Purity of Crystalline Strata

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ABSTRACT. Let p be a prime. Let $n \in \mathbb{N}^*$. Let $\mathcal{C}$ be an F^n-crystal over a locally noetherian $\mathbb{F}_p$-scheme S. Let $(a,b) \in \mathbb{N}^2$. We show that the reduced locally closed subscheme of S whose points are exactly those $x \in S$ such that (a,b) is a break point of the Newton polygon of the fiber $\mathcal{C}_x$ of $\mathcal{C}$ at x is pure in S, i.e., it is an affine S-scheme. This result refines and reobtains previous results of de Jong–Oort, Vasiu, and Yang. As an application, we show that for all $m \in \mathbb{N}$ the reduced locally closed subscheme of S whose points are exactly those $x \in S$ for which the p-rank of $\mathcal{C}_x$ is m is pure in S; the case $n = 1$ was previously obtained by Deligne (unpublished) and the general case $n \geq 1$ refines and reobtains a result of Zink.

KEY WORDS: $\mathbb{F}_p$-scheme, F-crystal, Newton polygon, p-rank, purity.

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1 Introduction

For a reduced locally closed subscheme Z of a locally noetherian scheme Y, let $\bar{Z}$ be the schematic closure of Z in Y. We recall from [NVW], Definition 1.1 that Z is called *pure* in Y if it is an affine Y-scheme. The paper [NVW] also uses a weaker variant of this purity which in [L] is called *weakly pure*: we say Z is weakly pure in Y if each non-empty irreducible component of the complement $\bar{Z} - Z$ is of pure codimension 1 in $\bar{Z}$. It is well-known that if Z is pure in Y, then Z is also weakly pure in Y (for instance, cf. Proposition 3 of Subsection 4.4).

Let n and r be natural numbers. Let p be a prime. Let S be a locally noetherian $\mathbb{F}_p$-scheme. Let $\Phi_S : S \to S$ be the Frobenius endomorphism of S.

Let $\mathcal{M}$ be a *crystal* of the gross absolute crystalline site $CRIS(S/\mathrm{Spec}\,(\mathbb{Z}_p))$ introduced in [B], Chapter III, Example 1.1.3 and Definition 4.1.1 in locally free $\underline{\mathcal{O}}_{S/\mathrm{Spec}\,(\mathbb{Z}_p)}$-modules of rank r. We assume that we have an *isogeny* $\phi_{\mathcal{M}} : (\Phi_S^n)^*(\mathcal{M}) \to \mathcal{M}$; thus the pair $\mathcal{C} = (\mathcal{M}, \phi_{\mathcal{M}})$ is an F^n-crystal of $CRIS(S/\mathrm{Spec}\,(\mathbb{Z}_p))$. If the $\mathbb{F}_p$-scheme $S = \mathrm{Spec}\, A$ is affine, then the pair $\mathcal{C} = (\mathcal{M}, \phi_{\mathcal{M}})$ is canonically identified with a $\sigma^n - F$-crystal on A in the sense of [K], Subsection (2.1).

Let $\nu : [0, r] \to [0, \infty)$ be a *Newton polygon*, i.e., a nondecreasing piecewise linear continuous function such that $\nu(0) = 0$ and the coordinates of all its *break points* are natural numbers. For $x \in S$, let ν_x be the *Newton polygon* of the fiber $\mathcal{C}_x$ of $\mathcal{C}$ at x. Let S_ν be the reduced locally closed subscheme of S whose points are exactly those $x \in S$ such that we have $\nu_x = \nu$, cf. Grothendieck–Katz Theorem (see [K], Corollary 2.3.2); if non-empty, S_ν is a *stratum* of the Newton polygon *stratification* of S defined by $\mathcal{C}$.

Let $a, b \in \mathbb{N}$ be such that $0 \leq a \leq r$. Let $T = T_{(a,b)}(\mathcal{C})$ be the reduced locally closed subscheme of S whose points are those $x \in S$ such that (a, b) is a break point of ν_x. The end break point $(r, \nu_x(r))$ remains constant under specializations of $x \in S$. Thus locally in the Zariski topology of S, we can assume that there exists $d \in \mathbb{N}$ such that for all $x \in S$ we have $\nu_x(r) = d$ and this implies that T is the reduced locally closed subscheme of S which is a finite union $\bigcup_{\nu \in N_{r,d,a,b}} S_\nu$ of Newton polygon strata S_ν indexed by the set $N_{r,d,a,b}$ of all Newton polygons $\nu : [0, r] \to [0, \infty)$ with the two properties that $\nu(r) = d$ and (a, b) is a break point of ν.

It is known that T is weakly pure in S, cf. [Y], Theorem 1.1. It is also known that S_ν is pure in S, cf. [V1], Main Theorem B. This last result implies the celebrated result of de Jong–Oort which asserts that S_ν is weakly pure in S, cf. [dJO], Theorem 4.1. Strictly speaking, the references of this paragraph work with $n = 1$ but their proofs apply to all $n \in \mathbb{N}^*$.

In general, a finite union of locally closed subschemes of S which are pure in S is not pure in S. Therefore the following purity result which refines and reobtains the mentioned results of de Jong–Oort, Vasiu, and Yang comes as a surprise.

Theorem 1 *With the above notations, T is pure in S.*

In Section 2 we gather few preliminary steps that are required to prove Theorem 1 in Section 3. We have the following two direct consequences of Theorem 1, the first one for $n = 1$ just reobtains [V1], Main Theorem B in the locally noetherian case.

Corollary 1 *Each Newton polygon stratum S_ν is pure in S.*

The p-rank $\chi(x)$ of $\mathcal{C}_x$ is the multiplicity of the Newton polygon slope 0 of ν_x. Equivalently, $\chi(x)$ is the unique natural number such that $(0,0)$ and $(\chi(x),0)$ are the only break points of ν_x on the horizontal axis (i.e., which have the second coordinate 0).

Corollary 2 *Let $m \in \mathbb{N}$. We consider the reduced locally closed subscheme S_m of S whose points are exactly those $x \in S$ such that the p-rank $\chi(x)$ of $\mathcal{C}_x$ is m. Then S_m is pure in S.*

If $m > 0$, then we have $S_m = T_{(m,0)}(\mathcal{C})$ and if $m = 0$, then we have $S_0 = T_{(1,0)}(\mathcal{C} \oplus \mathcal{E}_0)$ where $\mathcal{E}_0$ is the pull back to S of the F^n-crystal over Spec $(\mathbb{F}_p)$ of rank 1 and Newton polygon slope 0 which has a Frobenius invariant global section; therefore, regardless of what m is, Corollary 2 follows from Theorem 1.

For $n = 1$ Corollary 2 was first obtained by Deligne and more recently by Vasiu and Li (see [D], [V3], and [L]). Corollary 2 also refines and reobtains a prior result of Zink which asserts that S_m is weakly pure in S (see [Z], Proposition 5).

In Section 4 we first follow [L] to show that Corollary 1 follows directly from Theorem 1 and then we follow [V3] to include a second proof of Corollary 2 in the more general context provided by a functorial version of the *Artin–Schreier stratifications* introduced in [V2], Definition 2.4.2 which is simpler, does not rely on Theorem 1, and is based on Theorem 2 of Subsection 4.2.

Theorem 1 is due to the first author, cf. [L]. While the proof of [Y], Theorem 1.1 follows the proof of [dJO], Theorem 4.1, the proof of Theorem 1 presented follows [L] and thus the proofs of [V1], Main Theorem B and Theorem 6.1. It is known (cf. [NVW], Example 7.1) that in general S_m is not strongly pure in S in the sense of [NVW], Definition 7.1 and therefore Theorem 1 and Corollary 2 cannot be improved in general (i.e., are optimal).

We refer to $T_{(a,b)}(\mathcal{C})$, S_ν, and S_m as crystalline strata of S associated to $\mathcal{C}$ and certain (basic) discrete invariants of F^n-crystals. Cases of non-discrete invariants stemming from isomorphism classes are also studied in the literature (for instance, see [V1], Subsection 5.3 and [NVW], Theorem 1.2 and Corollary 1.5). Crystalline strata have applications to the study in positive characteristic of different moduli spaces and schemes such as special fibers of Shimura varieties of Hodge type (for instance, see [V1] and [NVW]).

2 Standard reduction steps

The above notations p, S, Φ_S, $\bar{Z}$, n, r, $\mathcal{C} = (\mathcal{M}, \phi_{\mathcal{M}})$, $\mathcal{C}_x$, ν_x, $(a,b) \in \mathbb{N}^2$, $T = T_{(a,b)}(\mathcal{C})$, S_ν, m, S_m, $\chi(x)$, and $\mathcal{E}_0$ will be used throughout the paper. For a fixed Newton polygon ν, let $S_{\geq\nu}$ be the reduced closed subscheme of S whose points are exactly those $x \in S$ such that the Newton polygon ν_x is above ν, cf. [K], Corollary 2.3.2.

In what follows by an étale cover we mean a surjective finite étale morphism of schemes. For basic properties of excellent rings we refer to [M], Chapter 13. If $V \to Y$ is a morphism of $\mathbb{F}_p$-schemes and if $\mathcal{F}$ (or $\mathcal{F}_Y$) is an F^n-crystal over Y, let $\mathcal{F}_V$ be the pull back of $\mathcal{F}$ (or $\mathcal{F}_Y$) to an F^n-crystal over V, i.e., of $CRIS(V/\text{Spec }(\mathbb{Z}_p))$. Let $k(y)$ be the residue field of a point $y \in Y$. If $V = \text{Spec }(k(y)) \to Y$ is the natural morphism, then we denote $\mathcal{F}_V = \mathcal{F}_{\text{Spec }(k(y))}$ simply by $\mathcal{F}_y$ (the fiber of $\mathcal{F}$ at y).

For an $\mathbb{F}_p$-algebra R, let $W(R)$ be the ring of p-typical Witt vectors with coefficients in R. Let $\mathbb{W}(R) = (\text{Spec } R, \text{Spec }(W(R)), \text{can})$ be the thickening in which 'can' stands for the canonical divided power structure of the kernel of the epimorphism $W(R) \to W_1(R) = R$. For $s \in \mathbb{N}^*$, let $W_s(R)$ be the ring of p-typical Witt vectors of length s with coefficients in R. Let $\mathbb{W}_s(R) = (\text{Spec } R, \text{Spec }(W_s(R)), \text{can})$ be the thickening defined naturally by $\mathbb{W}(R)$. Let Φ_R be the Frobenius endomorphism of either $W(R)$ or $W_s(R)$.

The property of a reduced locally closed subscheme being pure in S is local for the faithfully flat topology of S, and thus until the end we will also assume that $S = \text{Spec } A$ is an affine $\mathbb{F}_p$-scheme and that there exists $d \in \mathbb{N}$ such that for all $x \in S$ we have $\nu_x(r) = d$. As the scheme S is locally noetherian and affine, it is noetherian. To prove Theorem 1, we have to prove that T is an affine scheme.

2.1 Some abelian categories

Let $\mathcal{M}(W_s(R))$ be the abelian category whose objects are pairs (O, ϕ_O) comprising from a $W_s(R)$-module O and a Φ_R^n-linear endomorphism $\phi_O : O \to O$ (i.e., ϕ_O is additive and for all $z \in O$ and $\sigma \in W_s(R)$ we have $\phi_O(\sigma z) = \Phi_R^n(\sigma)\phi_O(z)$) and whose morphisms $f : (O_1, \phi_{O_1}) \to (O_2, \phi_{O_2})$ are $W_s(R)$-linear maps $f : O_1 \to O_2$ satisfying $f \circ \phi_{O_1} = \phi_{O_2} \circ f$. If $t \in \{0, \ldots, s-1\}$, then by a *quasi-isogeny* of $\mathcal{M}(W_s(R))$ whose cokernel is annihilated by p^t we mean a morphism $f : (O_1, \phi_{O_1}) \to (O_2, \phi_{O_2})$ of $\mathcal{M}(W_s(R))$ which has the following two properties: (i) both O_1 and O_2 are projective $W_s(R)$-modules which locally in the Zariski topology of Spec $(W_s(R))$ have the same positive rank, and (ii) the cokernel $O_2/f(O_1)$ is annihilated by p^t. An object (O, ϕ_O) of $\mathcal{M}(W_s(R))$ is called *divisible* by $t \in \{1, \ldots, s-1\}$ if O is a projective $W_s(R)$-module such that $\mathrm{Im}(\phi_O) \subseteq p^t O_2$.

For $l \in \mathbb{N}^*$ we have a natural functor $\mathcal{M}(W_{s+l}(R)) \to \mathcal{M}(W_s(R))$ to be referred by abuse of language as the reduction modulo p^s functor.

If Y is a Spec $(\mathbb{F}_p)$-scheme, in a similar way we define the scheme $W_s(Y)$, its Frobenius endomorphism Φ_Y, and the abelian category $\mathcal{M}(W_s(Y))$ and speak about quasi-isogenies of $\mathcal{M}(W_s(Y))$ whose cokernels are annihilated by p^t with $t \in \{0, \ldots, s-1\}$, about objects of $\mathcal{M}(W_s(Y))$ divisible by $t \in \{1, \ldots, s-1\}$, and about reduction modulo p^s functors $\mathcal{M}(W_{s+l}(Y)) \to \mathcal{M}(W_s(Y))$. We have canonical identifications $\mathcal{M}(W_s(R)) = \mathcal{M}(W_s(\mathrm{Spec}\ R))$.

For homomorphisms $R \to R_1$ and morphisms $Y_1 \to Y$ we have natural pull back functors $\mathcal{M}(W_s(R)) \to \mathcal{M}(W_s(R_1))$ and $\mathcal{M}(W_s(Y)) \to \mathcal{M}(W_s(Y_1))$.

To prove that T is an affine scheme, we can also assume that the *evaluation* M of $\mathcal{M}$ at the thickening $\mathbb{W}_1(A)$ is a free A-module of rank r. The evaluation of $\phi_{\mathcal{M}}$ at this thickening is a Φ_A^n-linear endomorphism $\phi_M : M \to M$.

In what follows we will apply twice the following elementary general fact which can be also deduced easily from the elementary divisor theorem.

Fact 1 *Let D be a discrete valuation ring and let $\pi \in D$ be a uniformizer of it. Let $s, t \in \mathbb{N}$ be such that $s > t$. Let $D_s = D/(\pi^s)$. Let $g_s : D_s^r \to D_s^r$ be a D_s-linear endomorphism such that its cokernel is annihilated by π^t. Then for each $x \in D_s^r - \pi D_s^r$, we have $g_s(x) \in D_s^r - \pi^{t+1} D_s$.*

Proof: Let $g : D^r \to D^r$ be a D-linear endomorphism which lifts g_s. Let $E = \mathrm{Im}(g) + \pi^s D^r$ (one can easily check that $E = \mathrm{Im}(g)$ but we will not stop to argue this). It is a free D-module of rank r which (as $\pi^t \mathrm{Coker}(g_s) = 0$) contains $\pi^t D^r$. Thus $\pi^s D^r \subseteq pE$ and therefore $\mathrm{Im}(g)$ surjects onto the D_1-vector space $E/\pi E$ of rank r. Hence a D_s-basis of D_s^r maps via g to a D_1-basis of E. From this and the fact that $\pi^{t+1} D^r \subseteq \pi E$ we get that no element of a D_s-basis of D_s^r is mapped by g to $\pi^{t+1} D^r$. Thus the fact holds.

2.2 On (a, b)

If (a, b) is $(0, 0)$ or (r, d), then $T = S$. If $a = 0$ and $b > 0$ or if $a = r$ and $b \neq d$, then $T = \emptyset$. Thus, to prove that T is an affine scheme we can assume that $1 \leq a \leq r - 1$.

Lemma 1 *Let k be a field of characteristic p. Let $\nu : [0, r] \to [0, \infty)$ be the Newton polygon of an F^n-crystal $\mathcal{F}$ over k of rank r. Let $a, b \in \mathbb{N}$ be such that $1 \leq a \leq r - 1$. Then (a, b) is a break point of ν if and only if $(1, b)$ is a break point of the Newton polygon $\bigwedge^a(\nu)$ of the F^n-crystal over k of rank $\binom{r}{a}$ which is the exterior power $\bigwedge^a(\mathcal{F})$ of $\mathcal{F}$.*

Proof: Let $\alpha_1 \leq \cdots \leq \alpha_r$ be the Newton polygon slopes of ν. Let $\beta_1 \leq \cdots \leq \beta_{\binom{r}{a}}$ be the Newton polygon slopes of $\bigwedge^a(\nu)$. We have

$$\beta_1 = \sum_{i=1}^{a} \alpha_i \quad \text{and} \quad \beta_2 = (\sum_{i=1}^{a-1} \alpha_i) + \alpha_{a+1} = \beta_1 + \alpha_{a+1} - \alpha_a.$$

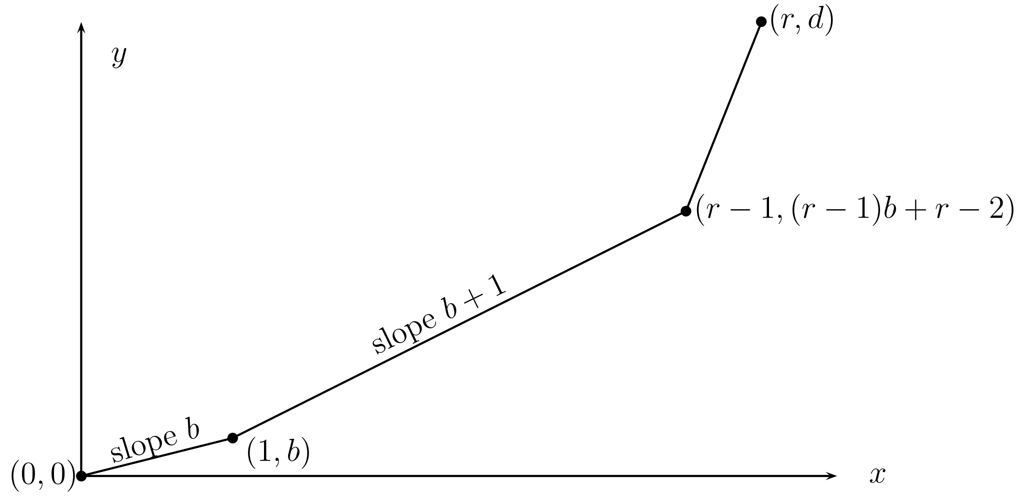
Thus $\beta_1 < \beta_2$ if and only if $\alpha_a < \alpha_{a+1}$. Moreover, (a, b) is a break point of ν if and only if we have $\alpha_a < \alpha_{a+1}$, and $(1, b)$ is a break point of the Newton polygon $\bigwedge^a(\nu)$ if and only if we have $\beta_1 < \beta_2$. The lemma follows from the last two sentences. $\square$

Based on Lemma 1, to prove that T is an affine scheme by replacing $\mathcal{C}$ with its exterior power $\bigwedge^a(\mathcal{C})$ we can assume that $a = 1$.

2.3 A description of T

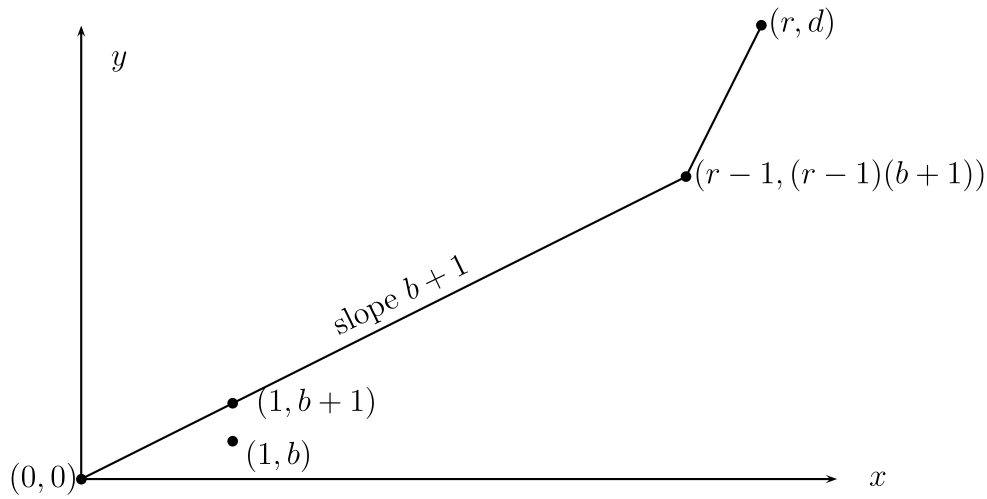
Let $q \in \mathbb{N}^*$ be such that for each $x \in S$ the Newton polygon slopes of the F^{nq}-crystal over Spec $(k(x))$ which is the q-th iterate of $\mathcal{C}_x$ are all integers. For instance, as each Newton polygon slope of $\mathcal{C}_x$ is a rational number whose denominator is a natural number at most equal to r, we can take $q = r!$. Thus by replacing n by nq and $\mathcal{C}$ by its q-th iterate, we can assume that for each $x \in S$ the Newton polygon slopes of $\mathcal{C}_x$ are natural numbers.

We consider the Newton polygon $\nu_1 : [0, r] \to [0, \infty)$ whose graph is:



If $x \in T$, then as all Newton polygon slopes of $\mathcal{C}_x$ are natural numbers, these Newton polygon slopes are $\alpha_1 = b$, $\alpha_2 \geq b+1$, $\alpha_{r-1} \geq b+1$, and $\alpha_r = d - \sum_{i=1}^{r-1} \alpha_i \geq b+1$. Therefore, if $x \in T$ then we have $x \in S_{\geq \nu_1}$. This implies that T is a subscheme of the closed subscheme $S_{\geq \nu_1}$ of S. By replacing S with $S_{\geq \nu_1}$ we can assume that $S = S_{\geq \nu_1}$. Thus S is reduced.

If $r(b+1) > d$, then $S = S_{\geq \nu_1} = S_{\nu_1} = T$ and thus T is affine. Thus we can assume that $r(b+1) \leq d$ and therefore there exists a Newton polygon $\nu_2 : [0, r] \to [0, \infty)$ whose graph is:



If $x \in S - T = S_{\geq \nu_1} - T$, then all Newton polygon slopes of $\mathcal{C}_x$ are natural numbers $\alpha_1 \geq b+1$, $\alpha_2 \geq b+1$, $\alpha_{r-1} \geq b+1$, and $\alpha_r = d - \sum_{i=1}^{r-1} \alpha_i \geq b+1$ and thus ν_x is above ν_2. If ν_x is not above ν_2, then as ν_x is above ν_1 (as $S = S_{\geq \nu_1}$) we get that we have $\alpha_1 = b$ and $\alpha_i \geq b+1$ for $i \in \{2, \ldots, r\}$.

From the last two sentences we get that we have identities

$$T = T_{(1,b)} = S - S_{\geq \nu_2} = S_{\geq \nu_1} - S_{\geq \nu_2}.$$

Thus, under all the above reduction steps, T is an open subscheme of S.

2.4 On S

The statement that T is an affine scheme is local in the faithfully flat topology of S and therefore until the end of Section 3 we will assume that A is a complete local reduced noetherian ring. Thus A is also excellent and therefore its normalization in its ring of fractions is a finite product of normal complete local noetherian integral domains. Based on [V1], Lemma 2.9.2 which is a standard application of Chevalley's theorem of [G1], Chapter II, (6.7.1), to prove that T is an affine scheme we can replace A by one of the factors of the last product. Thus we can assume that A is a normal complete local noetherian integral domain. We can also assume that T is non-empty and therefore it is an open dense subscheme of S. Let K be the field of factions of A and let $\bar{K}$ be an algebraic closure of it.

3 Proof of Theorem 1

In this section we complete the proof of Theorem 1, i.e., we prove that T is an affine scheme when $a = 1 < r$, for each $x \in S$ all Newton polygon slopes of $\mathcal{C}_x$ are natural numbers, we have $S = S_{\geq \nu_1} = \operatorname{Spec} A$ with A a normal complete local noetherian integral domain, and $T = T_{(1,b)} = S - S_{\geq \nu_2}$ is open dense in S. Let $\mathcal{E}_b = (\mathcal{M}_b, \phi_{\mathcal{M}_b})$ be the pull back to S of the F^n-crystal over $\operatorname{Spec}(\mathbb{F}_p)$ of rank 1 and Newton polygon slope b defined by the pair $(\mathbb{Z}_p, p^b 1_{\mathbb{Z}_p})$. Let η be the generic point $\operatorname{Spec} K \to S$ of S. Let $s, l \in \mathbb{N}^*$.

In Subsection 3.1 we consider commutative affine group schemes $\mathbb{H}_s$ over S of morphisms between certain evaluations of $\mathcal{E}_b$ and $\mathcal{C}$. In Sections 3.2 we glue morphisms between different such evaluations in order to introduce good sections above T of the morphisms $\mathbb{H}_s \to S$ in Subsection 3.3. In Subsection 3.4 we complete the proof of Theorem 1. The key idea (the plan) can be summarized as follows: under suitable reductions, for $s >> 0$ via such good sections above T we can identify T with a closed subscheme of $\mathbb{H}_s$ and therefore we can conclude that T is an affine scheme.

If R is a reduced perfect ring of characteristic p, following [K] we say that an F^n-crystal $\mathcal{F}$ over Spec R is divisible by b if its evaluation at the endomorphism Φ_R^n of the thickening $\mathbb{W}(R)$ is defined by a Φ_R^n-linear endomorphism whose q-th iterate for all $q \in \mathbb{N}^*$ is congruent to 0 modulo p^{bq}. Thus if $y \in$ Spec R, then the Hodge polygon slopes of $\mathcal{F}_y$ are all greater or equal to b.

3.1 Moduli group schemes of morphisms

For an A-algebra B and an F^n-crystal $\mathcal{F}$ over B, let $\mathbb{E}_s(\mathcal{F})$ be the evaluation of $\mathcal{F}$ at the thickening $\mathbb{W}_s(B)$; it is an object of the category $\mathcal{M}(W_s(B))$. In particular, we write $\mathbb{E}_s(\mathcal{C}_B) = (M_{s,B}, \phi_{M_{s,B}})$ and let $\mathbb{E}_s(\mathcal{E}_{b,B}) = (N_{s,B}, \phi_{N_{s,B}})$. Thus we have $M = M_{1,A}$, $\phi_M = \phi_{M_{1,A}}$, and $N_{s,B} = W_s(B)$. Moreover $\phi_{N_{s,B}} : N_{s,B} \to N_{s,B}$ is the Φ_B^n-linear endomorphism which maps 1 to p^b and $\phi_{M_{s,B}} : M_{s,B} \to M_{s,B}$ is a Φ_B^n-linear endomorphism and we have $M_{s,B} = W_s(B) \otimes_{W_s(A)} M_{s,A}$. The kernel of the epimorphism $W_s(B) \to W_1(B) = B$ is a nilpotent ideal. Based on this and the fact that M is a free A-module of rank r, we get that each $M_{s,B}$ is a free $W_s(B)$-module of rank r.

We consider the commutative affine group scheme $\mathbb{H}_s$ over S which represents the following functor: for an A-algebra B, the abelian group

$$\mathbb{H}_s(B) = \mathrm{Hom}_{\mathcal{M}(W_s(B))}(\mathbb{E}_s(\mathcal{E}_{b,B}), \mathbb{E}_s(\mathcal{C}_B))$$

is the group of all $W_s(B)$-linear maps $f : N_{s,B} \to M_{s,B}$ which satisfy the identity $f \circ \phi_{N_{s,B}} = \phi_{M_{s,B}} \circ f$. The S-scheme $\mathbb{H}_s$ is of finite presentation (for $n = 1$, see [V1], Lemma 2.8.4.1; the proof of loc. cit. applies to all $n \in \mathbb{N}^*$).

Let $x \in S$ be a point of codimension 1. Thus the local ring $D_x := \mathcal{O}_{S,x}$ of S at x is a discrete valuation ring. Let E_x be a complete discrete valuation ring which dominates D_x and has a residue field which is algebraically closed. Let P_x be the perfection of E_x. We recall that $\mathcal{C}_{P_x}$ is the pull back of $\mathcal{C}$ via the natural morphism Spec $P_x \to S$. As $S = S_{\geq \nu_1}$, the Newton polygon slopes of the two fibers of $\mathcal{C}_{P_x}$ are greater or equal to b. Thus from [K], Theorem 2.6.1 we get the existence of an F^n-crystal $\mathcal{D}$ over Spec P_x which is divisible by b and which is equipped with an isogeny

$$\psi_x : \mathcal{D} \to \mathcal{C}_{P_x}$$

whose cokernel is annihilated by p^t for some $t \in \mathbb{N}$. Based on the proof of loc. cit. we can assume that

$$t = (r-1)b$$

depends only on r and b.

Proposition 1 *We assume that the point $x \in S$ of codimension 1 belongs to T. Then there exists a unique F^n-subcrystal $\mathcal{D}_b$ of $\mathcal{D}$ which is isomorphic to the pull back $\mathcal{E}_{b,P_x}$ of $\mathcal{E}_b$. Moreover, $\mathcal{D}_b$ has a unique direct supplement in $\mathcal{D}$.*

Proof: We know that for $y \in \text{Spec } P_x$ all Hodge polygon slopes of $\mathcal{D}_y$ are at least b. If all Hodge polygon slopes of $\mathcal{D}_y$ are at least $b+1$, then all Newton polygon slopes of $\mathcal{D}_y$ are at least $b+1$. As under the morphism $\text{Spec } P_x \to S$, the point y maps to either $x \in T$ or $\eta \in T$ and as ψ_x is an isogeny, $(1, b)$ is a break point of the Newton polygon of $\mathcal{D}_y$. From the last three sentences we get that $(1, b)$ is a point of the Hodge polygon of $\mathcal{D}_y$.

Thus for each point $y \in \text{Spec } P_x$, $(1, b)$ is a break point of the Newton polygon of $\mathcal{D}_y$ and is a point of the Hodge polygon of $\mathcal{D}_y$. Due to this, from [K], Theorem 2.4.2 we get that there exists a unique direct sum decomposition

$$\mathcal{D} = \mathcal{D}_b \oplus \mathcal{D}_{>b}$$

into F^n-crystals over $\text{Spec } P_x$, where $\mathcal{D}_b$ is of rank 1 and each fiber of it at a point $y \in \text{Spec } P_x$ has all Hodge and Newton polygon slopes equal to b and where $\mathcal{D}_{>b}$ is of rank $r-1$ and each fiber of it at a point $y \in \text{Spec } P_x$ has all Newton polygon slopes greater than b (and has all Hodge polygon slopes greater or equal to b).

As $\mathcal{D}$ is divisible by b, $\mathcal{D}_b$ and $\mathcal{D}_{>b}$ are also divisible by b.

As P_x is perfect, for each $l \in \mathbb{N}^*$ we have $W(P_x)/(p^l) = W_l(P_x)$ and the module of differentials $\Omega^1_{W_l(P_x)}$ is 0. Thus, from [BM], Proposition 1.3.3 we get that an F^n-crystal over $\text{Spec } P_x$ is uniquely determined by its evaluation at the thickening $\mathbb{W}(P_x)$. The evaluation of $\mathcal{E}_{b,P_x}$ at the thickening $\mathbb{W}(P_x)$ is canonically identified with $(W(P_x), p^b\Phi^n_{P_x})$ and the evaluation of $\mathcal{D}_b$ at the thickening $\mathbb{W}(P_x)$ can be identified with $(W(P_x), p^b\Phi_b)$, where $\Phi_b : W(P_x) \to W(P_x)$ is a $\Phi^n_{P_x}$-linear endomorphism such that $\Phi_b(1)$ generates $W(P_x)$.

As P_x is the perfection of E_x and as E_x is complete and has an algebraically closed residue field, the rings $W(P_x)$ and $W_l(P_x)$ are strictly henselian and p-adically complete. We check that these properties imply that there exists a unit υ of $W(P_x)$ such that we have

$$\Phi_b(\upsilon) = \Phi_{P_x}^n(\upsilon)\Phi_b(1) = \upsilon.$$

If $n = 1$, then from [BM], Proposition 2.4.9 we get that for each $l \in \mathbb{N}^*$ there exists a unit $\upsilon_l \in W(P_x)$ such that we have $\Phi_b(\upsilon_l) - \upsilon_l \in p^l W(P_x)$ and the proof of loc. cit. checks that we can assume that $\upsilon_{l+1} - \upsilon_l \in p^l W(P_x)$. Thus for $n = 1$ we can take υ to be the p-adic limit of the sequence $(\upsilon_l)_{l\geq 1}$. This argument applies entirely for $n > 1$.

The multiplication by u defines an isomorphism

$$(W(P_x), p^b\Phi_{P_x}^n) \to (W(P_x), p^b\Phi_b)$$

which defines an isomorphim $\mathcal{E}_{b,P_x} \to \mathcal{D}_b$. $\square$

From now we will assume that $x \in T$. We consider a composite morphism

$$j_x[s] : \mathbb{E}_s(\mathcal{E}_{b,P_x}) \to \mathbb{E}_s(\mathcal{D}_b) \to \mathbb{E}_s(\mathcal{D}) = \mathbb{E}_s(\mathcal{D}_b) \oplus \mathbb{E}_s(\mathcal{D}_{>b})$$

in which the first arrow is an isomorphism and the second arrow is the split monomorphism associated to the direct sum decomposition.

Let

$$i_x(s) : \mathbb{E}_s(\mathcal{E}_{b,P_x}) \to \mathbb{E}_s(\mathcal{C}_{P_x})$$

be the composite of $j_x[s]$ with the morphism $\psi_x[s] : \mathbb{E}_s(\mathcal{D}) \to \mathbb{E}_s(\mathcal{C}_{P_x})$ which is the evaluation of the isogeny ψ_x at the thickening $\mathbb{W}_s(P_x)$ (i.e., which is the reduction modulo p^s of ψ_x). From now on, we will take $s > t = (r-1)b$. We note that $\psi_x[s]$ is a quasi-isogeny whose cokernel is annihilated by p^t and whose domain is divisible by b.

3.2 Gluing morphisms

For each point $x \in T$ of codimension 1 (i.e., whose local ring D_x is a discrete valuation ring), we follow [V1], Subsection 2.8.3 to show the existence of a finite field extension K_x of K and of an open subset T_x of the normalization of T in Spec K_x such that T_x has a local ring which is a discrete valuation ring D_x^+ that dominates D_x and moreover we have a morphism

$$i_{T_x}(s) : \mathbb{E}_s(\mathcal{E}_{b,T_x}) \to \mathbb{E}_s(\mathcal{C}_{T_x})$$

of the category $\mathcal{M}(W_s(T_x))$ which is the composite of a split monomorphism with a quasi-isogeny whose cokernel is annihilated by p^t and whose domain is divisible by b.

To check this, with the notations of Subsection 3.1 we consider four identifications $E_s(\mathcal{C}_{D_x}) = (W_s(D_x)^r, \phi_{s,x})$, $\mathbb{E}_s(\mathcal{E}_{b,D_x}) = (W_s(D_x), p^b\Phi^n_{D_x})$, $\mathbb{E}_s(\mathcal{D}_b) = (W_s(P_x), p^b\Phi^n_{P_x})$, and $\mathbb{E}_s(\mathcal{D}_{>b}) = (W_s(P_x)^{r-1}, p^b\phi_{s,>b,x})$. Now, the $W_s(P_x)$-linear map $\psi_{s,P_x} : W_s(P_x)^r \to W_s(P_x)^r$ defining $\psi_x[s]$ and the $\Phi^n_{P_x}$-linear map $\phi_{s,>b,x} : W_s(P_x)^{r-1} \to W_s(P_x)^{r-1}$ involve a finite number of coordinates of Witt vectors of length s and therefore are defined over $W_s(B_x)$, where B_x is a finitely generated D_x-subalgebra of P_x. We can choose B_x such that the resulting $W_s(B_x)$-linear map $\psi_{s,B_x} : W_s(B_x)^r \to W_s(B_x)^r$ has a cokernel annihilated by p^t. The faithfully flat morphism Spec $B_x \to$ Spec D_x has quasi-sections (cf. [G2], Corollary (17.16.2)) and therefore there exists a finite field extension K_x of K and a discrete valuation ring D^+_x of the normalization T in K_x which dominates D_x and for which we have a D_x-homomorphism $B_x \to D^+_x$. The $W_s(D^+_x)$-linear map $\psi_{s,D^+_x} : W_s(D^+_x)^r \to W_s(D^+_x)^r$ which is the natural tensorization of ψ_{s,B_x} induces (via restriction to the first factor $W_s(D^+_x)$ of $W_s(D^+_x)^r$) a morphism $i_{D^+_x}(s) : \mathbb{E}_s(\mathcal{E}_{b,D^+_x}) \to \mathbb{E}_s(\mathcal{C}_{D^+_x})$ of the category $\mathcal{M}(W_s(D^+_x))$ which is the composite of a split monomorphism with a quasi-isogeny whose cokernel is annihilated by p^t and whose domain is divisible by b. It is easy to see that there exists an open subset T_x of the normalization of T in K_x which has D^+_x as a local ring and for which there exists a morphism $i_{T_x}(s) : \mathbb{E}_s(\mathcal{E}_{b,T_x}) \to \mathbb{E}_s(\mathcal{C}_{T_x})$ of the category $\mathcal{M}(W_s(T_x))$ that has all the desired properties and that extends the morphism $i_{D^+_x}(s)$ of the category $\mathcal{M}(W_s(D^+_x))$.

By working with $s+l$ instead of s, we can assume that there exists $l \in \mathbb{N}$, $l >> 0$ such that $i_{T_x}(s) : \mathbb{E}_s(\mathcal{E}_{b,T_x}) \to \mathbb{E}_s(\mathcal{C}_{T_x})$ is the reduction modulo p^s of a morphism

$$i_{T_x}(s+l) : \mathbb{E}_{s+l}(\mathcal{E}_{b,T_x}) \to \mathbb{E}_{s+l}(\mathcal{C}_{T_x})$$

of the category $\mathcal{M}(W_{s+l}(T_x))$.

Let I_s be the set of morphisms $\mathbb{E}_s(\mathcal{E}_{b,\bar{K}}) \to \mathbb{E}_s(\mathcal{C}_{\bar{K}})$ which lift to morphisms $\mathbb{E}_{s+l}(\mathcal{E}_{b,\bar{K}}) \to \mathbb{E}_{s+l}(\mathcal{C}_{\bar{K}})$ for some $l >> 0$. From [V1], Theorem 5.1.1 (a) (applied for $l >> 0$ which depends only on b and r) we get that each element of I_s is the evaluation at the thickening $\mathbb{W}_s(\bar{K})$ of a morphism of F^n-crystals $\mathcal{E}_{b,\bar{K}} \to \mathcal{C}_{\bar{K}}$ (strictly speaking loc. cit. is stated for $n = 1$ but its proof works for all $n \in \mathbb{N}^*$). This implies that I_s is a finite set whose elements are all pull backs of morphisms of $\mathcal{M}(W_s(L))$, where L is a suitable finite field extension of K contained in $\bar{K}$. By replacing S with its normalization in L, we can assume that $L = K$. As inside K_x we have an identity $D_x^+ \cap K = D_x$, inside $W_s(K_x)$ we have an identity $W_s(D_x^+) \cap W_s(K) = W_s(D_x)$. From the last three sentences we get that the pull back $i_{D_x^+}(s)$ of $i_{T_x}(s)$ to a morphism of $\mathcal{M}(W_s(D_x^+))$ is the pull back of a morphism of $\mathcal{M}(W_s(D_x))$. Based on this we can assume that there exists an open subscheme U_x of T which contains x and which has the property that there exists a morphism

$$i_{U_x}(s) : \mathbb{E}_s(\mathcal{E}_{b,U_x}) \to \mathbb{E}_s(\mathcal{C}_{U_x})$$

of the category $\mathcal{M}(W_s(U_x))$ such that $i_{T_x}(s)$ is the pull back of it.

We consider an identification $\mathcal{C}_{\bar{K}} = (Q, \phi_Q)$, where $Q = W(\bar{K})^r$ and $\phi_Q : Q \to Q$ is a $\Phi_{\bar{K}}^n$-linear endomorphism. The Newton polygon ν_η of $\mathcal{C}_{\bar{K}}$ has the Newton polygon slope b with multiplicity 1 and therefore there exists a unique non-zero direct summand Q_b of Q such that we have $\phi_Q(Q_b) = p^b Q_b$. The rank of the $W(\bar{K})$-module Q_b is 1. Let $z_b \in Q_b$ be such that $Q_b = W(\bar{K}) z_b$ and $\phi_Q(z_b) = p^b z_b$; it is unique up to multiplication by units of $W(\mathbb{F}_{p^n})$.

We have a canonical identification $\mathcal{E}_{b,\bar{K}} = (W(\bar{K}), p^b \Phi_K^n)$. The morphism $\mathbb{E}_s(\mathcal{E}_{b,\bar{K}}) \to \mathbb{E}_s(\mathcal{C}_{\bar{K}})$ defined by $i_{T_x}(s)$ is an element of I_s and therefore it is the reduction modulo p^s of a morphism $\lambda_x : (W(\bar{K}), p^b \Phi_K^n) \to (Q, \phi_Q)$ of F^n-crystals over $\bar{K}$. Clearly $\lambda_x(1) \in Q_b$ and thus there exists a unique element $\tau_x \in W(\mathbb{F}_{p^n})$ such that we have

$$\lambda_x(1) = \tau_x z_b.$$

As $i_{T_x}(s)$ is the composite of a split monomorphism with a quasi-isogeny whose cokernel is annihilated by p^t from Fact 1 applied with $D = W(\bar{K})$ we get that τ_x modulo p^{t+1} is a non-zero element of $W_{t+1}(\mathbb{F}_{p^n})$. Therefore we can write $\tau_x = p^{t_x} u_x$, where $u_x \in W(\mathbb{F}_{p^n})$ is a unit and where $t_x \in \{0, \ldots, t\}$.

From now on, we will take $s > 2t$. We consider the morphism

$$\theta_x := p^{t-t_x} u_x^{-1} i_{U_x}(s) : \mathbb{E}_s(\mathcal{E}_{b,U_x}) \to \mathbb{E}_s(\mathcal{C}_{U_x})$$

of the category $\mathcal{M}(W_s(U_x))$; its pull back to a morphism of $\mathcal{M}(W_s(T_x))$ is the composite of a split monomorphism with a quasi-isogeny whose cokernel is annihilated by p^{t+t_x} and thus also by p^{2t} and whose domain is divisible by b. The pull back of θ_x to a morphism of $\mathcal{M}(W_s(\bar{K}))$ is the reduction modulo p^s of the morphism $p^{t-t_x} u_x^{-1} \lambda_x : (W(\bar{K}), p^b \Phi_K^n) \to (Q, \phi_Q)$ which maps 1 to $p^t z_b$ and which does not depend on the point $x \in T$ of codimension 1.

Let U be the open subscheme of T which is the union of all U_x's. From the previous paragraph we get that the θ_x's glue together to define a morphism

$$\theta : \mathbb{E}_s(\mathcal{E}_{b,U}) \to \mathbb{E}_s(\mathcal{C}_U)$$

of the category $\mathcal{M}(W_s(U))$.

By replacing S with its normalization in anyone of the finite field extensions K_x of K, we can assume that there exists an open dense subscheme U_0 of U such that the pull back $\theta_{U_0} : \mathbb{E}_s(\mathcal{E}_{b,U_0}) \to \mathbb{E}_s(\mathcal{C}_{U_0})$ of θ to a morphism of $\mathcal{M}(W_s(U_0))$ is the composite of a split monomorphism with a quasi-isogeny whose cokernel is annihilated by p^{2t} and whose domain is divisible by b: under such a replacement, we can take U_0 to be T_x itself.

3.3 Good section of $\mathbb{H}_s$

We have $\mathrm{codim}_T(T-U) \geq 2$ and the morphism θ is defined by a section $\theta : U \to \mathbb{H}_s$ denoted in the same way.

Let $\mathbb{I}_s$ be the schematic closure $\overline{\theta(U)}$ of $\theta(U)$ in $\mathbb{H}_s$. As the scheme $\mathbb{H}_s$ is affine and noetherian and as U is an integral scheme, the scheme $\mathbb{I}_s$ is also affine, noetherian, and integral. We have a commutative diagram:

$$\begin{tikzcd} & & \mathbb{I}_s \arrow[d, "\text{affine}"] \\ U \arrow[urr, "\text{open}\ \theta"] \arrow[r, hook] & T \arrow[r, hook] & S. \end{tikzcd}$$

We consider the pullback $\mathbb{J}_s$ of $\mathbb{I}_s$ to T:

$$\begin{tikzcd} & \mathbb{J}_s \arrow[r, hook, "\text{open}"] \arrow[d, "\xi"', "\text{affine}"] & \mathbb{I}_s \arrow[d, "\text{affine}"] \\ U \arrow[ur, hook, "\text{open}"] \arrow[r, hook, "\text{open}"'] & T \arrow[r, hook, "\text{open}"'] & S. \end{tikzcd}$$

Lemma 2 *The affine morphism $\xi : \mathbb{J}_s \to T$ is an isomorphism.*

Proof: To prove that ξ is an isomorphism, we can assume that $T = S = \mathrm{Spec}\ A$ is an affine scheme. As ξ is an affine morphism, $\mathbb{J}_s = \mathrm{Spec}\ B$ is also an affine scheme. Since U is open dense in both T and $\mathbb{I}_s$, T and $\mathbb{J}_s$ have the same field of fractions K. As $\mathrm{codim}_T(T-U) \geq 2$ and as U is an open subscheme of both T and $\mathbb{J}_s$, we have $A_{\mathfrak{p}} = B_{\mathfrak{p}}$ for each prime $\mathfrak{p} \in S = T$ of height 1. As A is a noetherian normal domain, inside K we have

$$A \subseteq B \subseteq \bigcap_{\mathfrak{q} \in \mathrm{Spec}\, B \text{ of height } 1} B_{\mathfrak{q}} \subseteq \bigcap_{\mathfrak{p} \in \mathrm{Spec}\, A \text{ of height } 1} A_{\mathfrak{p}} = A$$

(cf. [M], (17.H), Theorem 38 for the equality part; the first inclusion is defined by ξ). Therefore $A = B$. $\square$

This Lemma 2 allows us in what follows to identify T itself with an open dense subscheme of $\mathbb{I}_s$ (i.e., with $\mathbb{J}_s$).

3.4 End of the proof

In this subsection we will show that for $s >> 0$, we have $T = \mathbb{I}_s$. This will complete the proof of Theorem 1 as $\mathbb{I}_s$ is an affine scheme.

We are left to show that the assumption that for $s >> 0$ we have $T \neq \mathbb{I}_s$ leads to a contradiction. This assumption implies that there exists an algebraically closed field k of characteristic p and a morphism $\zeta_0 : \mathrm{Spec}\,(k[[X]]) \to \mathbb{I}_s$ with the properties that under it the generic point of $\mathrm{Spec}\,(k[[X]])$ maps to U_0 and its special point maps to $\mathbb{I}_s - T$.

Let $P = k[[X]]^{\mathrm{perf}}$ be the perfection of $k[[X]]$, let κ be the perfect field which is the field of fractions of P, and let $\zeta : \mathrm{Spec}\, P \to \mathbb{I}_s$ be the morphism defined naturally by ζ_0. To the composite of ζ with the closed embedding $\mathbb{I}_s \to \mathbb{H}_s$ corresponds a morphism

$$\omega : \mathbb{E}_s(\mathcal{E}_{b,P}) \to \mathbb{E}_s(\mathcal{C}_P)$$

of the category $\mathcal{M}(W_s(P))$ whose pull back ω_κ to a morphism of $\mathcal{M}(W_s(\kappa))$ is equal to the pull back $\theta_\kappa : \mathbb{E}_s(\mathcal{E}_{b,\kappa}) \to \mathbb{E}_s(\mathcal{C}_\kappa)$ of θ.

We have a natural identification $\mathbb{E}_s(\mathcal{E}_{b,P}) = (W_s(P), p^b\Phi_P^n)$ and we consider an identification $\mathbb{E}_s(\mathcal{C}_P) = (W_s(P)^r, \phi)$. Thus we have a $W_s(P)$-linear map

$$\omega : W_s(P) \to W_s(P)^r$$

such that $\omega \circ p^b\Phi_P^n = \phi \circ \omega$. We consider an isogeny $\mathcal{D} \to \mathcal{C}_P$ whose cokernel is annihilated by p^t and with $\mathcal{D}$ divisible by b, again cf. [K], Theorem 2.6.1 (here $t = (r-1)b$ is as before Proposition 1). Thus we also have an isogeny $\iota : \mathcal{C}_P \to \mathcal{D}$ whose cokernel is annihilated by p^t. We consider its evaluation

$$\iota[s] : \mathbb{E}_s(\mathcal{C}_P) \to \mathbb{E}_s(\mathcal{D})$$

at the thickening $\mathbb{W}_s(P)$. Under an identification $\mathbb{E}_s(\mathcal{D}) = (W_s(P)^r, p^b\varphi)$ with $\varphi : W_s(P)^r \to W_s(P)^r$ as a Φ_P^n-linear endomorphism, we get a $W_s(P)$-linear endomorphism $\iota[s] : W_s(P)^r \to W_s(P)^r$ such that we have $\iota[s] \circ \phi = p^b\varphi \circ \iota[s]$. We consider the composite morphism

$$\rho = \iota[s] \circ \omega : \mathbb{E}_s(\mathcal{E}_{b,P}) \to \mathbb{E}_s(\mathcal{D})$$

identified with a $W_s(P)$-linear map $\rho : W_s(P) \to W_s(P)^r$ such that we have $\rho \circ p^b\Phi_P^n = p^b\varphi \circ \rho$. Let

$$\gamma = \rho(1) = (\gamma_1, \ldots, \gamma_r) \in W_s(P)^r.$$

From the identity $\rho \circ p^b\Phi_P = p^b\varphi \circ \rho$ we get that the image of $\varphi(\gamma) - \gamma$ in $W_{s-b}(P)^r$ is 0. Writing $\gamma = p^u\delta$, where $u \in \mathbb{N}$ and $\delta \in W_s(P)^r - pW_s(P)^r$, we get that the image of $\varphi(\delta) - \delta$ in $W_{s-b-u}(P)^r$ is 0. Let $\bar{\delta} \in P^r - 0$ be the image in $P^r = W_1(P)$ of δ (i.e., the reduction modulo p of δ).

Lemma 3 *If $s \geq 3t + 1$, then we have $u \leq 3t$. Therefore, if moreover we have $s \geq 3t + b + 1$, then the image of $\varphi(\delta) - \delta$ in $W_{s-b-3t}(P)^r$ is 0.*

Proof: To check this we can work over $W_s(\kappa)$. As the generic point of Spec P maps to U_0, $\omega_\kappa = \theta_\kappa : \mathbb{E}_s(\mathcal{E}_{b,\kappa}) \to \mathbb{E}_s(\mathcal{C}_\kappa)$ is the pull back of the morphism θ_{U_0}. The pull back ρ_κ of ρ to $\mathcal{M}(W_s(\kappa))$ is a composite morphism

$$\rho_\kappa = \iota[s]_\kappa \circ \theta_\kappa : \mathbb{E}_s(\mathcal{E}_{b,\kappa}) \to \mathbb{E}_s(\mathcal{D}_\kappa)$$

and therefore it is the composite of a split monomorphism with a quasi-isogeny whose cokernel is annihilated by p^{2t} (as θ_{U_0} has this property) and with a quasi-isogeny whose cokernel is annihilated by p^t (as ι is an isogeny whose cokernel is annihilated by p^t). Therefore, ρ_κ is also the composite of a split monomorphism with a quasi-isogeny whose cokernel is annihilated by p^{3t}. This implies that the image of γ in $W_{3t+1}(\kappa)$ is non-zero (cf. Fact 1 applied with $D = W(\kappa)$) and therefore we have $u \leq 3t$. $\square$

Lemma 4 *If $s \geq 3t+b+1$, then the image of $\bar{\delta}$ in $k^r = W_1(k)^r$ is non-zero.*

Proof: We show that the assumption that the image of $\bar{\delta} \in P^r - 0$ in $k^r = W_1(k)^r$ is 0 leads to a contradiction. This assumption implies that there exists a largest positive number c of denominator a power of p such that we have

$$\bar{\delta} \in X^c P^r \subset P^r = (k[[X]]^{\text{perf}})^r.$$

Let $\bar{\varphi} : P^r \to P^r$ be the P-linear endomorphism which is the reduction modulo p of φ. From Lemma 3 we get that $\bar{\delta} = \bar{\varphi}(\bar{\delta})$. Thus $\bar{\delta} \in \bar{\varphi}(X^c P^r) \subseteq X^{p^n c} P^r$ and this implies that $p^n c \leq c$ which is a contradiction. $\square$

From the inequality $u \leq 3t$ (see Lemma 3) and from Lemma 4 we get that for $s \geq 3t + b + 1$ the pull back ω_k of ω to a morphism of $\mathcal{M}(W_s(k))$ is such that its reduction modulo p^{3t+1} is non-zero. For $s > 3t + b + 1 + l$ with $l \in \mathbb{N}^*$ large enough but depending only on b and r, the reduction of ω_k modulo p^{s-l} lifts to a morphism $\mathcal{E}_{0,k} \to \mathcal{D}_k$ (cf. [V1], Theorem 5.1.1 (a); again loc. cit. stated for $n = 1$ applies to all $n \in \mathbb{N}^*$) which is non-zero. Thus $\mathcal{D}_k$ has Newton polygon slope b with multiplicity at least 1. From this and the existence of the isogeny ι we get that $\mathcal{C}_k$ has Newton polygon slope b with multiplicity at least 1. This implies that the special point of Spec $(k[[X]])$ under the composite of ζ_0 : Spec $(k[[X]]) \to \mathbb{I}_s$ with the morphism $\mathbb{I}_s \to S$ does not map to a point of $S_{\nu_2} = S - T$ and therefore it maps to a point of T. Contradiction. This ends the proof of Theorem 1. $\square$

4 Applications of Theorem 1

In Subsection 4.1 we prove Corollary 1. In Subsection 4.2 we follow [V2] to introduce generalized Artin–Schreier systems of equations and Artin–Schreier stratifications. In Subsection 4.3 we refine and reobtain Corollary 2 in the context of these stratifications. Subsection 4.4 contains some complements, including Proposition 3 which prove that 'pure in' implies 'weakly pure in'. Until the end let A be an arbitrary $\mathbb{F}_p$-algebra.

4.1 Proof of Corollary 1

To prove Corollary 2, in this subsection we can assume that $S = \operatorname{Spec} A$ and $d \in \mathbb{N}$ are as in the paragraph before Subsection 2.1. We can also assume that $\nu(r) = d$ as otherwise $S_\nu = \emptyset$ is pure in S. Let $l \in \mathbb{N}$ be such that the Newton polygon ν has exactly $l + 1$ breaking points denoted as $(a_0, b_0) = (0, 0), \ldots, (a_l, b_l) = (r, d)$.

We have obvious identities

$$S_\nu = [S_{\geq\nu} \bigcap_{i=0}^{l} T_{(a_l,b_l)}(\mathcal{C})]_{\text{red}} = [S_{\geq\nu} \times_S (T_{(a_0,b_0)}(\mathcal{C}))_S \times \cdots \times_S T_{(a_l,b_l)}(\mathcal{C})]_{\text{red}}.$$

From Theorem 1 we get that each $T_{(a_l,b_l)}(\mathcal{C})$ is an affine scheme. We recall that $S_{\geq\nu}$ is a reduced closed subscheme of S. From the last three sentences we get that S_ν is an affine scheme, i.e., is pure in S. $\square$

4.2 Artin–Schreier stratifications

Let x_0, x_1,..., x_r be free variables. For $i, j \in \{1, \ldots, r\}$ let $P_{i,j}(x_0) \in A[x_0]$ be a polynomial which is a linear combination with coefficients in A of the monomials x_0^q with $q \in \mathbb{N}$ either 0 or a power of p. By a generalized Artin–Schreier system of equations in r variables over A we mean a system of equations of the form

$$x_i = \sum_{j=1}^{r} P_{i,j}(x_j^p) \quad i \in \{1, \ldots, r\}$$

to which we associate the A-algebra

$$B = A[x_1, \ldots, x_r]/(x_1 - \sum_{j=1}^{r} P_{1,j}(x_j^p), x_2 - \sum_{j=1}^{r} P_{2,j}(x_j^p), \ldots, x_r - \sum_{j=1}^{r} P_{r,j}(x_j^p)).$$

Each equation of the form $x_i = \sum_{j=1}^r P_{i,j}(x_j^p)$ will be called as a generalized Artin–Schreier equation, and its degree $e_i \in \mathbb{N}$ is defined as follows. We have $e_i = 0$ if and only if for all $j \in \{1, \ldots, r\}$ the polynomial $P_{i,j}(x_0)$ is a constant, and if $e_i > 0$ then e_i is the largest integer such that there exists a $j \in \{1, \ldots, r\}$ with the property that the degree of $P_{i,j}(x_j^p)$ is p^{e_i}.

Let $e = \max\{e_1, \ldots, e_r\}$; we call it the degree of the generalized Artin–Schreier system of equations in r variables over A. Following [V2], when $e \leq 1$ we drop the word 'generalized'.

Proposition 2 *The morphism ϵ : Spec $B \to$ Spec A is étale and surjective and its geometric fibers have a number of points equal to a power of p.*

Proof: If $e_i > 1$, then by adding for each $j \in \{1, \ldots, r\}$ such that the degree of $P_{i,j}(x_j^p)$ is p^{e_i} an extra variable $y_{i,j}$ and an equation of the form $y_{i,j} = x_j^p$, the generalized Artin–Schreier equation $x_i = \sum_{j=1}^r P_{i,j}(x_j^p)$ gets replaced by several generalized Artin–Schreier equations of degrees less than e_i. By repeating this process of adding extra variables and equations which (up to isomorphisms between Spec A-schemes) does not change the morphism

$\epsilon : \text{Spec } B \to \text{Spec } A$, we can assume that $e \le 1$. Thus the proposition follows from [V2], Theorem 2.4.1 (a) and (b). $\square$

The below definition is a natural extrapolation of [V2], Definition 2.4.2 which applies to étale morphisms $\epsilon : \text{Spec } B \to \text{Spec } A$ as in Proposition 2.

Definition 1 *Let $\varepsilon : \mathit{Spec}\, \mathcal{B} \to \mathit{Spec}\, A$ be an étale morphism between affine $\mathbb{F}_p$-schemes.*

(a) *We assume that A is noetherian. Then by the Artin–Schreier stratification of Spec A associated to $\varepsilon : \mathit{Spec}\, \mathcal{B} \to \mathit{Spec}\, A$ in reduced locally closed subschemes $\mathcal{V}_1, \ldots, V_q$ we mean the stratification defined inductively by the following property: for each $l \in \{1, \ldots, q\}$ the scheme V_l is the maximal open subscheme of the reduced scheme of $(\mathit{Spec}\, A) - (\cup_{q=1}^{l-1} V_q)$ which has the property that the morphism $\epsilon_{V_l} : (\mathit{Spec}\, B) \times_{\mathit{Spec}\, A} V_l \to V_l$ is an étale cover.*

(b) *Let $\mu_1 > \mu_2 > \cdots > \mu_v$ be the shortest sequence of strictly decreasing natural numbers such that each fiber of the morphism $\epsilon : \mathit{Spec}\, B \to \mathit{Spec}\, A$ has a number of geometric points equal to μ_l for some $l \in \{1, \ldots, v\}$. Then by the functorial Artin–Schreier stratification of Spec A associated to $\varepsilon : \mathit{Spec}\, \mathcal{B} \to \mathit{Spec}\, A$ we mean the stratification of Spec A in reduced locally closed subschemes $U_1, \ldots, U_v$ defined inductively by the following property: for each $l \in \{1, \ldots, v\}$ the scheme U_l is the maximal open subscheme of the reduced scheme of $(\mathit{Spec}\, A) - (\cup_{q=1}^{l-1} U_q)$ which has the property that the morphism $\epsilon_{U_l} : (\mathit{Spec}\, B) \times_{\mathit{Spec}\, A} U_l \to U_l$ is an étale cover whose all fibers have a number of geometric points equal to μ_l.*

The existence of the stratification $V_1, \ldots, V_q$ of Spec A is a standard piece of algebraic geometry. The existence of the sequence $\mu_1 > \mu_2 > \cdots > \mu_v$ follows from the facts that each étale morphism is locally quasi-finite and that Spec $\mathcal{B}$ is quasi-compact. The existence of the stratification $U_1, \ldots, U_v$ of Spec A is implied by [G2], Proposition 18.2.8 and Corollary 18.2.9 which show that one can define U_l directly and functorially as follows: each U_l is the set of all points $x \in \text{Spec } A$ such that the fiber of ε at x has exactly μ_l geometric points.

Theorem 2 *Let $\varepsilon : \mathit{Spec}\, \mathcal{B} \to \mathit{Spec}\, A$ be an étale morphism between affine $\mathbb{F}_p$-schemes. Then the functorial Artin–Schreier stratification of Spec A associated to $\varepsilon : \mathit{Spec}\, \mathcal{B} \to \mathit{Spec}\, A$ in reduced locally closed subschemes $U_1, \ldots, U_v$ is pure, i.e., for each $l \in \{1, \ldots, v\}$ the stratum U_l is pure in Spec A.*

Proof: As the étale morphism $\varepsilon : \mathrm{Spec}\ \mathcal{B} \to \mathrm{Spec}\ A$ is of finite presentation and due to the functorial part, we can assume that A is a finitely generated $\mathbb{F}_p$-algebra and thus an excellent ring. We follow [V3]. By replacing Spec A by its closed subscheme $(\mathrm{Spec}\ A) - (\cup_{q=1}^{l-1} U_q)$ endowed with the reduced structure, we can assume that $l = 1$ and that A is reduced. Thus U_1 is an open dense subscheme of Spec A. Based again on [V1], Lemma 2.9.2 to prove that U_1 is an affine scheme, we can replace A by its normalization in its ring of fractions. Thus by passing to connected components of Spec A, we can assume that A is an excellent normal domain. Thus $B = \prod_{l=1}^{w} B_l$ is a finite product of excellent normal domain which are étale A-algebras. Let K_l be the field of fractions of B_l. Let L be the finite Galois extension of the field of fractions K of A generated by the finite separable extensions K_l's of K. By replacing A by its normalization in L (again based on [V1], Lemma 2.9.2), we can assume that we have $K = K_1 = \cdots = K_w$. This implies that each Spec (B_l) is an open subscheme of Spec A and thus

$$U_1 = \cap_{l=1}^{w} \mathrm{Spec}\ (B_l) = (\mathrm{Spec}\ (B_1)) \times_{\mathrm{Spec}\ A} (\mathrm{Spec}\ (B_2)) \times_{\mathrm{Spec}\ A} \cdots \times_{\mathrm{Spec}\ A} (\mathrm{Spec}\ (B_w))$$

is the affine scheme Spec $(B_1 \otimes_A \otimes \cdots \otimes_A B_w)$. $\square$

4.3 A second proof of Corollary 2

We will use Theorem 2 to obtain a second proof of Corollary 2 which is simpler and independent of Theorem 1. We can assume that $S = \mathrm{Spec}\ A$ is affine and let $\phi_M : M \to M$ be as in Subsection 2.1.

The identities $S_m = T_{(m,0)}(\mathcal{C})$ if $m > 0$ and $S_0 = T_{(1,0)}(\mathcal{C} \oplus \mathcal{E}_0)$ show that S_m is a reduced locally closed subscheme of S. Thus by replacing S by $\bar{S}_m$, we can assume that S_m is an open dense subscheme of $S = \bar{S}_m$.

We consider the equation

$$\phi_M(z) = z \tag{1}$$

in $z \in M$. For $x \in S$ we have $\chi(x) = \dim_{\mathbb{F}_{p^n}}(\vartheta_x)$, where ϑ_x is the $\mathbb{F}_{p^n}$-vector space of solutions of the tensorization of the Equation 1 over A with an algebraic closure of the residue field $k(x)$ of S at x.

From now on we will forget about $\mathcal{C}$ and just work with the free A-module M of rank r and its Φ_A^n-linear endomorphism $\phi_M : M \to M$ and we only assume that we have an open dense subset S_m of $S = \mathrm{Spec}\ A$ defined by the following property: for $x \in S$, we have $x \in S_m$ if and only if $\dim_{\mathbb{F}_{p^n}}(\vartheta_x) = m$.

With respect to a fixed A-basis $\{v_1, \ldots, v_r\}$ of M, by writing $z = \sum_{i=1}^{r} x_i v_i$ the Equation 1 defines a generalized Artin–Schreier system of equations in the r variables $x_1, \ldots, x_r$ of the form

$$x_i = L_i(x_1^{p^n}, \ldots, x_r^{p^n}) \quad i \in \{1, \ldots, r\},$$

where each L_i is a homogeneous polynomial of total degree at most 1. Let

$$B = A[x_1, \ldots, x_r]/(x_1 - L_1(x_1^{p^n}, \ldots, x_r^{p^n}), \ldots, x_r - L_r(x_1^{p^n}, \ldots, x_r^{p^n})),$$

let $\epsilon : \mathrm{Spec}\, B \to S$ and let $U_1, \ldots, U_v$ be the functorial Artin–Schreier stratification of S associated to $\epsilon : \mathrm{Spec}\, B \to S$. Let $p^{\mu_1} > p^{\mu_2} > \cdots > p^{\mu_v}$ be the shortest sequence of strictly decreasing of powers of p by natural numbers such that for each each $l \in \{1, \ldots, v\}$ every geometric fiber of the morphism $\epsilon_{U_l} : \mathrm{Spec}\, B \times_S U_l \to U_l$ has a number of geometric points equal to p^{μ_l}, cf. Proposition 2 and Definition 1 (b).

The fact that the morphism $\epsilon : \mathrm{Spec}\, B \to S$ is étale (cf. Proposition 2) is equivalent to [Z], Proposition 3. We consider the lower semi-continuous function (cf. [G2], Proposition 18.2.8)

$$\mu : S \to \mathbb{N}$$

defined by the rule: $\mu(x) = p^{n \dim_{\mathbb{F}_{p^n}}(\vartheta_x)}$ is the number of geometric points of $\epsilon : \mathrm{Spec}\, B \to S$ above x (i.e., is the number of elements of ϑ_x). We get that μ_l is divisible by n for all $l \in \{1, \ldots, v\}$ and (as S_m is dense in S) we have $\mu_1 = mn$. Moreover, for $x \in S$ and $q \in \mathbb{N}$ we have $\mu(x) = p^{nq}$ if and only if $x \in S_q$. We conclude that $S_m = U_1$ and therefore (cf. Theorem 2) S_m is an affine scheme. $\square$

4.4 Complements

For the sake of completeness, we include a proof of the following well-known result (to be compare with [V1], Remark 6.3 (a)).

Proposition 3 *Let Z be a reduced locally closed subscheme of a locally noetherian scheme Y. If Z is pure in Y, then Z is weakly pure in Y.*

Proof: We can assume that $Z \subsetneq \bar{Z} = Y$. By localizing Y at the generic point of an irreducible component of $\bar{Z} - Z$, we can assume that $Y = \bar{Z} = \mathrm{Spec}\, C$ is a local affine scheme of dimension at least 1 and Z is the complement in

Y of the closed point of Y and we have to prove that C has dimension 1. By passing to a connected component of the normalization of the reduced completion $\hat{C}_{\mathrm{red}}$ of C in the ring of fractions of $\hat{C}_{\mathrm{red}}$, we can assume that C is in fact an integral normal local ring which is not a field.

We show that the assumption that $\dim(C) \geq 2$ leads to a contradiction. As the open dense subscheme Z of Y is pure in in Y, Z is the spectrum of a C-subalgebra of the field of fractions of C which contains C and which is contained in the intersection of all the localizations of C at points of Y of codimension 1 in Y (as these points belong to Z). As $\dim(C) \geq 2$, from [M], (17H), Theorem 38 we get that this intersection is C and thus we have $Z = \mathrm{Spec}\, C = Y$. Contradiction. Thus $\dim(C) = 1$. $\square$

Remark 1 *Suppose A is a local noetherian $\mathbb{F}_p$-algebra of dimension at least 2. Let $\mathfrak{m}$ be the maximal ideal of A. Suppose $M = A^r$ is equipped with a Φ_A^n-linear endomorphism $\phi_M : M \to M$ such that for each non-closed point x of $S = Spec\, A$, with the notations of Subsection 4.3 we have $\dim_{\mathbb{F}_{p^n}}(\vartheta_x) = m$. Then $S_m = U_1$ being pure in S, it is also weekly pure in S (cf. Proposition 3) and thus $S - S_m$ cannot be $\mathfrak{m}$ as $codim_S(\mathfrak{m}) \geq 2$. Therefore we have $S_m = S$ and in this way we reobtain [Z], Proposition 5. One can view Theorem 2 as a generalization and a refinement of [Z], Proposition 5.*

Remark 2 *For $q \in \mathbb{N}^*$ we define recursively an A-linear map*

$$\phi_M^{(q)} : A \otimes_{F_A^{nq},A} M \to M$$

as follows: let $\phi_M^{(1)} : A \otimes_{F_A^n,A} M \to M$ be the A-linear map defined by ϕ_M and we have the recursive formula $\phi_M^{(q)} = \phi_M^{(1)} \circ (1_A \otimes_{F_A^n,A} \phi_M^{(q-1)})$. Deligne proved in [D] the case $n = 1$ of Theorem 2 using ranks of images of $\phi_M^{(q)}$ for $q >> 0$ at points $x \in S = Spec\, A$ and properties of Grassmannians.

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