Shimura Varieties and the Mumford–Tate conjecture, part two

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ABSTRACT. We prove the Mumford–Tate conjecture for those abelian varieties over number fields, whose simple factors of their adjoint Mumford–Tate groups have (over $\mathbb{R}$) few non-compact factors of non isotypic Lie type.

Key words: abelian and Shimura varieties, reductive and p-divisible groups, Galois representations and Hodge cycles.

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Contents

§1. Introduction

1.0. This part two is a natural extension of [Va3]. Not to make it too long, we will rely heavily on it. In particular, we will use the conventions and notations of [Va3, 1.3] and, at the appropriate moments, of other parts of [Va3].

1.1. Let A be an abelian variety over a number field E. From [Va2, 5.4 8) and 5.1.2], we deduce that in order to prove the Mumford–Tate conjecture for A, we can assume that the Mumford–Tate group H_A of A has a simple adjoint, and that the Shimura pair (H_A, X_A) attached to A (cf. [Va1, 2.12 3)]) is of some A_n, C_n or $D_n^{\mathbb{H}}$ type.

Inspired from [LP] and [Pi, 5.14], the following definition looks natural:

Definition 1. We say that the adjoint Shimura pair $(H_A^{\mathrm{ad}}, X^{\mathrm{ad}})$ is of MT isotypic type, if one of the following conditions is satisfied:

– H_A^{ad} is of A_n Lie type, with $n+1$ of the form C_r^s, with $r, s \in \mathbb{N}$, $2 \leq s \leq r-1$, or of the form m^k, with $k > 1$ and $m > 2$;

– H_A^{ad} is of C_n type, with $2n$ of the form m^k with $k > 2$ odd and $m \geq 2$, or of the form C_{2k}^k with $k \geq 3$ an odd number;

– $(H_A^{\mathrm{ad}}, X_A^{\mathrm{ad}})$ is of $D_n^{\mathbb{H}}$ with trivial involution, with $2n$ of the form C_{4k}^{2k}, or of the form m^{2k} with $m, k \in \mathbb{N}$, $(m, k) \neq (2, 1)$, or of the form m^{2k+1}, with $m, k \in \mathbb{N}$, $m > 2$;

– $(H_A^{\mathrm{ad}}, X_A^{\mathrm{ad}})$ is of $D_n^{\mathbb{H}}$ type with non-trivial involution, and $2n$ is of the form m^{2k} with $m, k \in \mathbb{N}$, $(m, k) \neq (2, 1)$, or of the form m^{2k+1}, with $m, k \in \mathbb{N}$, $m > 2$.

1.2. The remaining (to be proved) part of the Mumford–Tate conjecture for A, from the point of view of difficulty, can be divided into two parts (cases). The first part refers to the cases when $(H_A^{\mathrm{ad}}, X_A^{\mathrm{ad}})$ is not of MT isotypic type, while the second one refers to the cases when $(H_A^{\mathrm{ad}}, X_A^{\mathrm{ad}})$ is of MT isotypic type. We think the first part is not so hard and should be entirely provable using our work on the Langlands–Rapoport conjecture (see [Va5] for a quick reference), while the second one it is very hard, but we still believe that it can be attacked using [Va2, 3.4]. In this paper we deal only with the first part: we mainly motivate why the first part is not so hard.

1.3. To formulate better the main results of this paper, we need an extra definition.

Definition 2. Let $n \in \mathbb{N}$. We say that the adjoint Shimura pair $(H_A^{\mathrm{ad}}, X^{\mathrm{ad}})$ is n-elementary (resp. n-quasi-elementary), if $H_{A\mathbb{R}}^{\mathrm{ad}}$ has precisely n simple non-compact factors (resp. if H_A^{ad} is of A_n Lie type and the set C defined as in [Va3, 5.1.2.2] has precisely n elements. When $n = 1$, we speak about elementary (resp. quasi-elementary) instead of 1-elementary (resp. 1-quasi-elementary). When the adjoint Shimura pair $(H_A^{\mathrm{ad}}, X^{\mathrm{ad}})$ is n-elementary and n-quasi-elementary, we say that it is strongly n-elementary.

We have:

Theorem. *A. The Mumford–Tate conjecture is true for A if one of the following three conditions is satisfied:*

a) the adjoint Shimura pair $(H_A^{\mathrm{ad}}, X^{\mathrm{ad}})$ is elementary and is not of MT isotypic type;

b) the adjoint Shimura pair $(H_A^{\mathrm{ad}}, X^{\mathrm{ad}})$ is 2-elementary and is not of MT isotypic type, and $[EF_1(H_A^{\mathrm{ad}}, X^{\mathrm{ad}}) : F(H_A^{\mathrm{ad}}, X^{\mathrm{ad}})] > n(H_A^{\mathrm{ad}}, X^{\mathrm{ad}})$, where $n(H_A^{\mathrm{ad}}, X^{\mathrm{ad}}) \in \mathbb{N}$ is a suitable number;

c) the adjoint Shimura pair $(H_A^{\mathrm{ad}}, X^{\mathrm{ad}})$ is strongly n-elementary, is not of MT isotypic type, and moreover $[EF_1(H_A^{\mathrm{ad}}, X^{\mathrm{ad}}) : F(H_A^{\mathrm{ad}}, X^{\mathrm{ad}})] > n(H_A^{\mathrm{ad}}, X^{\mathrm{ad}})$, where $n(H_A^{\mathrm{ad}}, X^{\mathrm{ad}}) \in \mathbb{N}$ is a suitable number.

B. We have $n(H_A^{\mathrm{ad}}, X^{\mathrm{ad}}) = 1$ if H_A^{ad} is of C_{2p} or D_{2p} Lie type, with p a prime.

Here we are using the notations of [Va3, 2.5]. Similar criteria to c) above, can be proved for any $n \in \mathbb{N}$, but it is much more involved work to write them done (see 3.???; see 4.??? for examples).

1.4. Part a) of the above Theorem represents the generalization (by capturing the very essence of) of the previously results on the Mumford–Tate conjecture not covered by [Va3], and mentioned in [Va3, 5.4 6)]. Also, to our knowledge, the above Theorem handles for the first time non-trivial situations not covered by [Va3, 5.1.2], and where we have more than one non-compact factor and at least one compact factor at the same time.

1.5. The proof of a) is a straight forward consequence of the ideas of [Va3], when put together with the isotypic ideas of [LP]. The guiding principle is (see 3.???):

Fact 1. *We use the notations of [Va3, 4.0]. If* $(H_A^{\mathrm{ad}}, X_A^{\mathrm{ad}})$ *is* n*-elementary, than the set* $\mathcal{V}$ *of values of the dimensions of the irreducible subrepresentations of a simple factor of* $Lie(G_{\mathbb{C}})$ *of the representation of it we get via the standard representation of a simple factor of* $Lie(H_{A\mathbb{C}}^{\mathrm{ad}})$*, has a number of elements* $m(\mathcal{V})$ *not bigger than* n *(here we use an arbitrary embedding* $\mathbb{Q}_p \hookrightarrow \mathbb{C}$*).*

The proofs of b) and c) need some refinements: assuming the contrary we build up (starting from the mentioned ideas) some reductive subgroups of H_A^{ad} (or of its extension to suitable totally real number fields), which would lead to a contradiction with the hypotheses made on the expression of the reflex field $E(H_A^{\mathrm{ad}}, X^{\mathrm{ad}})$.

All the three parts of the above Theorem rely heavely on the unitary trick (see [Va4, 2.3.8] and [Va3, §3]), and (in some cases of $D_n^{\mathbb{H}}$ type) on the ideas pertaining to the Langlands–Rapoport conjecture, which can be found in [Va5]. In other words, we use that in suitable situations, the Frobenius tori of the reduction of A w.r.t. some primes of E, are subtori of H_A. Moreover, in essentially all the situations we can get the same information on Frobenius tori, by just knowing that they are subtori of some other reductive group H'_A, closely in nature to H_A, but obtained via the mentioned unitary trick (in the C_n and $D_n^{\mathbb{H}}$ type situations; so $H_A'^{\mathrm{ad}}$ is $\mathbb{Q}$-simple of A_{2n} Lie type), or via some inner twist of H_A (in the A_n type situation; to be compared with [Va3, 3.2]); this has the advantage that the arguments are much more elementary.

The starting (guiding) idea to reach the above mentioned contradiction is (see 3.???):

Fact 2. *Different Frobenius tori are naturally isogeneous to a product of* $m(\mathcal{V})$ *tori.*

1.6. The paper is structured as follows. Some complements to [Va3, 2.5 and §3] and [Va4, 6.2.3.0] are gathered in §2. The proofs of the above Theorem (and so of the above two Facts) and of some of its generalizations are carried on in §3, while §4 presents lots of examples. To the 77 percent of [Va3, 1.8.1], the generalized version of c) of Theorem above adds 3 percent; however the theorem itself qualifies as 0 percent.

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§2. Preliminaries

§3. The main results

§4. Examples

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