local and strictly henselian. Let $\mathfrak{g} := \mathfrak{l}(Y)$ be the Lie algebra over A of global sections of $\mathfrak{l}$.

As U is connected, based on [DG, Vol. III, Exp. XXII, Prop. 2.8] we can speak about the split, adjoint group scheme S over Y of the same Lie type as all geometric fibres of G_U. Let $\mathfrak{s} := \mathrm{Lie}(S)$. Let $\mathrm{Aut}(S)$ be the group scheme over Y of automorphisms of S. We have a short exact sequence $1 \to S \to \mathrm{Aut}(S) \to C \to 1$, where C is a finite, étale, constant group scheme over Y (cf. [DG, Vol. III, Exp. XXIV, Thm. 1.3]). Let $\gamma \in H^1(U, \mathrm{Aut}(S)_U)$ be the class that defines the form G_U of S_U.

We recall that $\boldsymbol{GL}_{\mathfrak{g}}$ and $\boldsymbol{GL}_{\mathfrak{s}}$ are the reductive group schemes over Y of linear automorphisms of $\mathfrak{g}$ and $\mathfrak{s}$ (respectively). The adjoint representations define closed embedding homomorphisms $j_U : G_U \hookrightarrow \boldsymbol{GL}_{\mathfrak{g},U}$ and $i : S \hookrightarrow \boldsymbol{GL}_{\mathfrak{s}}$ and moreover i extends naturally to a closed embedding homomorphism $\mathrm{Aut}(S) \hookrightarrow \boldsymbol{GL}_{\mathfrak{s}}$, cf. Lemma 2.5.1. Let $\delta \in H^1(U, (\boldsymbol{GL}_{\mathfrak{s},U}))$ be the image of γ via the homomorphism $\mathrm{Aut}(S)_U \hookrightarrow \boldsymbol{GL}_{\mathfrak{s},U}$.

We recall that the quotient sheaf for the faithfully flat topology of Y of the action of S on $\boldsymbol{GL}_{\mathfrak{s}}$ via right translations, is representable by an Y-scheme $\boldsymbol{GL}_{\mathfrak{s}}/S$ that is affine and that makes $\boldsymbol{GL}_{\mathfrak{s}}$ to be a right torsor of S over $\boldsymbol{GL}_{\mathfrak{s}}/S$ (cf. [CTS, Cor. 6.12]). Thus $\boldsymbol{GL}_{\mathfrak{s}}/S$ is a smooth, affine Y-scheme. The finite, étale, constant group scheme C acts naturally (from the right) on $\boldsymbol{GL}_{\mathfrak{s}}/S$ and this action is free (cf. Lemma 2.5.1). From [DG, Vol I, Exp. V, Thm. 4.1] we get that the quotient Y-scheme $(\boldsymbol{GL}_{\mathfrak{s}}/S)/C$ is affine and that the quotient epimorphism $\boldsymbol{GL}_{\mathfrak{s}}/S \twoheadrightarrow (\boldsymbol{GL}_{\mathfrak{s}}/S)/C$ is a finite étale cover. Thus $(\boldsymbol{GL}_{\mathfrak{s}}/S)/C$ is a smooth, affine scheme over Y that represents the quotient sheaf for the faithfully flat topology of Y of the action of $\mathrm{Aut}(S)$ on $\boldsymbol{GL}_{\mathfrak{s}}$ via right translations. From constructions we get that $\boldsymbol{GL}_{\mathfrak{s}}$ is a right torsor of $\mathrm{Aut}(S)$ over $\boldsymbol{GL}_{\mathfrak{s}}/\mathrm{Aut}(S) := (\boldsymbol{GL}_{\mathfrak{s}}/S)/C$.

The twist of i_U via the class γ is j_U. This implies that the class δ defines the torsor that parametrizes isomorphisms between the pull backs to U of the vector group schemes over Y defined by $\mathfrak{s}$ and $\mathfrak{g}$. Therefore, as the A-modules $\mathfrak{s}$ and $\mathfrak{g}$ are isomorphic (being free of equal ranks), the class δ is trivial. Thus γ is the coboundary of a class in $H^0(U, \boldsymbol{GL}_{\mathfrak{s},U}/\mathrm{Aut}(S)_U)$. But $H^0(U, \boldsymbol{GL}_{\mathfrak{s},U}/\mathrm{Aut}(S)_U) = H^0(Y, \boldsymbol{GL}_{\mathfrak{s}}/\mathrm{Aut}(S))$ (cf. Proposition 2.2 (b)) and therefore γ is the restriction of a class in $H^1(Y, \mathrm{Aut}(S))$. As Y is strictly henselian, each class in $H^1(Y, \mathrm{Aut}(S))$ is trivial. Thus γ is the trivial class. Therefore the group schemes G_U and S_U are isomorphic. Thus G_U extends to an adjoint group scheme G over Y isomorphic to S. This ends the proof of Theorem 1.4 (a). $\square$

4.2. Proof of Theorem 1.4 (b). Let $\eta : \bar{K} \to U$ be the geometric point of U which is the composite of the natural morphisms $\mathrm{Spec}\, \bar{K} \to \mathrm{Spec}\, K$ and $\mathrm{Spec}\, K \to U$. We denote also by $\eta : \bar{K} \to Y$ the resulting geometric point of Y. As Y (resp. U) is normal and locally noetherian, from [DG, Vol. II, Exp. X, Thms. 5.16 and 7.1] we get that there exists an antiequivalence of categories between the category of tori over Y (resp. over U) and the category of continuous $\pi_1(Y, \eta)$-representations (resp. continuous $\pi_1(U, \eta)$-representations) on free $\mathbb{Z}$-modules of finite rank. As the pair $(Y, Y \setminus U)$ is quasi-pure, we have a canonical identification $\pi_1(U, \eta) = \pi(Y, \eta)$. From the last two sentences we get that there exists a unique torus H^{ab} over Y which extends H_U^{ab}.

Let H^{ad} be the adjoint group scheme over Y that extends H_U^{ad}, cf. Theorem 1.4 (a). Let $F \to H^{\mathrm{ad}} \times_Y H^{\mathrm{ab}}$ be the central isogeny over Y that extends the central isogeny $H_K \to H_K^{\mathrm{ad}} \times_{\mathrm{Spec}\, K} H_K^{\mathrm{ab}}$, cf. Lemma 2.3.1 (a). Both F_U and H_U are the normalization of $H_U^{\mathrm{ab}} \times_U H_U^{\mathrm{ad}}$ in H_K, cf. Lemma 2.3.1 (b). Thus $H_U = F_U$ extends uniquely to a reductive group scheme $H := F$ over Y (cf. the first paragraph of Subsection 4.1 for the uniqueness part). This ends the proof of Theorem 1.4 (b) and thus also of the Theorem 1.4. $\square$

4.3. Remarks. (a) Let $Y_1 \to Y$ be a finite, non-étale morphism between normal, noetherian, integral $\mathrm{Spec}\,\mathbb{Z}_{(2)}$-schemes such that there exists an open subscheme U of Y with the properties that: (i) $Y \setminus U$ has codimension in Y at least 2, and (ii) $Y_1 \times_Y U \to U$ is a Galois cover of degree 2. Let H_U be the rank 1 non-split torus over U that splits over $Y_1 \times_Y U$. Then H_U does not extend to a smooth, affine group scheme over Y. If moreover $Y = \mathrm{Spec}\, A$ is an affine $\mathrm{Spec}\,\mathbb{F}_2$-scheme, then we have $Lie(H_U)(U) = A$ and therefore $Lie(H_U)$ extends to a Lie algebra $\mathcal{O}_Y$-module which as an $\mathcal{O}_Y$-module is free. Thus the quasi-pure part of the hypotheses of Theorem 1.4 (b) is needed in general.

(b) Suppose that $Y = \mathrm{Spec}\, A$ is local, strictly henselian, regular, and of dimension $n \geq 3$. Let $K := K_Y$. Let $d \in \mathbb{N}^*$ be such that there exists an A-submodule M of K^d that contains A^d, that is of finite type, that is not free, and that satisfies the identity $M = \cap_{V \in \mathcal{D}(Y)} M \otimes_A V$ (inside $M \otimes_A K$). A typical example (communicated to us by Serre): $d = n - 1$ and $M \overset{\sim}{\to} \mathrm{Coker}(f)$, where the A-linear map $f : A \to A^n$ takes 1 to an n-tuple $(x_1, \ldots, x_n) \in A^n$ of regular parameters of A.

Let $\mathcal{F}$ be the coherent $\mathcal{O}_Y$-module defined by M. Let U be an open subscheme of Y such that $Y \setminus U$ has codimension in Y at least 2 and the restriction $\mathcal{F}_U$ of $\mathcal{F}$ to U is a locally free $\mathcal{O}_U$-module. Let H_U be the reductive group scheme over U of linear automorphisms of $\mathcal{F}_U$. We recall the reason why the assumption that H_U extends to a reductive group scheme H over Y leads to a contradiction. The group scheme H is isomorphic to $\boldsymbol{GL}_{d,A}$ (as A is strictly henselian) and therefore there exists a free A-submodule L of K^d of rank d such that we can identify $H = \boldsymbol{GL}_L$. As A is a unique factorization domain (being local and regular), it is easy to see that there exists an element $f \in K$ such that the identity $M \otimes_A V = fL \otimes_A V$ holds for each $V \in \mathcal{D}(Y)$. This implies that $M = fL$. Thus M is a free A-module. Contradiction.

As H_U does not extend to a reductive group scheme over Y and as the pair $(Y, Y \setminus U)$ is quasi-pure, from Subsection 4.2 we get that H_U^{ad} also does not extend to an adjoint group scheme over Y. Thus the Lie part of the hypotheses of Theorem 1.4 (a) is needed in general.

5. Extending homomorphisms via schematic closures

In this Section we prove four results on extending homomorphisms of reductive group schemes via taking (normalizations of) schematic closures. The first one complements Theorem 1.4 (b) and Proposition 2.5.2 (see Proposition 5.1) and the other three refine parts of [V1] (see Subsections 5.2 to 5.6). Theorem 1.5 is proved in Subsection 5.6 based on the Theorem 5.4.

5.1. Proposition. *Let Y be a normal, noetherian, integral scheme. Let $K := K_Y$. Let U be an open subscheme of Y such that the codimension of $Y \setminus U$ in Y is at least 2. Let H_U be a reductive group scheme over U and let G be a reductive group scheme over Y. We assume we have a finite homomorphism $\rho_U : H_U \to G_U$ whose generic fiber over $\mathrm{Spec}\, K$ is a closed embedding. We assume that one of the following two properties holds:*

(i) *H_U extends to a reductive group scheme H over Y;*

(ii) *$Y = Spec\,R$ is a local regular scheme of dimension 2 (thus U is the complement in Y of the closed point of Y).*

Then the following three properties hold:

(a) *There exists a unique reductive group scheme H over Y which extends H_U.*

(b) *The homomorphism ρ_U extends uniquely to a finite homomorphism $\rho : H \to G$ between reductive group schemes over Y.*

(c) *If there exists a point of $Y \setminus U$ of characteristic 2, we assume that H_K has no normal subgroup that is adjoint of isotypic B_n Dynkin type for some $n \in \mathbb{N}^*$. Then $\rho : H \to G$ is a closed embedding.*

Proof: If the property (i) holds, then the uniqueness of H_U follows from Proposition 2.2 (b). Thus to prove (a) we can assume that the property (ii) holds. As (ii) holds, the pair $(Y, Y \setminus U)$ is quasi-pure (see Section 1) and the Lie algebra $\mathcal{O}_U$-module $Lie(H_U)$ extends to a Lie algebra $\mathcal{O}_Y$-module which is a free $\mathcal{O}_Y$-module (cf. Proposition 2.2 (c) and the fact that Y is local). Thus the hypotheses of Theorem 1.4 (b) hold and therefore from Theorem 1.4 (b) we get that there exists a unique reductive group scheme H over Y that extends H_U. Thus (a) holds.

To prove (b) and (c) we can assume that $Y = \operatorname{Spec} R$ is an affine scheme. We write $H = \operatorname{Spec} R_H$ and $G = \operatorname{Spec} R_G$. As $\mathcal{D}(H) = \mathcal{D}(H_U)$ and $\mathcal{D}(G) = \mathcal{D}(G_U)$, from Proposition 2.2 (a) we get that R_H and R_G are the R-algebras of global functions of H_U and G_U (respectively). Let $R_G \to R_H$ be the R-homomorphism defined by ρ_U and let $\rho : H \to G$ be the morphism of Y-schemes it defines. The morphism ρ is a homomorphism as it is so generically. To check that ρ is finite, we can assume that R is complete. Thus R_H and R_G are excellent rings, cf. [M, Sect. 34]. Therefore the normalization $H' = \operatorname{Spec} R_{H'}$ of the schematic closure of H_K in G is a finite, normal G-scheme.

The identity components of the reduced geometric fibres of ρ are trivial groups, cf. Proposition 2.5.2 (a) or (b). Thus ρ is a quasi-finite morphism. From Zariski Main Theorem (see [G1, Thm. (8.12.6)]) we get that H is an open subscheme of H'. But from Proposition 2.5.2 (b) we get that the morphism $H \to H'$ satisfies the valuative criterion of properness with respect to discrete valuation rings which contain R. As each local ring of H' is dominated by such a discrete valuation ring, we get that the morphism $H \to H'$ is surjective. Therefore the open, surjective morphism $H \to H'$ is an isomorphism. Thus ρ is finite i.e., (b) holds.

We prove (c). The pull back of the homomorphism $\rho : H \to G$ via each dominant morphism $\operatorname{Spec} V \to Y$, with V a discrete valuation ring, is a closed embedding (cf. Proposition 2.5.2 (c)). This implies that the fibres of ρ are closed embeddings. Thus the homomorphism ρ is a closed embedding, cf. Theorem 2.5. $\square$

We have the following refinement of [V1, Lemma 3.1.6].

5.2. Proposition. *Let G be a reductive group scheme over a reduced, affine scheme $Y = Spec\,A$. Let K be a localization of A. Let $s \in \mathbb{N}^*$. For $j \in \{1, \ldots, s\}$ let $G_{j,K}$ be a reductive, closed subgroup scheme of G_K. We assume that the group subschemes $G_{j,K}$'s commute among themselves and that one of the following two conditions holds:*

(i) *either the direct sum* $\oplus_{j=1}^{s} Lie(G_{j,K})$ *is a Lie subalgebra of* $Lie(G_K)$*, or*

(ii) $s = 2$*,* $G_{1,K}$ *is a torus, and* $G_{2,K}$ *is a semisimple group scheme.*

Then the closed subgroup scheme $G_{0,K}$ *of* G_K *generated by* $G_{j,K}$*'s exists and is reductive. Moreover, we have:*

(a) *If the condition (i) holds, then* $Lie(G_{0,K}) = \oplus_{j=1}^{s} Lie(G_{j,K})$*.*

(b) *We assume that for each* $j \in \{1, \ldots, s\}$ *the schematic closure* G_j *of* $G_{j,K}$ *in* G *is a reductive group scheme over* Y*. Then the schematic closure* G_0 *of* $G_{0,K}$ *in* G *is a reductive, closed subgroup scheme of* G*.*

Proof: Let Λ be the category whose objects $Ob(\Lambda)$ are finite subsets of K and whose morphisms are the inclusions of subsets. For $\alpha \in Ob(\Lambda)$, let K_α be the $\mathbb{Z}$-subalgebra of K generated by α and let $A_\alpha := A \cap K_\alpha$. We have $K = \mathrm{ind.lim.}_{\alpha \in Ob(\Lambda)} K_\alpha$ and $A = \mathrm{ind.lim.}_{\alpha \in Ob(\Lambda)} A_\alpha$. The reductive group schemes $G_{j,K}$ are of finite presentation. Based on this and [G1, Thms. (8.8.2) and (8.10.5)], one gets that there there exists $\beta \in Ob(\Lambda)$ such that each $G_{j,K}$ is the pull back of a closed subgroup scheme G_{j,K_β} of G_{K_β}. For $\alpha \supseteq \beta$, the set $C(\alpha)$ of points of $\mathrm{Spec}\, K_\alpha$ with the property that the fibres over them of all morphisms $G_{j,K_\alpha} \to \mathrm{Spec}\, K_\alpha$ are (geometrically) connected, is a constructible set (cf. [G1, Thm. (9.7.7)]). We have $\mathrm{proj.lim.}_{\alpha \in Ob(\Lambda)} C(\alpha) = \mathrm{Spec}\, K$. From this and [G1, Thm. (8.5.2)], we get that there exists $\beta_1 \in Ob(\Lambda)$ such that $\beta_1 \supseteq \beta$ and $C(\beta_1) = \mathrm{Spec}\, K_{\beta_1}$. Thus by replacing β with β_1, we can assume that the fibres of all morphisms $G_{j,K_\beta} \to \mathrm{Spec}\, K_\beta$ are connected. A similar argument shows that, by enlarging β, we can assume that all morphisms $G_{j,K_\beta} \to \mathrm{Spec}\, K_\beta$ are smooth and their fibres are reductive groups (the role of [G1, Thm. (9.7.7)] being replaced by [G1, Prop. (9.9.5)] applied to the $\mathcal{O}_{G_{j,K_\alpha}}$-module $Lie(G_{j,K_\alpha})$ and respectively by [DG, Vol. III, Exp. XIX, Cor. 2.6]). Thus each G_{j,K_β} is a reductive closed subgroup scheme of G_{K_β}. The smooth group schemes G_{j,K_β}'s commute among themselves as this is so after pull back through the dominant morphism $\mathrm{Spec}\, K \to \mathrm{Spec}\, K_\beta$. By enlarging β, we can also assume that either condition (i) or condition (ii) holds for the G_{j,K_β}'s and that K_β is a localization of A_β. By replacing A with the local ring of $\mathrm{Spec}\, A_\beta$ dominated by A, to prove the Proposition we can assume that A is a localization of a reduced, finitely generated $\mathbb{Z}$-algebra.

Using induction on $s \in \mathbb{N}^*$, it suffices to prove the Proposition for $s = 2$. Moreover, we can assume that $K = K_Y$. For the sake of flexibility, in what follows we will only assume that A is a reduced, noetherian $\mathbb{Z}$-algebra; thus K is a finite product of fields. As all the statements of the Proposition are local for the étale topology of Y, it suffices to prove the Proposition under the extra assumption that G_1 and G_2 are split (cf. Proposition 2.3). Let $C_K := G_{1,K} \cap G_{2,K}$. It is a closed subgroup scheme of $G_{j,K}$ that commutes with $G_{j,K}$, $j \in \{1, 2\}$. The Lie algebra $\mathrm{Lie}(C_K)$ is included in $\mathrm{Lie}(G_{1,K}) \cap \mathrm{Lie}(G_{2,K})$ and therefore it is trivial if the condition (i) holds. Thus if the condition (i) holds, then C_K is a finite, étale, closed subgroup scheme of $Z(G_{j,K})$. If the condition (ii) holds, then C_K is a closed subgroup scheme of both $G_{1,K} = Z(G_{1,K})$ and $Z(G_{2,K})$ and thus (as K is a finite product of fields) it is a finite group scheme of multiplicative type.

Let C be the schematic closure of C_K in G. Let T_j be a maximal torus of G_j. We have $C_K \leqslant Z(G_{1,K}) \cap Z(G_{2,K}) \leqslant T_{1,K} \cap T_{2,K} \leqslant G_{1,K} \cap G_{2,K} = C_K$ and thus $C_K = T_{1,K} \cap T_{2,K}$. Let $T_1 \times_Y T_2 \to G$ be the product homomorphism. The kernel $\mathfrak{K}$ of this product homomorphism is a group scheme over Y of multiplicative type (cf. Lemma 2.3.2 (a)) isomorphic to $T_1 \cap T_2$. But $\mathfrak{K}_K \tilde{\to} C_K$ is a finite group scheme over $\operatorname{Spec} K$ and therefore $\mathfrak{K}$ is a finite, flat group scheme over Y of multiplicative type (cf. Lemma 2.3.2 (b)). Thus $T_1 \cap T_2$ is a finite, flat group scheme over Y. From this, the identity $C_K = (T_1 \cap T_2)_K$, and the definition of C we get that $C = T_1 \cap T_2$. We conclude that C is a finite, flat group scheme over Y of multiplicative type contained in the center of both G_1 and G_2. We embed C in $G_1 \times_Y G_2$ via the natural embedding $C \hookrightarrow G_1$ and via the composite of the inverse isomorphism $C \tilde{\to} C$ with the natural embedding $C \hookrightarrow G_2$. Let $G_{1,2} := (G_1 \times_Y G_2)/C$; it is a reductive group scheme over Y. We have a natural product homomorphism $q : G_{1,2} \to G$ whose pull back to $\operatorname{Spec} K$ can be identified with the closed embedding homomorphism $G_{0,K} \hookrightarrow G_K$. Therefore $G_{0,K}$ is a reductive group scheme over $\operatorname{Spec} K$. Moreover, if the condition (i) holds, then as C_K is étale we have natural identities $\operatorname{Lie}(G_{1,K}) \oplus \operatorname{Lie}(G_{2,K}) = \operatorname{Lie}(G_{1,2,K}) = \operatorname{Lie}(G_{0,K})$. Thus (a) holds. If q is a closed embedding, then q induces an isomorphism $G_{1,2} \tilde{\to} G_0$ and therefore G_0 is a reductive, closed subgroup scheme of G. Thus to end the proof of (b), we only have to show that the homomorphism q is a closed embedding.

To check that q is a closed embedding, it suffices to check that the fibres of q are closed embeddings (cf. Theorem 2.5). For this we can assume that A is a complete discrete valuation ring which has an algebraically closed residue field k; this implies that G_0 is a flat, closed subgroup scheme of G. Let $\mathfrak{n} := \operatorname{Lie}(\operatorname{Ker}(q_k))$. From Proposition 2.5.2 (a) and Lemma 2.4 we get that: either (iii) $\mathfrak{n} = 0$ or (iv) $\operatorname{char}(k) = 2$ and there exists a normal subgroup F_k of $G_{1,2,k}$ which is isomorphic to $\boldsymbol{SO}_{2n+1,k}$ for some $n \in \mathbb{N}^*$ and for which we have $\operatorname{Lie}(F_k) \cap \mathfrak{n} \neq 0$. We show that the assumption that the condition (iv) holds leads to a contradiction. Let F be a normal, closed subgroup scheme of $G_{1,2}$ that lifts F_k and that is isomorphic to $\boldsymbol{SO}_{2n+1,A}$ (cf. last paragraph of the proof of Proposition 2.5.2 (c)). Let $j_0 \in \{1, 2\}$ be such that $F \lhd G_{j_0} \lhd G_{1,2}$ (if the condition (ii) holds, then $j_0 = 2$). As G_{j_0} is a closed subgroup scheme of G, we have $\operatorname{Lie}(G_{j_0,k}) \cap \mathfrak{n} = 0$ and therefore also $\operatorname{Lie}(F_k) \cap \mathfrak{n} = 0$. Contradiction. Thus the condition (iv) does not hold and therefore the condition (iii) holds. Thus $\operatorname{Ker}(q_k)$ has a trivial Lie algebra and therefore it is a finite, étale, normal subgroup of $G_{1,2,k}$. Thus $\operatorname{Ker}(q_k)$ is a subgroup of $Z(G_{1,2,k})$ and therefore also of each maximal torus of $G_{1,2,k}$. From this and Proposition 2.5.2 (a) we get that $\operatorname{Ker}(q_k)$ is trivial. Therefore q_k is a closed embedding. Thus q is a closed embedding. $\square$

In the last part of Section 5 we present significant refinements and simplifications to the fundamental results [V1, Prop. 4.3.10 and Rm. 4.3.10.1 1)].

5.3. Definitions. (a) Let $p \in \mathbb{N}^*$ be a prime and let $\mathcal{S}$ be a subset of $\mathbb{Z}$. We say $\mathcal{S}$ is of p-type 1, if the natural map $\mathcal{S} \to \mathbb{Z}/p\mathbb{Z}$ is injective. We say $\mathcal{S}$ is of p-type 2 (resp. of p-type 3), if $\mathcal{S}$ (resp. if $2\mathcal{S}$) is a subset of $\{-p+1, -p+2, \dots, p-2, p-1\}$.

(b) Let $Y = \operatorname{Spec} A$ be a non-empty affine scheme. Let T be a split torus over Y of rank 1. We fix an isomorphism $i_T : T \tilde{\to} \mathbb{G}_{m,Y}$. By a diagonal character of T with respect to i_T we mean the composite of an endomorphism of $\mathbb{G}_{m,Y}$ with i_T. We identify the group of diagonal characters of T with respect to i_T with $\mathbb{Z} = \operatorname{End}(\mathbb{G}_{m,\mathbb{Z}})$. If Y is connected, then each character of T is a diagonal character with respect to i_T.

5.3.1. On left $\mathfrak{sl}_2$-modules. Let $p \in \mathbb{N}^*$ be a prime. Let A be a commutative $\mathbb{Z}_{(p)}$-

algebra. Let M be a projective, finitely generated A-module. Let $\mathfrak{g}$ be an $\mathfrak{sl}_2$ Lie algebra over A equipped with a Lie homomorphism $\mathfrak{g} \to \mathfrak{gl}_M$. We consider an A-basis $\{h, x, y\}$ for $\mathfrak{g}$ such that the formulas $[h, x] = 2x$, $[h, y] = -2y$, and $[x, y] = h$ hold. For $s \in \mathbb{N}$, we will view x^s and y^s as A-linear endomorphisms of M (here $x^0 = y^0 := 1_M$).

Let $j \in \{1, \ldots, p-1\}$. Let $z_0 \in M$ be such that we have $h(z_0) = jz_0$ and $x(z_0) = 0$. For $i \in \{1, \ldots, j\}$, let $z_i := \frac{1}{i!} y^i(z_0)$. Then the following formulas hold (see [H1, Ch. II, Subsect. 7.2, Lemma]; the arguments of loc. cit. hold over any base ring A):

(i) for $i \in \{1, \ldots, j\}$, we have $h(z_i) = (j - 2i)z_i$ and $x(z_i) = (j - i + 1)z_{i-1}$.

From the property (i) we get that:

(ii) for each $i \in \{0, \ldots, j\}$, the element $x^i(y^i(z_0))$ is a multiple of z_0 by an invertible element of $\mathbb{Z}_{(p)}$ and thus also of A.

We have the following general form of [V1, Claim 3, p. 465].

5.3.2. Theorem. *Let $p \in \mathbb{N}^*$ be a prime. Let $Y = Spec\, A$ be a local scheme whose residue field k has characteristic p. Let $K := K_Y$. Let M be a free A-module of finite rank. Let T be a split torus of rank 1 over Y equipped with an isomorphism $i_T : T \tilde{\to} \mathbb{G}_{m,Y}$ and a homomorphism $f_K : T_K \to \boldsymbol{GL}_{M \otimes_A K}$. We assume that we have a direct sum decomposition $M \otimes_A K = \oplus_{i \in \mathcal{S}} M_{i,K}$ such that each projective K-module $M_{i,K}$ has constant rank and T_K acts on each $M_{i,K}$ via a diagonal character i of T_K with respect to $i_T \times_Y Spec\, K : T_K \tilde{\to} \mathbb{G}_{m,K}$. We also assume that one of the following two conditions holds:*

(i) *the subset $\mathcal{S}$ of $\mathbb{Z}$ is of p-type 1 and the Lie homomorphism $Lie(T_K) \to \mathfrak{gl}_M \otimes_A K$ is the tensorization with K over A of a Lie homomorphism $Lie(T) \to \mathfrak{gl}_M$;*

(ii) *the subset $\mathcal{S}$ of $\mathbb{Z}$ is of p-type 2 and there exists a semisimple group scheme G over Y which is isomorphic to $\boldsymbol{SL}_{2,Y}$, which has T as a maximal torus, and for which there exists a Lie homomorphism $Lie(G) \to \mathfrak{gl}_M$ whose tensorization with K over A is a Lie homomorphism $Lie(G_K) \to \mathfrak{gl}_{M \otimes_A K}$ that is also a morphism of left T_K-modules and that extends the natural Lie homomorphism $Lie(T_K) \to \mathfrak{gl}_{M \otimes_A K}$.*

Then the homomorphism $\rho_K : T_K \to \boldsymbol{GL}_{M \otimes_A K}$ extends to a homomorphism $\rho : T \to \boldsymbol{GL}_M$. If ρ_K is a closed embedding, then ρ is also a closed embedding.

Proof: Let $\mathfrak{m}$ be the maximal ideal of A; we have $k = A/\mathfrak{m}$. Let $M_i := M \cap M_{i,K}$, the intersection being taken inside $M \otimes_A K$. We show that the natural injective A-linear map $\mathcal{J} : \oplus_{i \in \mathcal{S}} M_i \hookrightarrow M$ is an A-linear isomorphism. It suffices to show that for each $i_0 \in \mathcal{S}$ there exists a projector π_{i_0} of M on M_{i_0} along $M \cap (\oplus_{i \in \mathcal{S} \setminus \{i_0\}} M_{i,K})$. Let h be the standard generator of $\mathrm{Lie}(T) = \mathrm{Lie}(\mathbb{G}_{m,A})$; it acts on M_i as the multiplication with $i \in \mathcal{S} \subseteq \mathbb{Z}$.

We first assume that the condition (i) holds. As $\mathcal{S}$ is of p-type 1, the elements $i \in \mathcal{S}$ are not congruent modulo p. Thus for each $i_0 \in \mathcal{S}$ there exists a polynomial $f_{i_0}(x) \in \mathbb{Z}_{(p)}[x]$ that is a product of linear factors and such that $\pi_{i_0} := f_{i_0}(h)$ is a projector of M on M_{i_0} along $M \cap (\oplus_{i \in \mathcal{S} \setminus \{i_0\}} M_{i,K})$.

Suppose that the condition (ii) holds. To ease the notations, for $i \in \mathbb{Z} \setminus \mathcal{S}$ let $M_i := 0$ and let $\pi_i : M \to M$ be the zero map. Thus we can assume that $\mathcal{S} = \{-p+1, \ldots, p-1\}$. As G is isomorphic to $\boldsymbol{SL}_{2,Y}$ and as T is a maximal torus of G, there exists an A-basis

$\{h, x, y\}$ for $\mathrm{Lie}(G)$ which contains the standard generator h of $\mathrm{Lie}(T)$ and for which the three identities $[h, x] = 2x$, $[h, y] = -2y$, and $[x, y] = h$ hold. We view M as a left $\mathrm{Lie}(G)$-module. As the Lie homomorphism $\mathrm{Lie}(G_K) \to \mathfrak{gl}_{M \otimes_A K}$ is also a homomorphism of left T_K-modules, we can assume that $x, y \in \mathrm{Lie}(G)$ are such that for $i \in \mathcal{S}$ we have $x(M_{i,K}) \subseteq M_{i+2,K}$ and $y(M_{i,K}) \subseteq M_{i-2,K}$ (for $p > 2$ these inclusions are implied by the three identities; this is so as for $p > 2$ the A-modules Ax and Ay are uniquely determined by the three identities). This implies that for $i \in \mathcal{S}$ we have $x(M_i) \subseteq M_{i+2}$ and $y(M_i) \subseteq M_{i-2}$.

We will show the existence of the projectors π_{i_0} of M by induction on the rank M. Let $r \in \mathbb{N}^*$. If the rank of M is at most 1, then the existence of the projectors π_{i_0} of M is obvious. Suppose that the projectors π_{i_0} of M exist if the rank of M is less than r. We show that the projectors π_{i_0} of M exist even if the rank of M is r. For $i \in \{1, \ldots, p-1\}$, we define $N_i := M \cap (M_{i,K} + M_{-p+i,K})$ and we consider the following two statements:

$$Q(i) \qquad \text{if } (z_i, z_{-p+i}) \in (M_i \setminus \mathfrak{m}M_i) \times M_{-p+i}, \text{ then } z_i + z_{-p+i} \notin \mathfrak{m}N_i;$$

$$Q(-p+i) \quad \text{if } (z_i, z_{-p+i}) \in M_i \times (M_{-p+i} \setminus \mathfrak{m}M_{-p+i}), \text{ then } z_i + z_{-p+i} \notin \mathfrak{m}N_i.$$

Let $i_0 \in \{0, \ldots, p-1\}$. If $i_0 = 0$, then as above we argue that there exists a polynomial $f_{i_0}(x) \in \mathbb{Z}_{(p)}[x]$ that is a product of linear factors and such that $\pi_{i_0} := f_{i_0}(h)$ is a projector of M on M_{i_0} along $M \cap (\oplus_{i \in \mathcal{S} \setminus \{i_0\}} M_{i,K})$; thus π_{i_0} exists if $i_0 = 0$.

Suppose that $i_0 > 0$. We consider a polynomial $f_{i_0}(x) \in \mathbb{Z}_{(p)}[x]$ such that $f_{i_0}(h)$ is an A-linear endomorphism of M whose kernel is $M \cap (\oplus_{i \in \mathcal{S} \setminus \{i_0, -p+i_0\}} M_{i,K})$ and whose restriction to N_{i_0} is a scalar automorphism of N_{i_0}. We have $\mathrm{Im}(f_{i_0}(h)) = N_{i_0}$ and therefore there exists a projector $\pi_{i_0,-p+i_0}$ of M on N_{i_0} along $M \cap (\oplus_{i \in \mathcal{S} \setminus \{i_0, -p+i_0\}} M_{i,K})$. Thus N_{i_0} is a direct summand of M and therefore (as Y is local) it is a free A-module of finite rank. Thus the projectors π_{i_0} and π_{-p+i_0} of M exist if and only if the natural injective A-linear map $\mathcal{J}_{i_0} : M_{i_0} \oplus M_{-p+i_0} \hookrightarrow N_{i_0}$ is an A-linear isomorphism (i.e., it is onto). But $\mathcal{J}_{i_0}$ is onto if and only if $\mathcal{J}_{i_0} \otimes 1_k$ is onto.

In this paragraph we check that to prove that all A-linear maps $\mathcal{J}_{i_0}$ with $i_0 \in \{1, \ldots, p-1\}$ are isomorphisms, we can assume that A is as well noetherian. We do not know a priori that the A-modules $M_{i_0} \oplus M_{-p+i_0}$ are finitely generated. Due to this, the A-linear maps $\mathcal{J}_{i_0}$ maps are not a priori defined over the localization of a finitely generated $\mathbb{Z}$-subalgebra A_0 of A. However, as M is a free A-module of finite rank and as the set $\{h, x, y\} \cup \mathcal{S}$ is finite, we can choose such an A_0 with the property that the A-linear maps $\mathcal{J}_{i_0}$ have natural analogues $\mathcal{J}_{i_0,A_0}$ over A_0 (in particular, the direct sum decomposition $M \otimes_A K = \oplus_{i \in \mathcal{S}} M_{i,K}$ is defined over a certain localization of A_0 and thus its analogue over the ring of fractions K_0 of A_0 is well defined). If all the A_0-linear maps $\mathcal{J}_{i_0,A_0}$ are isomorphisms, then we have $\mathcal{J}_{i_0} = \mathcal{J}_{i_0,A_0} \otimes_{A_0} 1_A$ and therefore all the A-linear maps $\mathcal{J}_{i_0}$ are isomorphisms. Thus by replacing A with A_0 we can assume that A is a local, noetherian $\mathbb{Z}_{(p)}$-algebra. Therefore each A-module $M_{i_0} \oplus M_{-p+i_0}$ is finitely generated.

In this paragraph we check that the k-linear map $\mathcal{J}_{i_0} \otimes 1_k$ is onto if and only if it is injective. If $\mathcal{J}_{i_0} \otimes 1_k$ is onto, then $\mathcal{J}_{i_0}$ is an A-linear isomorphism (see above) and therefore the k-linear map $\mathcal{J}_{i_0} \otimes 1_k$ is injective. If the k-linear map $\mathcal{J}_{i_0} \otimes 1_k$ is injective, then $\dim_k([M_{i_0} \oplus M_{-p+i_0}] \otimes_A k)$ is at most equal to $\dim_k(N_{i_0} \otimes_A k)$ and thus it is at most equal to the constant rank of the projective K-module $N_{i_0,K} = M_{i_0,K} \oplus M_{-p+i_0,K}$. Thus from Nakayama's lemma we get that the finitely generated A-module $M_{i_0} \oplus M_{-p+i_0}$ is the quotient of a free A-module of the same (constant) rank as the projective K-module $M_{i_0,K} \oplus M_{-p+i_0,K}$. As $M_{i_0} \oplus M_{-p+i_0} \subseteq M_{i_0,K} \oplus M_{-p+i_0,K}$, we easily get that $M_{i_0} \oplus M_{-p+i_0}$ is a free A-module of the same rank as N_{i_0}. As the k-linear map $\mathcal{J}_{i_0} \otimes 1_k$ is injective, by reasons of dimensions we conclude that it is also onto.

Based on the last three paragraphs we conclude that for each $i_0 \in \{1, \ldots, p-1\}$, the existence of the projectors π_{i_0} and π_{-p+i_0} of M is equivalent to the fact that both statements $Q(i_0)$ and $Q(-p+i_0)$ hold.

We will check that the statement $Q(i_0)$ holds by decreasing induction on $i_0 \in \{1, \ldots, p-1\}$. Thus we can assume that the statement $Q(i_1)$ holds for all numbers $i_1 \in \{i_0+1, \ldots, p-1\}$. As the statement $Q(i_1)$ holds, the k-linear map $M_{i_1}/\mathfrak{m}M_{i_1} \to M/\mathfrak{m}M$ is injective. We will show that the assumption that the statement $Q(i_0)$ does not holds leads to a contradiction. This assumption implies that there exist elements $z_{i_0} \in M_{i_0} \setminus \mathfrak{m}M_{i_0}$ and $z_{-p+i_0} \in M_{-p+i_0}$ such that we have $z_{i_0} + z_{-p+i_0} \in \mathfrak{m}N_{i_0}$.

Let $j_0 \in \mathbb{N}$ be the greatest number such that $i_0+2j_0 \in \mathcal{S}$ and $M_{i_0+2j_0} \neq 0$. Depending on the value of j_0, to reach the desired contradiction we consider two cases as follows.

Case 1: $j_0 > 0$. As $i_0+2j_0 > i_0 > 0$ and $M_{i_0+2j_0} \neq 0$, we have $p > 2$ and the k-linear map $M_{i_0+2j_0}/\mathfrak{m}M_{i_0+2j_0} \to M/\mathfrak{m}M$ is injective and non-zero. Therefore there exists a non-zero element $z_0 \in M_{i_0+2j_0}$ such that $\tilde{M}_{i_0+2j_0} := Az_0$ is a direct summand of both $M_{i_0+2j_0}$ and M. We have $x(\tilde{M}_{i_0+2j_0}) = 0$ as otherwise $M_{i_0+2j_0+2} \neq 0$. For $s \in \{0, \ldots, i_0+2j_0\}$ let $\tilde{M}_{i_0+2j_0-2s} := y^s(\tilde{M}_{i_0+2j_0}) = \frac{1}{s!}y^s(\tilde{M}_{i_0+2j_0}) \subseteq M_{i_0+2j_0-2s}$. Let

$$\tilde{M} := \oplus_{s=0}^{i_0+2j_0} \tilde{M}_{i_0+2j_0-2s}.$$

From properties 5.3.1 (i) and (ii) we easily get that for $1 \leq s \leq i_0+2j_0$ we have an identity $x(\tilde{M}_{i_0+2j_0-2s}) = \tilde{M}_{i_0+2j_0-2s+2}$. As $x(\tilde{M}_{i_0+2j_0}) = 0 = M_{i_0+2j_0+2} =: \tilde{M}_{i_0+2j_0+2}$, the identity of the last sentence holds even if $s = 0$. These properties of $\tilde{M}_{i_0+2j_0-2s}$, $\tilde{M}$, and z_0 imply that $x(\tilde{M}) \subseteq \tilde{M}$, that $h(\tilde{M}) \subseteq \tilde{M}$, and that each $\tilde{M}_{i_0+2j_0-2s}$ with $0 \leq s \leq i_0+2j_0$ is a direct summand of M and thus also of $M_{i_0+2j_0-2s}$. As $p > 2$, the numbers i_0+2j_0-2s with $0 \leq s \leq i_0+2j_0 \leq p-1$ are not congruent modulo p. We easily get that $\tilde{M}$ is a direct summand of M. From the property 5.3.1 (i) we get that $\tilde{M} \cap M_{-p+i_0} = 0$.

We check that $\tilde{M}$ is a left $\mathrm{Lie}(G)$-module. It suffices to show that $y^{i_0+2j_0+1}(z_0) = 0$. To check this, we can assume that $M = \oplus_{s=0}^{[\frac{i_0+2j_0+p}{2}]} y^s(\tilde{M}_{i_0+2j_0})$. We show that the assumption that $y^{i_0+2j_0+1}(z_0) \neq 0$ leads to a contradiction. Let $s_0 \in \mathbb{N}^*$ be the greatest number such that $y^{i_0+2j_0+s_0}(z_0) \in M_{-i_0-2j_0-2s_0} \setminus \{0\}$. We have $-i_0-2j_0-2s_0 \in \mathcal{S}$. Thus $-p+1 \leq -i_0-2j_0-2s_0 \leq -1$. By applying the property 5.3.1 (ii) to the sextuple $(y^{i_0+2j_0+s_0}(z_0), -h, y, x, i_0+2j_0+2s_0, i_0+2j_0+2s_0)$ instead of (z_0, h, x, y, j, i), we get that $w_0 := x^{i_0+2j_0+2s_0}(y^{i_0+2j_0+s_0}(z_0)) \in M \setminus \{0\}$. Therefore the element $u_0 := x^{i_0+2j_0+s_0}(y^{i_0+2j_0+s_0}(z_0))$ is a multiple of z_0 by a non-zero element of A. As $w_0 = x^{s_0}(u_0) \neq 0$ and $s_0 \geq 1$, we have $x(u_0) \neq 0$. Thus there exists a multiple of $x(z_0)$ by an element of A which is non-zero. This contradicts the fact that $x(z_0) = 0$. Therefore $\tilde{M}$ is a left $\mathrm{Lie}(G)$-module.

As $\tilde{M}_{i_0}$ is a direct summand of M, it is also a direct summand of M_{i_0}. Let $\tilde{M}'_{i_0}$ be a direct supplement of $\tilde{M}_{i_0}$ in M_{i_0}. We write $z_{i_0} = \tilde{z}_{i_0} + \tilde{z}'_{i_0}$, where $\tilde{z}_{i_0} \in \tilde{M}_{i_0}$ and $\tilde{z}'_0 \in \tilde{M}'_{i_0}$.

The rank of $M/\tilde{M}$ is less than r and thus by our induction on ranks we know that the analogues of the projectors π_{i_0} exist for $M/\tilde{M}$. Therefore the analogues of the statements $Q(i)$ with $i \in \{1, \ldots, p-1\}$ hold in the context of the left Lie(G)-module $M/\tilde{M}$. As $\tilde{M}'_{i_0} \oplus M_{-p+i_0}$ is naturally an A-submodule of $M/\tilde{M}$, we conclude that $\tilde{z}'_{i_0} \in \mathfrak{m}\tilde{M}'_{i_0}$. Thus we can assume that $z_{i_0} = \tilde{z}_{i_0} \in \tilde{M}_{i_0} \setminus \mathfrak{m}\tilde{M}_{i_0}$ i.e., $Az_{i_0} = \tilde{M}_{i_0}$. As $z_{i_0} + z_{-p+i_0} \in \mathfrak{m}N_{i_0}$, we have $x(z_{i_0} + z_{-p+i_0}) \in \mathfrak{m}N_{i_0+2}$. As $Ax(z_{i_0}) = x(\tilde{M}_{i_0}) = \tilde{M}_{i_0+2}$ is a direct summand of M and thus of M_{i_0+2}, we have $x(z_{i_0}) \in M_{i_0+2} \setminus \mathfrak{m}M_{i_0+2}$. As the statement $Q(i_0+2)$ holds, we conclude that $x(z_{i_0}) + x(z_{-p+i_0}) \notin \mathfrak{m}N_{i_0+2}$. Thus, as $x(z_{i_0}) + x(z_{-p+i_0}) \in x(\mathfrak{m}N_{i_0}) \subseteq \mathfrak{m}N_{i_0+2}$, we reached a contradiction.

Case 2: $j_0 = 0$. We have $y^{i_0}(z_{-p+i_0}) = 0$ as $y^{i_0}(M_{-p+i_0,K}) = 0$. Moreover we get that $y^{i_0}(\mathfrak{m}N_{i_0}) = \mathfrak{m}y^{i_0}(M \cap (M_{i_0,K} \oplus M_{-p+i_0,K}))$ is included in

$$\mathfrak{m}[M \cap (y^{i_0}(M_{i_0,K}) \oplus y^{i_0}(M_{-p+i_0,K})))] \subseteq \mathfrak{m}(M \cap M_{-i_0,K}) = \mathfrak{m}M_{-i_0}.$$

Thus we have $x^{i_0}(y^{i_0}(z_{i_0})) = x^{i_0}(y^{i_0}(z_{i_0} + z_{-p+i_0})) \in x^{i_0}(\mathfrak{m}M_{-i_0}) \subseteq \mathfrak{m}M_{i_0}$. But from the property 5.3.1 (ii) we get that $x^{i_0}(y^{i_0}(z_{i_0}))$ is a multiple of z_{i_0} by an invertible element of A. Thus $z_{i_0} \in \mathfrak{m}M_{i_0}$. Contradiction.

Thus the statement $Q(i_0)$ holds if $i_0 > 0$. This ends our decreasing induction on $i_0 \in \{1, \ldots, p-1\}$. As statement $Q(i)$ holds for all $i \in \{1, \ldots, p-1\}$, the statement $Q(-p+i)$ also holds for all $i \in \{1, \ldots, p-1\}$. This is so as Lie(G) has an automorphism that takes the triple (h, x, y) to the triple $(-h, y, x)$ (under this isomorphism the statement $Q(-p+i)$ gets replaced by the statement $Q(p-i)$). This implies that the projectors π_i and π_{-p+i} of M exist for all $i \in \{1, \ldots, p-1\}$.

Therefore all the projectors π_i of M with $i \in \mathcal{S}$ exist. Thus, regardless of which one of the two conditions (i) and (ii) holds, the injective A-linear map $\mathcal{J} : \oplus_{i \in \mathcal{S}} M_i \hookrightarrow M$ is an A-linear isomorphism. As Y is local, each M_i with $i \in \mathcal{S}$ is a free A-module. Thus we have a unique homomorphism $\rho : T \to \boldsymbol{GL}_M$ such that T acts on the direct summand M_i of M via the diagonal character i of T with respect to i_T; it extends ρ_K. The last part of the Theorem is implied by Lemma 2.3.2 (b) and (c). $\square$

5.3.3. Example. Let p be a prime, let $A = \mathbb{Z}_p$, let $d \geq p$ be an integer, and $\mathcal{S}_d = \{-d, -d+2, \ldots, d-2, d\}$. Let $G = \boldsymbol{SL}_{2,\mathbb{Z}_p}$ and let M_{d+1} be the irreducible left G-module of rank $d+1$ that is the d-th symmetric power of the standard left G-module M_2 of rank 2. We can identify $M_1 = \oplus_{i=0}^{d} \mathbb{Z}_p u^i v^{d-i}$, where u and v are viewed as independent variables as well as forming the standard $\mathbb{Z}_p$-basis of M_2. Let T be the split torus of G which normalizes both $\mathbb{Z}_p u$ and $\mathbb{Z}_p v$; it has rank 1 and we fix an identification $T = \mathbb{G}_{m,\mathbb{Z}_p}$. We consider the $\mathbb{Z}_p$-submodule of $M_1[\frac{1}{p}]$ generated by M and by the elements $w_i = \frac{1}{p}(u^i v^{d-i} + u^{i-p} v^{d+p-i})$ with $i \in \{p, \ldots, d\}$. It is easy to see that the $\mathbb{Z}_p$-module M is a left Lie(G)-module. But M is not a left T-module and thus it is also not a left G-module. The characters of the action of $T = \mathbb{G}_{m,\mathbb{Z}_p}$ on M_1 are the j-th powers of the identity character of $\mathbb{G}_{m,\mathbb{Z}_p}$, where $j \in \mathcal{D}_d$ is an arbitrary element. This implies that the conditions 5.3.2 (i) and (ii) are needed in general.

5.4. Theorem. *Let $Y = Spec\, A$ be a local, reduced scheme. Let $K := K_Y$ and let k be the residue field of the closed point y of Y. Let M be a free A-module of finite rank.*

Let G_K be a reductive, closed subgroup scheme of $\boldsymbol{GL}_{M\otimes_A K}$. Let $\mathfrak{h} := Lie(G_K^{\mathrm{der}}) \cap \mathfrak{gl}_M$, the intersection being taken inside $\mathfrak{gl}_M \otimes_A K$. Let $Z^0(G)$, G^{der}, and G be the schematic closures of $Z^0(G_K)$, G_K^{der}, and G_K (respectively) in $\boldsymbol{GL}_M$. We assume that the following three properties hold:

(i) *$Z^0(G)$ is a closed subgroup scheme of $\boldsymbol{GL}_M$ that is a torus;*

(ii) *the Lie algebra $\mathfrak{h}$ is (as an A-module) a direct summand of $\mathfrak{gl}_M$ and there exists a semisimple group scheme H over Y which extends G_K^{der} and for which we have an identity $Lie(H) = \mathfrak{h}$ that extends the identities $Lie(H_K) = Lie(G_K^{\mathrm{der}}) = \mathfrak{h} \otimes_A K$;*

(iii) *if H is non-trivial, then locally in the étale topology of Y, there exist a maximal split torus T of H and a family of rank 1 split subtori $(T_i)_{i\in I}$ of T which generate T and for which the following four things hold:*

(iii.a) *if $\tilde{T}_K$ is the torus of G_K generated by T_K and $Z^0(G_K)$, then the isogeny $Z^0(G_K)\times_{\operatorname{Spec} K} T_K \to \tilde{T}_K$ extends to an isogeny $Z^0(G)\times_Y T \to \tilde{T}$ of split tori over Y (whose restriction to T is a closed embedding homomorphism $T \hookrightarrow \tilde{T}$);*

(iii.b) *for each $i \in I$, there exists a subtorus $\tilde{T}_i$ of $\tilde{T}$ equipped with an isomorphism $i_{\tilde{T}_i} : \tilde{T}_i \tilde{\to} \mathbb{G}_{m,Y}$ which has the same image in $G_K^{\mathrm{ad}} = H_K^{ad}$ as $T_{i,K}$ and for which we have a direct sum decomposition $M\otimes_A K = \oplus_{\gamma_i\in \mathcal{S}_i} M_{i,\gamma_i,K}$ such that each projective K-module $M_{i,\gamma_i,K}$ has constant rank and moreover $\tilde{T}_{i,K}$ acts on it via a diagonal character γ_i of $\tilde{T}_{i,K}$ with respect to $i_{\tilde{T}_{i,K}}$ (thus $\mathcal{S}_i$ is a finite subset of the group of diagonal characters of $\tilde{T}_{i,K}$ with respect to $i_{\tilde{T}_{i,K}}$, viewed additively and identified with $\mathbb{Z}$);*

(iii.c) *if $char(k)$ is a prime p, then for each element $i \in I$ either:*

– the set $\mathcal{S}_i$ is of p-type 1 or

– $\mathcal{S}_i$ is of p-type 2 (resp. of p-type 3) and $\tilde{T}_i$ is T_i and is a torus of a semisimple, closed subgroup scheme S_i of H that is isomorphic to $\boldsymbol{SL}_{2,Y}$ (resp. to $\boldsymbol{PGL}_{2,Y}$);

(iii.d) *if Y is not normal and if $char(k)$ is a prime p, then for each rank 1 free A-submodule $\mathfrak{n}$ of $\mathfrak{h}$ on which T acts via a non-trivial character of T, the order of nilpotency of every endomorphism $u \in \mathfrak{n} \subseteq End_A(M)$ is at most p and moreover the exponential map $Exp_{\mathfrak{n}\otimes_A K} : \mathfrak{n}\otimes_A K \to \boldsymbol{GL}_{M\otimes_A K}(K)$ that takes $u \in \mathfrak{n}\otimes_A K$ to $Exp_{\mathfrak{n}\otimes_A K}(u) := \sum_{l=0}^{p-1} \frac{u^l}{l!}$, factors through the group of K-valued points of a $\mathbb{G}_{a,K}$-subgroup of G_K^{der} normalized by T_K.*

Then G (resp. G^{der}) is a reductive (resp. semisimple), closed subgroup scheme of $\boldsymbol{GL}_M$.

Proof: We consider the following statement $\mathfrak{S}$: G^{der} is a semisimple, closed subgroup scheme of $\boldsymbol{GL}_M$. The closed subgroup schemes $G_{1,K} := Z^0(G_K)$ and $G_{2,K} := G_K^{\mathrm{der}}$ of $\boldsymbol{GL}_{M\otimes_A K}$ commute and the condition 5.2 (ii) holds. Thus (cf. Proposition 5.2 and the property (i)), to prove the Theorem it suffices to show that the statement $\mathfrak{S}$ is true. We recall that a connected, étale scheme over a normal scheme, is a normal, integral scheme. If the scheme Y is (resp. is not) normal, then as the statement $\mathfrak{S}$ is local for the étale topology of Y, we can assume that A is strictly henselian. As $G_{1,K}$ and $G_{2,K}$ are of finite presentation over $\operatorname{Spec} K$ and as the A-module $\mathfrak{h}$ is a direct summand of $\mathfrak{gl}_M$, as in the first paragraph of the proof of Proposition 5.2 we argue that to prove the Theorem we can also assume A is the strict henselization of a local ring of a finitely generated, reduced $\mathbb{Z}$-algebra. Thus A is an excellent, local, strictly henselian, reduced ring (cf. [M, Sect. 34]); we emphasize that below we will use only these four properties of A. As A is strictly henselian, from Proposition 2.3 we get that each torus over Y is split.

Let $U := Y \setminus \{y\}$. As properties (i) to (iii) are stable under localization, by localizing and passage to the strict henselization, to check that statement $\mathfrak{S}$ holds we can also assume that G_U^{der} is a semisimple, closed subgroup scheme of $\boldsymbol{GL}_{M,U}$.

As $\mathfrak{h}$ has a perfect Killing form, the central isogeny $H^{\mathrm{sc}} \to H^{\mathrm{ad}}$ has a kernel which is a finite étale group scheme over Y (cf. Proposition 3.6 (b) applied to the geometric fibers of this central isogeny) and therefore, being of multiplicative type, its order is invertible in A. Let $\tilde{T}_i'$ be the subtorus of $Z^0(G) \times_Y T$ whose image in $\tilde{T}$ is $\tilde{T}_i$. The degree of the isogeny $\tilde{T}_i' \to \tilde{T}_i$ divides the order of $Z(H)$ and thus it is also invertible in A. Thus we can identify $\mathrm{Lie}(\tilde{T}_i') = \mathrm{Lie}(\tilde{T}_i)$. Thus $\mathrm{Lie}(\tilde{T}_i) \subseteq (\mathfrak{gl}_M \otimes_A K) \cap (\mathrm{Lie}(Z^0(G)) \oplus \mathfrak{h})$ and this implies that M is a left $\mathrm{Lie}(\tilde{T}_i)$-module.

In this paragraph we check that the closed embedding homomorphism $\tilde{T}_{i,K} \hookrightarrow \boldsymbol{GL}_{M\otimes_A K}$ extends to a homomorphism $\tilde{T}_i \to \boldsymbol{GL}_M$ which is a closed embedding. We consider four disjoint cases as follows. The case when $\mathrm{char}(k) = p$ and $\mathcal{S}_i$ is of p-type 1 follows from Theorem 5.3.2 applied with $(T, i_T) = (\tilde{T}_i, i_{\tilde{T}_i})$, cf. property (iii.c). The case when $\mathrm{char}(k) = p$ and $\mathcal{S}_i$ is of p-type 2 follows from Theorem 5.3.2 applied with $(G, T, i_T) = (S_i, T_i = \tilde{T}_i, i_{\tilde{T}_i})$, cf. property (iii.c). The case when $\mathrm{char}(k) = p$ and $\mathcal{S}_i$ is of p-type 3 follows from Theorem 5.3.2 applied with $(G, T, i_T) = (S_i', T_i' = \tilde{T}_i', i_{\tilde{T}_i'})$, where S_i' is the simply connected semisimple group scheme cover of S_i, where $\tilde{T}_i'$ is the inverse image of $\tilde{T}_i$ to S_i' and where $i_{\tilde{T}_i'} : \tilde{T}_i' \tilde{\to} \mathbb{G}_{m,Y}$ is the isomorphism that is naturally compatible with $i_{\tilde{T}_i}$ (cf. property (iii.c)); we add that the degree of the isogeny $\tilde{T}_i' \to \tilde{T}_i$ is 2 and thus the set of characters of the action of $\tilde{T}_i'$ on $M \otimes_A K$ is naturally identified with $2\mathcal{S}_i$ and therefore (as $\mathcal{S}_i$ is of p-type 3) it is of p-type 2. The case when $\mathrm{char}(k) = 0$ is well known: in this case we have a direct sum decomposition $M = \oplus_{\gamma_i \in \mathcal{S}_i} M \cap M_{i,\gamma_i,K}$ of free A-modules and thus M is naturally a left $\tilde{T}_i$-module; from Lemma 2.3.2 (b) and (c) we get that the homomorphism $\tilde{T}_i \to \boldsymbol{GL}_M$ is a closed embedding.

Let $h : Z^0(G) \times_Y \times_{i\in I} \tilde{T}_i \to \boldsymbol{GL}_M$ be the natural product homomorphism. The kernel $\mathrm{Ker}(h)$ is a group scheme of multiplicative type, cf. Lemma 2.3.2 (a). The quotient torus $(Z^0(G) \times_Y \times_{i\in I} \tilde{T}_i)/\mathrm{Ker}(h)$ is a torus of $\boldsymbol{GL}_M$ (cf. Lemma 2.3.2 (c)) that is also a quotient of $Z^0(G) \times_Y T$ which extends $\tilde{T}_K$. This implies that $\tilde{T} = (Z^0(G) \times_Y \times_{i\in I} \tilde{T}_i)/\mathrm{Ker}(h)$. Thus we can identify naturally T with a maximal (automatically split) torus of H as well as with a torus of $\boldsymbol{GL}_M$ (or of $\tilde{T}$) contained in G^{der}.

The role of T is that of an arbitrary maximal torus of H. Thus for the rest of the proof that the statement $\mathfrak{S}$ holds, we will only use the following property:

(iv) *each maximal torus of H is naturally a torus of $\boldsymbol{GL}_M$ contained in G^{der}, the Lie algebra $\mathrm{Lie}(H) = \mathfrak{h}$ is (as an A-module) a direct summand of $\mathfrak{gl}_M$, G_U^{der} is a semisimple, closed subgroup scheme of $\boldsymbol{GL}_{M,U}$, and the property (iii.d) holds.*

5.4.1. Notations. Let $\mathrm{Aut}(\mathfrak{h})$ be the group scheme of Lie automorphisms of $\mathfrak{h}$. The adjoint representation defines a closed embedding homomorphism $H^{\mathrm{ad}} \hookrightarrow \mathrm{Aut}(\mathfrak{h})$, cf. Lemma 2.5.1. As $\mathfrak{h}$ is (as an A-module) a direct summand of the left $\boldsymbol{GL}_M$-module $\mathfrak{gl}_M$, we can speak about the normalizer N of $\mathfrak{h}$ in $\boldsymbol{GL}_M$. Obviously G^{der} is a closed subscheme of N and thus we have a natural morphism $G^{\mathrm{der}} \to \mathrm{Aut}(\mathfrak{h})$ that factors through the closed subscheme H^{ad} of $\mathrm{Aut}(\mathfrak{h})$ (as this happens after pull back to $\mathrm{Spec}\, K$). Thus we have a morphism $\rho : G^{\mathrm{der}} \to H^{\mathrm{ad}}$ that (cf. property (iv)) extends the central isogeny $\rho_U : G^{\mathrm{der}}_U \to H^{\mathrm{ad}}_U$. From the uniqueness part of Lemma 2.3.1 (a), we get that ρ_U can be identified with the central isogeny $H_U \to H^{\mathrm{ad}}_U$ and thus we can identify $G^{\mathrm{der}}_U = H_U$.

Let $\mathfrak{h} = \mathrm{Lie}(T) \bigoplus_{\alpha\in\Phi} \mathfrak{h}_\alpha$ be the root decomposition of $\mathfrak{h} = \mathrm{Lie}(H)$ with respect to the (split) maximal torus T of H. Thus Φ is a root system of characters of T and each $\mathfrak{h}_\alpha$ is a free A-module of rank 1. We choose a basis Δ of Φ. Let Φ^+ (resp. Φ^-) be the set of roots of Φ that are positive (resp. are negative) with respect to Δ. For $\alpha \in \Phi$, let $\mathbb{G}_{a,\alpha}$ be the $\mathbb{G}_{a,Y}$ subgroup scheme of H that is normalized by T and that has $\mathfrak{h}_\alpha$ as its Lie algebra (cf. [DG, Vol. III, Exp. XXII, Thm. 1.1]). It is known that the product morphism

$$\omega : \times_{\alpha\in\Phi^+} \mathbb{G}_{a,\alpha} \times_Y T \times_Y \times_{\alpha\in\Phi^-} \mathbb{G}_{a,\alpha} \to H$$

is an open embedding regardless of the (fixed) orders in which the two products of $\mathbb{G}_{a,Y}$ group schemes are taken (cf. loc. cit.). Let $\Omega := \mathrm{Im}(\omega)$; it is an open subscheme of H. As A is strictly henselian, each point $b \in H(k)$ lifts to a point $a \in H(A)$; let $a\Omega$ be the open subscheme of H that is the left translation of Ω by a.

We consider the following statement $\mathfrak{H}$: we have a closed embedding homomorphism $f : H \to \boldsymbol{GL}_M$ that extends the closed embedding $G^{\mathrm{der}}_K = H_K \hookrightarrow \boldsymbol{GL}_{M\otimes_A K}$. Obviously $\mathfrak{H}$ implies $\mathfrak{S}$. Thus to end the proof of the Theorem it suffices to check that statement $\mathfrak{H}$ is true. For this, we will consider the following three possible types of the excellent, local, strictly henselian, reduced ring A.

5.4.2. The discrete valuation ring case. Suppose that A is also a discrete valuation ring. Thus G^{der} is a closed subgroup scheme of $\boldsymbol{GL}_M$ and $\rho : G^{\mathrm{der}} \to H^{\mathrm{ad}}$ is a homomorphism. We check that ρ is a quasi-finite morphism whose fibres are surjective. Obviously ρ is of finite type. Thus it suffices to check that the special fibre $\rho_k : G^{\mathrm{der}}_k \to H^{\mathrm{ad}}_k$ of ρ over y is a quasi-finite, surjective morphism. Each maximal torus of H_k lifts to a split torus of H, cf. Proposition 2.3 and the fact that A is strictly henselian. From this and the property (iv) we get that the image of ρ_k contains all split, maximal tori of H^{ad}_k. As such tori generate H^{ad}_k (the field k being infinite as k is separably closed), ρ_k is surjective. We have equalities $\dim(H^{\mathrm{ad}}_k) = \dim(H^{\mathrm{ad}}_K) = \dim(G^{\mathrm{der}}_K) = \dim(G^{\mathrm{der}}_k)$ (the last one as G^{der} is the schematic closure of G^{der}_K in $\boldsymbol{GL}_M$). By reasons of dimensions, we get that the surjective homomorphism ρ_k is an isogeny. Thus ρ_k is a quasi-finite, surjective morphism.

As ρ is quasi-finite, the normalization X^{der} of G^{der} is an open subscheme of the normalization of H^{ad} in G_K (cf. Zariski Main Theorem). But this normalization of H^{ad} is H, cf. Lemma 2.3.1 (a). Let $X^{\mathrm{der}} \to \boldsymbol{GL}_M$ be the finite morphism defined by the closed embedding $G^{\mathrm{der}} \hookrightarrow \boldsymbol{GL}_M$. Obviously X^{der} contains H_K and the generic point of the special fibre of H; thus $H \setminus X^{\mathrm{der}}$ has codimension in H at least 2. The morphism $X^{\mathrm{der}} \to \boldsymbol{GL}_M$ extends to a morphism $f : H \to \boldsymbol{GL}_M$ (cf. [BLR, Ch. 4, Sect. 4.4, Thm. 1]) that is a homomorphism as its pull back to $U = \mathrm{Spec}\, K$ is so. As $\mathfrak{h} = \mathrm{Lie}(H)$ is (as an A-module) a direct summand of $\mathfrak{gl}_M$, the fibre f_k of f over y has a kernel whose Lie algebra is trivial. Thus $\mathrm{Ker}(f_k)$ is a finite, étale, normal subgroup of H_k and therefore also of $Z(H_k)$ and of T_k. As T is a torus of $\boldsymbol{GL}_M$, we get that $\mathrm{Ker}(f_k)$ is the trivial group. Thus the fibres of f are closed embeddings. Thus f is a closed embedding homomorphism, cf. Theorem 2.5. Therefore the statement $\mathfrak{H}$ holds.

5.4.3. The normal case. Suppose that A is also a normal ring of dimension at least 2. We identify $\mathbb{G}_{a,\alpha,U}$ with a closed subgroup scheme of G_U^{der} and thus also of $\boldsymbol{GL}_{M,U}$. The closed embedding homomorphism $\mathbb{G}_{a,\alpha,U} \hookrightarrow \boldsymbol{GL}_{M,U}$ extends to a morphism $\mathbb{G}_{a,\alpha} \to \boldsymbol{GL}_M$ (cf. [BLR, Ch. 4, Sect. 4.4, Thm. 1]) that is automatically a homomorphism. Using such homomorphisms and the closed embedding $T \hookrightarrow \boldsymbol{GL}_M$, we get that we have a natural morphism $\Omega \to \boldsymbol{GL}_M$ that extends the (locally closed) embedding $\Omega_U \hookrightarrow \boldsymbol{GL}_{M,U}$ and that is compatible with the closed embedding homomorphism $H_K = G_K^{\mathrm{der}} \hookrightarrow \boldsymbol{GL}_{M\otimes_A K}$. As $H \setminus (\Omega \cup H_K)$ has codimension in H at least 2, from loc. cit. we get that the morphisms $\Omega \to \boldsymbol{GL}_M$ and $H_K \to \boldsymbol{GL}_M$ extend to a morphism $f : H \to \boldsymbol{GL}_M$ that is automatically a homomorphism. As $G_U^{\mathrm{der}} = H_U$, the fibres of f over points of U are closed embeddings. As $\mathfrak{h} = \mathrm{Lie}(H)$ is (as an A-module) a direct summand of $\mathfrak{gl}_M$, the fibre f_k of f over y has a kernel whose Lie algebra is trivial. Thus $\mathrm{Ker}(f_k)$ is a finite, étale, normal subgroup of H_k; as in the end of Subsubsection 5.4.2 we argue that $\mathrm{Ker}(f_k)$ is the trivial group. Thus the fibres of f are closed embeddings and therefore f is a closed embedding homomorphism, cf. Theorem 2.5. Thus the statement $\mathfrak{H}$ is true.

5.4.4. The general case. We recall that A is excellent, local, strictly henselian, and reduced. Let $Y^{\mathrm{n}} = \mathrm{Spec}\, A^{\mathrm{n}}$ be the normalization of Y; it is a finite product of normal, local, strictly henselian rings and $K_{Y^{\mathrm{n}}} = K$ is a finite product of fields. The morphism $Y^{\mathrm{n}} \to Y$ is finite, as A is excellent. Based on Subsubsection 5.4.3 applied to local rings of Y^{n} that dominate Y, we have a natural closed embedding homomorphism $H_{Y^{\mathrm{n}}} \hookrightarrow \boldsymbol{GL}_{M,Y^{\mathrm{n}}}$. Thus we also have a natural finite morphism $f^{\mathrm{n}} : H_{Y^{\mathrm{n}}} \to \boldsymbol{GL}_M$ that extends the closed embedding $H_K = G_K^{\mathrm{der}} \hookrightarrow \boldsymbol{GL}_{M\otimes_A K}$. To check that f^{n} factors as a morphism $f : H \to \boldsymbol{GL}_M$, let $\mathfrak{h} = \mathrm{Lie}(T) \bigoplus_{\alpha\in\Phi} \mathfrak{h}_\alpha$, $\mathbb{G}_{a,\alpha}$, and Ω be as in Subsubsection 5.4.1.

Let c be $p-1$ if $\mathrm{char}(k)$ is a prime p and be ∞ if $\mathrm{char}(k) = 0$. Due to the property (iii.d) (see the property (iv)), the rule $\mathrm{Exp}(u) := \sum_{l=0}^{c} \frac{u^l}{l!}$ defined for $u \in \mathfrak{h}_\alpha$ defines naturally a homomorphism $\mathbb{G}_{a,\alpha} \to \boldsymbol{GL}_M$. Thus as in Subsubsection 5.4.3 we argue that we have a natural morphism $\Omega \to \boldsymbol{GL}_M$. In other words, f^{n} restricted to $\Omega_{Y^{\mathrm{n}}}$ factors as a morphism $\Omega \to \boldsymbol{GL}_M$. The product morphism $\Omega \times_Y \Omega \to H$ is surjective and smooth. Thus, as A is strictly henselian, the product map $\Omega(A) \times \Omega(A) \to H(A)$ is surjective. This and the existence of the morphism $\Omega \to \boldsymbol{GL}_M$ imply that for each point $a \in H(A)$, the restriction of f^{n} to $(a\Omega)_{Y^{\mathrm{n}}}$ factors also through a morphism $a\Omega \to \boldsymbol{GL}_M$. As k is infinite, $H(k)$ is Zariski dense in H_k (cf. [Bo, Ch. V, Cor. 18.3]). Thus H is the union of H_U and of its open subschemes of the form $a\Omega$. This implies the existence of the homomorphism $f : H \to \boldsymbol{GL}_M$. As in the end of Subsubsection 5.4.3 we argue that the fibres of f are closed embeddings and thus that f is a closed embedding. Thus the statement $\mathfrak{H}$ is true. This ends the proof of the Theorem 5.4. $\square$

5.5. Remarks on the Theorem 5.4. **(a)** If $\mathrm{char}(k) = 0$, Y is normal, and G_K is semisimple, then properties 5.4 (i) to (iii) hold.

(b) If Y is normal or if Y is strictly henselian and K is a field, then we can always choose the family of subtori $(T_i)_{i\in I}$ such that properties 5.4 (iii.a) and (iii.b) hold.

(c) Property 5.4 (iii.d) holds if Y is a $\operatorname{Spec}\mathbb{Z}[\frac{1}{\theta}]$-scheme, where $\theta \in \mathbb{N}^*$ is computable in terms of a fixed family $(T_i)_{i\in I}$ of tori for which properties 5.4 (iii.a) and (iii.b) hold.

(d) Properties 5.4 (i), (iii.a), and (iii.b) (resp. 5.4 (ii), (iii.c), and (iii.d)) are (resp. are not) implied by the assumption that G is a reductive subgroup scheme of $\boldsymbol{GL}_M$.

(e) Property 5.4 (iii.c) is needed in general, cf. Example 5.3.3.

5.6. Proof of Theorem 1.5. We check that the properties 5.4 (i) to (iii) hold in our present context. Obviously the property 5.4 (i) holds, cf. property 1.5 (i). From Proposition 3.6 (b) and the property 1.5 (ii) we get that $\mathrm{Lie}(G_K^{\mathrm{der}}) = \mathfrak{h}\otimes_A K$ is $\mathrm{Lie}(G_K^{\mathrm{ad}})$. Let H^{ad} be the adjoint group scheme over Y which extends G_K^{ad} and for which we have an identity $\mathrm{Lie}(H^{\mathrm{ad}}) = \mathfrak{h}$ that extends the identities $\mathrm{Lie}(H_K^{\mathrm{ad}}) = \mathrm{Lie}(G_K^{\mathrm{ad}}) = \mathfrak{h}\otimes_A K$, cf. Corollary 1.3 and property 1.5 (ii). The geometric fibres of the central isogeny $H^{\mathrm{sc}} \to H^{\mathrm{ad}}$ are étale isogenies, cf. Proposition 3.6 (b) and property 1.5 (ii). Let $H \to H^{\mathrm{ad}}$ be the central isogeny that extends the central isogeny $G_K^{\mathrm{der}} \to G_K^{\mathrm{ad}} = H_K^{\mathrm{ad}}$ (cf. Lemma 2.3.1 (a)); it is étale. We have $\mathrm{Lie}(H) = \mathrm{Lie}(H^{\mathrm{ad}}) = \mathfrak{h}$ and thus the property 5.4 (ii) holds.

To check that the property 5.4 (iii) holds as well, we can assume that H is non-trivial. If Y is normal, then locally in the étale topology of Y we can assume that all tori over Y under consideration are split. If Y is strictly henselian, then all tori over Y are split. Thus we can assume that H is split. Let T be a maximal split torus of H. Let $\tilde{T}_K$ be the maximal split torus of G_K generated by $Z^0(G_K)$ and T_K. From the property 1.5 (iii) we get that $\tilde{T}_K$ is generated by cocharacters that act on $M\otimes_A K$ via the trivial or the identity characters of $\mathbb{G}_{m,K}$. Let $(\tilde{T}_{i,K})_{i\in I}$ be the family of $\mathbb{G}_{m,K}$ subgroups of $\tilde{T}_K$ that are images of all such cocharacters of $\tilde{T}_K$ which have non-trivial images in $G_K^{\mathrm{ad}} = H_K$. Let T_i be the subtorus of T whose generic fibre $T_{i,K}$ has the same image in $G_K^{\mathrm{ad}} = H_K$ as $\tilde{T}_{i,K}$. Properties 5.4 (iii.a) and (iii.b) hold from constructions. Obviously, property 5.4 (iii.c) holds: each $\mathcal{S}_i$ as in the property 5.4 (iii.c) is in our present case a subset of $\{0,1\}$ and thus it is of p-type 1.

We consider the root decomposition $\mathrm{Lie}(G_K) = \mathfrak{h}\otimes_A K = \mathrm{Lie}(\tilde{T}_K)\bigoplus_{\alpha\in\Phi}\mathfrak{h}_{\alpha,K}$ with respect to the split, maximal torus $\tilde{T}_K$ of G_K. Based on the generation part of the property 1.5 (iii), for each $\alpha\in\Phi$ there exists an element $i_\alpha \in I$ such that $\tilde{T}_{i_\alpha,K}$ does not fix $\mathfrak{h}_{\alpha,K}$. From the action part of the property 1.5 (iii) applied to $\tilde{T}_{i_\alpha,K}$, we get that we have $u^2 = 0$ for each $u \in \mathfrak{h}_{\alpha,K} \subseteq \mathfrak{gl}_{M\otimes_A K}$ and that the exponential map $\mathrm{Exp}_{\mathfrak{h}_{\alpha,K}} : \mathfrak{h}_{\alpha,K} \to \boldsymbol{GL}_{M\otimes_A K}(K)$, which takes $u \in \mathfrak{h}_{\alpha,K}$ to $1_{M\otimes_A K} + u$, factors through the group of K-valued points of the $\mathbb{G}_{a,K}$-subgroup of G_K^{der} which is normalized by $\tilde{T}_K$ and whose Lie algebra is $\mathfrak{h}_{\alpha,K}$. Thus the property 5.4 (iii.d) holds if Y is not normal. We conclude that all properties 5.4 (i) to (iii) hold. Thus the Theorem 1.5 follows from Theorem 5.4. $\square$

5.7. Remark. Suppose $k = \bar{k}$. Let V be a finite, totally ramified, discrete valuation ring extension of the ring $W(k)$ of p-typical Witt vectors with coefficients in k. Let $e := [V : W(k)] \in \mathbb{N}^*$. Let R_e be the p-adic completion of the R-subalgebra of $W(k)[\frac{1}{p}][[x]]$ generated by $\frac{x^{ea}}{a!}$, with $a \in \mathbb{N}^*$; it is a $W(k)$-subalgebra of $W(k)[\frac{1}{p}][[x]]$ and thus it is an integral domain. The k-algebra R_e/pR_e is the inductive limit of its local, artinian k-subalgebras. This implies that R_e is a local, strictly henselian ring whose residue field is k. Let R_e^{n} be the normalization of R_e. We have a natural $W(k)$-epimorphism $R_e^{\mathrm{n}} \twoheadrightarrow W(k)$ that takes x to 0. It is well known that we also have a $W(k)$-epimorphism $q_\pi : R_e \twoheadrightarrow V$ that takes x to a uniformizer π of V (for instance, see [V6, Subsect. 2.2]). Thus we also have a natural $W(k)$-epimorphism $q_\pi^{\mathrm{n}} : R_e^{\mathrm{n}} \twoheadrightarrow V_{\mathrm{big}}$ that extends q_π, where V_{big} is a suitable integral, ind-finite V-algebra. Let $R_e^{\mathrm{n},PD}$ be the p-adic completion of the divided power hull of $\mathrm{Ker}(q_\pi^{\mathrm{n}})$. It is easy to see that one can redo [V1, Subsects. 5.2 and 5.3] entirely in the context of R_e^{n} and $R_e^{\mathrm{n},PD}$ instead of the rings R_e and $\tilde{R}_e$ which in loc. cit. were denoted respectively as Re and $\tilde{R}e$ (the simple details will be presented in a future work). Based on this one can get major shortcuts to [V1, Prop. 4.3.10 b)] and its crystalline applications of [V1, Subsections 5.2 and 5.3] in two possible ways:

(i) based on Subsubsection 5.4.3 applied with $Y = \mathrm{Spec}\, R_e^{\mathrm{n}}$ or

(ii) based on the proof of Theorem 1.5 applied with $Y = \mathrm{Spec}\, R_e$.

Our first motivation behind Theorems 1.5 and 5.4 stems from the possibility of getting such shortcuts which will be detailed as well in a future work.

Acknowledgments. We would like to thank University of Utah, University of Arizona, Binghamton University, and I. H. E. S., Bures-sur-Yvette for good working conditions. We would like to thank J.-P. Serre and A. Langer for pointing out the reference [CTS] and O. Gabber for an e-mail that independently outlined parts of [CTS]; these led to a shorter paper and to stronger forms of Theorems 1.2 and 1.4 and of Proposition 5.1, for those reductive group schemes whose adjoints have geometric fibres that have simple factors of exceptional Lie types. We would like to thank J.-L. Colliot-Thélène for some comments and J. E. Humphreys for pointing out the references quoted in Remark 3.7 (a). We would like to thank O. Gabber also for many other valuable comments and suggestions; Section 3 is mainly based on these suggestions. This research was partially supported by the NSF grant DMS #0900967.

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