Subtle invariants for p-divisible groups and Traverso's conjectures

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(joint works with Ofer Gabber and with Eike Lau and Marc-Hubert Nicole)

1. Notations

We report on the joint works [1] and [2] aimed towards the classification of p-divisible groups over an algebraically closed field k of positive characteristic p.

Let $W(k)$ be the p-typical Witt ring of k. Let σ be the Frobenius automorphism of $W(k)$. Let D be a p-divisible group over k of positive height r and let (M, ϕ, ϑ) be its (contravariant) Dieudonné module. Thus M is a free $W(k)$-module of rank r, $\phi : M \to M$ is a σ-linear endomorphism, and $\vartheta : M \to M$ is a σ^{-1}-linear endomorphism such that we have $\phi \circ \vartheta = \vartheta \circ \phi = p1_M$.

Let $d = \dim_k(M/\phi(M))$ and $c = \dim_k(M/\vartheta(M))$ be the dimension and the codimension (respectively) of D. We have $c + d = r$. Let $t := \min\{c, d\}$.

2. What one would like to achieve?

• Classify all D's. This means: (a) list all isomorphism classes with c and d fixed; (b) decide when another p-divisible group E over k specializes to D in the sense below; (c) understand the abstract groups $\mathrm{Hom}(E[p^m], D[p^m])$ for all $m \in \mathbb{N}^*$; and (d) identify good invariants of D (which go up or down under specializations).

• Refine the Newton polygon stratifications associated to p-divisible groups over $\mathbb{F}_p$-schemes using good invariants.

• Generalize to quadruples of the form (M, ϕ, ϑ, G), where G is an integral, closed subgroup subscheme of $\boldsymbol{GL}_M$ subject to the axioms of [1], Definition 2. [Here we will concentrate on the case $G = \boldsymbol{GL}_M$ which corresponds simply to D.]

Definition 1. We say that E specializes to D if there exists a p-divisible group $\mathcal{D}$ over $k[[x]]$ such that $\mathcal{D}_k = D$ and $\mathcal{D}_{\overline{k((x))}}$ is isomorphic to $E_{\overline{k((x))}}$.

3. Classical invariants

It is well known that we have a direct sum decomposition of F-isocrystals

$$(M[\frac{1}{p}], \phi) = \oplus_{\beta \in [0,1] \cap \mathbb{Q}} (I_\beta, \phi_\beta)^{m_D(\beta)},$$

where each (I_β, ϕ_β) is simple of Newton polygon slope β and each $m_D(\beta) \in \mathbb{N}$ is a multiplicity. With these multiplicities one builds up the Newton polygon of D: it is an increasing, concave up, piecewise linear, continuous function $\nu_D : [0, r] \to [0, d]$. Note that $\nu_D(c) \leq \frac{cd}{r}$. Let $a_D := \dim(\boldsymbol{Hom}(\boldsymbol{\alpha}_p, D[p]))$. Thus $\boldsymbol{\alpha}_p^{a_D} \oplus (\mathbb{Z}/p\mathbb{Z})^{m_D(0)} \oplus \boldsymbol{\mu}_p^{m(1)} \subset D[p]$. For $m \in \mathbb{N}$ let $\gamma_D(m) := \dim(\boldsymbol{Aut}(D[p^m])) = \dim(\boldsymbol{End}(D[p^m]))$.

4. Subtle invariants

In this section we introduce six geometric invariants of D.

(a) Isomorphism number. It is the smallest $n_D \in \mathbb{N}^*$ such that the isomorphism class of D is uniquely determined by $D[p^{n_D}]$. We have $n_D \leq cd+1$, cf. [5].

Conjecture 1 (Traverso [6]). *If $cd > 0$, then $n_D \leq t$.*

(b) Isogeny cutoff. It is the smallest $b_D \in \mathbb{N}^*$ such that the isogeny class (i.e., the Newton polygon ν_D) of D is uniquely determined by $D[p^{b_D}]$.

Conjecture 2 (Traverso [6]). If $cd > 0$, then $b_D \leq \lceil \frac{cd}{r} \rceil$.

Remark 1. Conjecture 2 is proved in [3]. Conjecture 1 is incorrect. Below we will provide corrected, refined, and optimal versions of these two conjectures.

(c) Minimal height. There exists a unique (up to isomorphism) p-divisible group D_0 over k such that $n_{D_0} = 1$ and $\nu_{D_0} = \nu_D$; it is called the minimal p-divisible group of Newton polygon ν_D. The minimal height of D is the smallest $q_D \in \mathbb{N}$ such that there exists an isogeny $D_0 \to D$ whose kernel is annihilated by p^{q_D}.

(d) Level torsion. We denote also by ϕ the σ-linear automorphism $\mathrm{End}(M[\frac{1}{p}]) \to \mathrm{End}(M[\frac{1}{p}])$ given by $x \mapsto \phi \circ x \circ \phi^{-1}$. We have a direct sum decomposition $(\mathrm{End}(M[\frac{1}{p}]), \phi) = (L_+, \phi) \oplus (L_0, \phi), \oplus (L_-, \phi)$ such that all Newton polygon slopes of (L_+, ϕ) are positive, all Newton polygon slopes of (L_0, ϕ) are zero, and all Newton polygon slopes of (L_-, ϕ) are negative. Let O_+ (resp. O_0 and O_-) be the largest $W(k)$-submodule of $\mathrm{End}(M)$ which is contained in L_+ (resp. L_0 and L_-) and for which we have $\phi(O_+) \subset O_+$ (resp. $\phi(O_0) = O_0$ and $\phi^{-1}(O_-) \subset O_-$). Thus $O := O_+ \oplus O_0 \oplus O_-$ is a $W(k)$-lattice of $\mathrm{End}(M[\frac{1}{p}])$ contained in $\mathrm{End}(M)$. The level torsion l_D of D is the smallest $l_D \in \mathbb{N}$ such that $p^{l_D}\mathrm{End}(M) \subset O \subset \mathrm{End}(M)$.

(e) Endomorphism number. Let $e_D \in \mathbb{N}$ be such that for all positive integers $m \geq s$, the images of the restriction homomorphisms $\tau_{\infty,s} : \mathrm{End}(D) \to \mathrm{End}(D[p^s])$ and $\tau_{m,s} : \mathrm{End}(D[p^m]) \to \mathrm{End}(D[p^s])$ are equal if and only if $m \geq s + e_D$.

(f) Coarse endomorphism number. Let $f_D \in \mathbb{N}$ be such that for all positive integers $m \geq s$, $\tau_{m,s}$ has finite image if and only if $m \geq s + f_D$.

Remark 2. We have natural variants for pairs $n_{D,E}$, $l_{D,E}$, $e_{D,E}$, and $f_{D,E}$. For instance, $n_{D,E} \in \mathbb{N}^*$ is the smallest number such that the natural restriction homomorphism $\mathrm{Ext}^1(D,E) \to \mathrm{Ext}^1(D[p^{n_{D,E}}], E[p^{n_{D,E}}])$ is injective. The equality $n_D = n_{D,D}$ provides a cohomological interpretation of n_D.

5. Our results

Theorem A ([2]). (a) *Let $j(\nu_D)$ be $\nu_D(c)+1$ if $(c, \nu_D(c))$ is a breakpoint of ν_D and be $\lceil \nu_D(c) \rceil$ otherwise. We have $b_D \leq j(\nu_D)$ and the equality holds if $a_D \leq 1$.*

(b) *If D is not ordinary, then $n_D \leq \lfloor 2\nu_D(c) \rfloor \leq \lfloor \frac{2cd}{r} \rfloor$; mooreover, the equality $n_D = \lfloor \frac{2cd}{r} \rfloor$ does hold for certain isoclinic p-divisible groups D with $a_D = 1$.*

(c) *We have $q_D \leq \lfloor \nu_D(c) \rfloor$ with equality if $a_D \leq 1$.*

Remark 3. In general, if $a_D = 1$ then n_D can vary. If $t > 0$ is fixed, then $\lfloor \frac{2cd}{r} \rfloor$ can be any integer in the interval $[t, 2t-1]$. Thus Conjecture 1 is in essence off by a factor 2. If $|c-d| \le 2$, then $\lfloor \frac{2cd}{r} \rfloor = t$ and the Conjecture 1 holds. The simplest case when Conjecture 1 fails is when $\{c,d\} = \{2,6\}$.

Theorem B ([2]). *If D is not ordinary, then $n_D = l_D = e_D = f_D$.*

Remark 4. The inequalities $e_D \le l_D \le f_D \le e_D$ are proved in [2] (the first one is easy, the second one is hard, and the third one is trivial), equality $n_D = f_D$ is a consequence of the next theorem, while in [7] it was first proved that $n_D \le l_D$.

Theorem C ([1]). For all $s \in \mathbb{N}^*$, the sequence $(\gamma_S(s+i) - \gamma_D(i))_{i\in\mathbb{N}}$ is decreasing. Moreover, if $cd > 0$, then we have $a_D^2 \le \gamma_D(1) < \cdots < \gamma_D(n_D) = \gamma_D(n_d + s) \le cd$.

Theorem D ([2]). *If $\mathcal{D}$ is a p-divisible group of constant Newton polygon over an $\mathbb{F}_p$-scheme S, then for all $m \in \mathbb{N}$ and $\Box \in \{b, n, q\}$, the set $\{s \in S | \Box_{\mathcal{D}_s} \le m\}$ is closed in S. Thus for each $\triangle \in \{b, n, q, bn, bq, nq, bnq\}$ we get a natural $\triangle$-stratification of S in a finite number of reduced, locally closed subschemes.*

Theorem E ([1]). *For $m \in \{1, 2, \ldots, n_D - 1\}$ there exist an infinite number of truncated Barsotti–Tate groups of level $m+1$ over k which are pairwise non-isomorphic and lift $D[p^m]$.*

Remark 5. The case $m = 1$ of Theorem E is a stronger form of [4], Theorem 4.

Theorem F ([1]). *Let $\Gamma_D(s)$ be the reduced group of the identity component of $\boldsymbol{Aut}(D[p^s])$. If $s \ge 2n_D$, then the unipotent group $\Gamma_D(s)$ is commutative.*

Example. We assume that $c = d > 0$; thus D could be the p-divisible group of an abelian variety of dimension d over k. (i) Then $n_D \le d$. (ii) For $m \in \mathbb{N}^*$, an endomorphism of $D[p^m]$ lifts to an endomorphism of D if and only if it lifts to an endomorphism of $D[p^{m+d}]$ (or $D[p^{m+\lfloor 2\nu_D(c)\rfloor}]$). (iii) The group $\Gamma_D(2d)$ (or $\Gamma_D(2\lfloor 2\nu_D(c)\rfloor)$ is commutative. (iv) E specializes to D if and only if $E[p^d]$ specializes to $D[p^d]$; the same holds with d replaced by $\max\{\lfloor 2\nu_D(c)\rfloor, \lfloor 2\nu_E(c)\rfloor\}$.

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