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Classical Affine Algebras

ALEX J. FEINGOLD*

AND

IGOR B. FRENKEL†

The Institute for Advanced Study, Princeton, New Jersey 08540

1. INTRODUCTION

The definitions of the classical Lie algebras as matrices, namely, the general linear, orthogonal, and symplectic series, implicitly assume their natural representations. However, one often uses another remarkable representation of the classical Lie algebras which is just as simple as matrices but contains more structural information. This realization, discussed in Section 2, is based on a Clifford or Weyl algebra rather than the algebra of matrices.

The Clifford and Weyl algebras are defined as associative algebras with generators a_i, a_i^*, $1 \leqslant i \leqslant l$, having, respectively, the anticommutation or commutation relations

$$[a_i, a_j]_\pm = 0 = [a_i^*, a_j^*]_\pm, \qquad [a_i, a_j^*]_\pm = \delta_{ij}. \tag{1.1}$$

Certain quadratic elements span the classical Lie algebras of types D_l and C_l, respectively. Including linear elements, one obtains the classical Lie algebra of type B_l and the superalgebra of type $B(0, l)$, which it is also natural to call "classical." The general linear algebra (whose commutant has type A_{l-1}) can be studied inside both D_l and C_l. It is spanned by those quadratic elements which are linear in both $\{a_i \mid 1 \leqslant i \leqslant l\}$ and $\{a_i^* \mid 1 \leqslant i \leqslant l\}$. We see in Table I the Dynkin diagrams of these classical finite-dimensional algebras, and in Table II their arrangement according to their constructions.

The Clifford and Weyl algebras have natural representations on the exterior and symmetric algebras of polynomials, respectively, in half of the

* Permanent address: The State University of New York, Binghamton, New York 13901.
† Permanent address: Rutgers University, New Brunswick, New Jersey 08903. The work of this author was partially supported by N.S.F. Grant MCS-8108814(A01).

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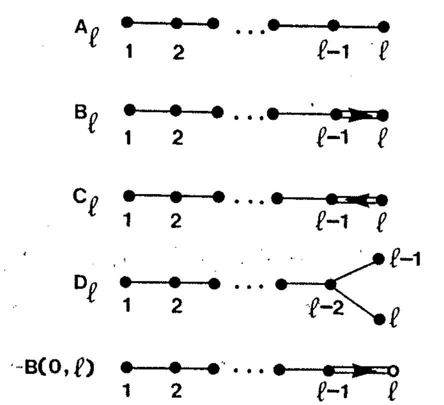
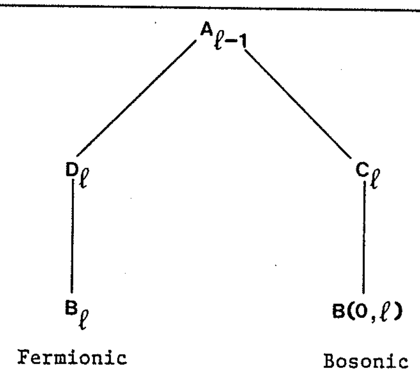
TABLE I

Classical Finite-Dimensional Algebras



generators. The corresponding representations of the classical algebras are often called, respectively, spinor and oscillator representations. (The oscillator representation is also known as the metaplectic or as the Segal–Shale–Weil representation.) The names of these representations indicate physical concepts, and, in fact, they play an important role in quantum and statistical mechanics. One can interpret a_i^*, a_i as operators which create or annihilate particles, satisfying Fermi or Bose statistics depending on the choice of anticommutation or commutation relations. In this paper, we will refer to these representations as fermionic or bosonic.

TABLE II



The complete list of affine root systems was given by Macdonald [6]. Later Kac [4] showed that to every root system on this list, one can canonically associate a Lie algebra or a superalgebra. These algebras, now called affine algebras, consist of 11 infinite series instead of 5 as in the finite-dimensional theory. Because of the direct connection between these affine algebras and the classical finite-dimensional algebras, we call them *classical affine algebras*. Their Dynkin diagrams are listed in Table III.

The classical affine algebras admit a representation by means of Toeplitz matrices, which can be compactly expressed using the ring of Laurent polynomials $\mathbb{C}[t, t^{-1}]$. For example, if $\mathbf{g}$ is an algebra from Table I, then

$$\mathbf{g}^{(1)} = \mathbf{g} \otimes \mathbb{C}[t, t^{-1}] \oplus \mathbb{C}c \tag{1.2}$$

is in Table III. The element c is central, and for $x, y \in \mathbf{g}$, $m, n \in \mathbb{Z}$, Lie brackets are defined by

$$[x \otimes t^m, y \otimes t^n] = [x, y] \otimes t^{m+n} + m\langle x, y\rangle \delta_{m,-n} c, \tag{1.3}$$

where $\langle \cdot, \cdot \rangle$ is a suitably normalized invariant bilinear form on $\mathbf{g}$. Unfortunately, these representations are not faithful since c acts trivially. To construct faithful representations, we will use the fermionic and bosonic constructions, in which c acts as $+1$ or -1, respectively. The main goal of this paper is to construct and study the fermionic and bosonic representations of *all* classical affine algebras.

To begin, we define Clifford and Weyl algebras with infinitely many generators $a_i(m)$, $a_i^*(m)$, $1 \leqslant i \leqslant l$, $m \in Z$ ($Z = \mathbb{Z}$ or $Z = \mathbb{Z} + \frac{1}{2}$) having, respectively, the anticommutation or commutation relations

$$\begin{aligned} [a_i(m), a_j(n)]_\pm &= 0 = [a_i^*(m), a_j^*(n)]_\pm, \\ [a_i(m), a_j^*(n)]_\pm &= \delta_{ij}\delta_{m,-n}. \end{aligned} \tag{1.4}$$

We choose the representation space to be the exterior or symmetric algebra of polynomials, respectively, over half of the generators (see Section 3). This space is usually called Fock space, and we already saw an elementary example of it in the finite-dimensional theory. The classical affine Lie algebras of types $D_l^{(1)}$ and $C_l^{(1)}$, respectively, are spanned by the identity operator and the quadratic elements forming the components of the generating functions,

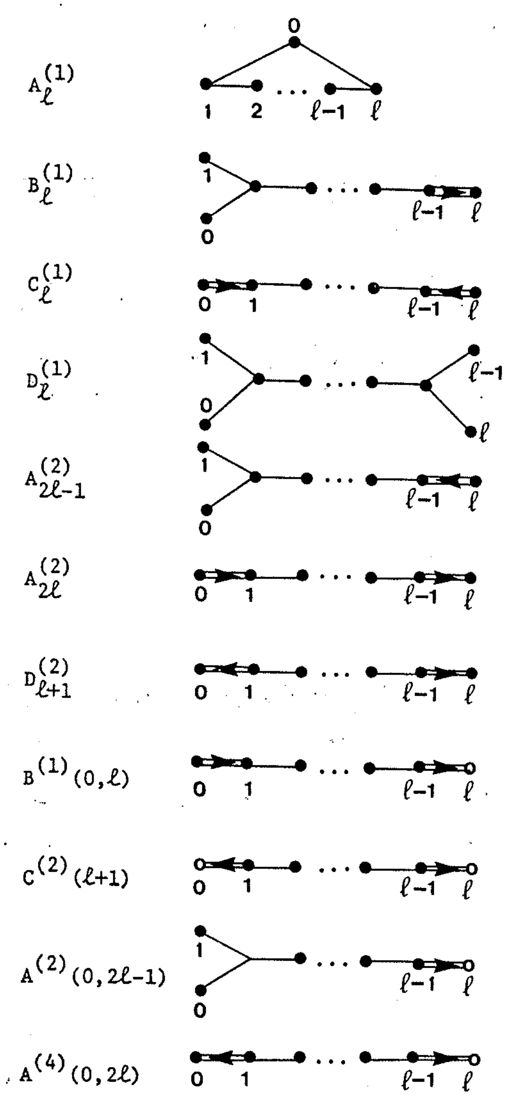
$$:a_i(w)\, a_j^*(w):, \quad :a_i(w)\, a_j(w):, \quad :a_i^*(w)\, a_j^*(w):, \quad 1 \leqslant i,j \leqslant l, \tag{1.5}$$

where

$$a_i(w) = \sum_{m \in Z} a_i(m)\, w^{-m} \tag{1.6}$$

TABLE III

Classical Affine Algebras



and : : denotes the normal ordering (3.4). To obtain the other affine algebras in the orthogonal and symplectic series, we add more Clifford generators $e(m)$, $m \in S$, where $S = Z$, $Z' = \frac{1}{2}\mathbb{Z} \setminus Z$ or $\frac{1}{2}\mathbb{Z}$ with the relations

$$[a_i(n), e(m)]_\pm = 0 = [a_i^*(n), e(m)]_\pm, \qquad [e(m), e(n)]_+ = \mp 2\delta_{m,-n}. \tag{1.7}$$

We use the generating function

$$e_S(w) = \sum_{n \in S} e(n)\, w^{-n} \tag{1.8}$$

to obtain the affine Lie algebras of types $B_l^{(1)}$ and $D_{l+1}^{(2)}$ in the fermionic case, and the superalgebras of types $B^{(1)}(0, l)$ and $C^{(2)}(l+1)$ in the bosonic case. If we add to (1.5) the functions

$$:a_i(w)\, e_S(w):, \qquad :a_i^*(w)\, e_S(w):, \quad 1 \leqslant i \leqslant l \tag{1.9}$$

for $S = Z$ (or Z'), then we get $B_l^{(1)}$ ($\approx B_l^{(2)}$) and $B^{(1)}(0, l)$ ($\approx B^{(2)}(0, l)$). With $S = \frac{1}{2}\mathbb{Z}$ and the components of $:e_Z(w)\, e_{Z'}(w):$ also included we obtain $D_{l+1}^{(2)}$ and $C^{(2)}(l+1)$.

The fermionic case which yields the affine orthogonal series has already been studied in detail in [1, 2] (see also [5]). The bosonic case which yields $C_l^{(1)}$ was first noted by H. Garland (unpublished) and, independently, later by M. Primc.

In contrast with the classical case, orthogonal and symplectic algebras do not exhaust the list of classical affine algebras (see Table III). To obtain the four missing A series of algebras, we need some "twist" of our construction. This twisted construction exists and provides us with representations of the general linear algebras in perfect analogy with the orthogonal and symplectic series. The generating functions whose components represent the algebra of type $A_{2l-1}^{(2)}$ differ from those of (1.5) only by the presence of minus signs as follows,

$$:a_i(w)\, a_j^*(w):, \qquad :a_i(w)\, a_j(-w):, \qquad a_i^*(w)\, a_j^*(-w):, \quad 1 \leqslant i, j \leqslant l. \tag{1.10}$$

It is remarkable that now the bosonic and fermionic constructions yield the same algebra. The extension of this representation to obtain the other three series of algebras again requires more generators $e(m)$, $m \in Z$, Z', or $\frac{1}{2}\mathbb{Z}$ having slightly different relations (cf. (1.7))

$$[a_i(n), e(m)]_\pm = 0 = [a_i^*(n), e(m)]_\pm, \tag{1.11}$$

$$[e(m), e(n)]_\pm = \mp 2(-1)^m\, \delta_{m,-n}. \tag{1.12}$$

The type of relation (±) on the left side of (1.12) is determined by the right side, i.e., we have + for $m, n \in \mathbb{Z}$, − for $m, n \in \mathbb{Z} + \frac{1}{2}$, and we choose − for $m + n \in \mathbb{Z} + \frac{1}{2}$ according to the definition of superalgebras. The type of relation (±) in (1.11) is then determined by the rule that two fermions anticommute, but two bosons or a fermion and a boson commute. (We call the linear elements of a Clifford (Weyl) algebra fermions (bosons).) We obtain both fermionic and bosonic constructions of the affine algebras of types $A_{2l}^{(2)}$, $A^{(2)}(0, 2l-1)$, $A^{(4)}(0, 2l)$ by adding to (1.10) the generating functions

$$:a_i(w)\, e_S(w):, \qquad :a_i^*(-w)\, e_S(w):,$$
$$:e_S(-w)\, e_S(w):, \qquad :e_S(w)\, e_S(-w):, \quad 1 \leqslant i \leqslant l. \tag{1.13}$$

Note that when $S = \frac{1}{2}\mathbb{Z}$ the components of the last two functions are not linearly dependent.

The constructions and identifications of all of the classical affine algebras are given in Sections 3 and 4 (Theorems A, B). The resulting arrangement of these algebras is given in Table IV. In each group of four algebras the top algebra can be extended three ways according to the addition of generators $e(m)$ for $m \in S$, where the left algebra is obtained when $S = Z$, the right algebra is obtained when $S = Z'$, and the bottom algebra is obtained when $S = \frac{1}{2}\mathbb{Z}$.

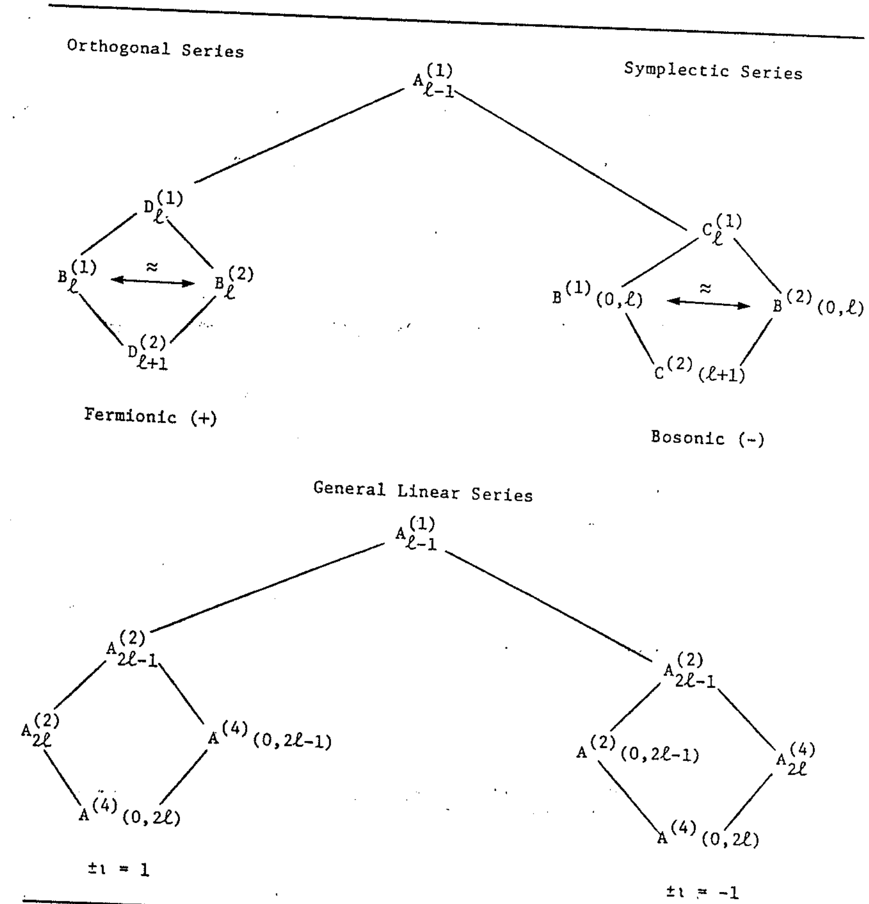
It is interesting to note that the four A series appear twice in bosonic and fermionic constructions but with two different $\mathbb{Z}$-gradings corresponding to opposite endpoints of the Dynkin diagram. The invariant which determines the grading is $\pm\iota$, where $\iota = +1$ for $Z = \mathbb{Z}$, $\iota = -1$ for $Z = \mathbb{Z} + \frac{1}{2}$, and ± is + for the fermionic construction, − for the bosonic construction. In particular, the superalgebras have their natural grading (integral levels consisting of Lie elements and semi-integral levels consisting of Jordan elements) when

$$\pm\iota = 1. \tag{1.14}$$

This fact is certainly a variant of the spin-statistics theorem well-known in quantum field theory. We want to emphasize again the deep connection of our constructions with quantum field theory, which is based on representations of anticommutation and commutation relations in a Fock space.

Every highest weight representation of each affine algebra can be extended to a semidirect product with the Virasoro algebra. The representation of the Virasoro algebra is given by means of the Segal operators [2]. In the bosonic and fermionic constructions of classical affine algebras the Virasoro algebra admits a considerable simplification. For example, in the nonextended cases when no $e(m)$ generators are present, the fermionic construction of the Virasoro algebra for type $D_l^{(1)}$ reduces to that of $(\mathbf{o}^{(1)}(2))^l$, while its bosonic

TABLE IV



construction for type $C_l^{(1)}$ reduces to that of $(\mathbf{sp}^{(1)}(2))^l$. We prove the operator identities which underlie this reduction in Section 5. These identities allow us to prove the irreducibility theorems (Theorems C, D). For each algebra except type $A_{l-1}^{(1)}$, these assert that the bosonic and fermionic representations we have constructed are either irreducible or decompose into two irreducible components according to the presence or absence of $e(0)$ in the Clifford or Weyl algebra. For type $A_{l-1}^{(1)}$, we have the decomposition of the representation spaces of $D_l^{(1)}$, $C_l^{(1)}$, and $A_{2l-1}^{(2)}$ into infinitely many components, each distinguished by the eigenvalue of a single operator from the Cartan subalgebra.

In Section 6, we identify the irreducible representations we have constructed by determining their highest weights. We show that every irreducible fermionic representation is a level 1 standard representation, while every irreducible bosonic representation has level -1; so it is nonstandard. Our bosonic constructions are the first known constructions of nonstandard irreducible highest weight representations of affine algebras. As a corollary we compute the characters of our representations.

Finally, we would like to remark on the canonical nature of these constructions, which distinguishes the classical affine algebras as analogous to the classical finite-dimensional algebras, and which justifies the title of our paper.

2. Finite-Dimensional Algebras

Let $\{\cdot,\cdot\}_\pm$ denote a nondegenerate bilinear form on $\mathbf{a}\approx\mathbb{C}^{2l}$ which is either symmetric (+) or antisymmetric (−). Let $\mathbf{a}=\mathbf{a}_1\oplus\mathbf{a}_2$, $\mathbf{a}_i\approx C^l$, be a polarization into maximal isotropic subspaces. If $u, v\in\mathbf{a}$, then define the "normally ordered" quadratic expression $:uv:=\frac{1}{2}(uv\mp vu)$. The associative algebra $A_\pm(2l)$ with unit 1 generated by the elements of $\mathbf{a}$ subject to the relations $uv\pm vu=\{u,v\}_\pm$ gives either a 2^{2l}-dimensional Clifford algebra (+) or an infinite-dimensional Weyl algebra (−). Both algebras are graded by degree, and using the identity

$$[AB, CD]=A\{B,C\}_\pm D\mp AC\{B,D\}_\pm+\{A,C\}_\pm DB\mp C\{A,D\}_\pm B, \tag{2.1}$$

we see that the subspaces of quadratic elements are closed under the Lie bracket $[\cdot,\cdot]$. The Lie algebra obtained by taking all normally ordered quadratic expressions $:uv:$ for $u,v\in\mathbb{C}^{2l}$ is just $\mathbf{o}(2l)$ of type D_l in the fermionic (+) case, and $\mathbf{sp}(2l)$ of type C_l in the bosonic (−) case. If the linear space $\mathbf{a}$ is added on to these, then using the identity

$$[AB, C]=A\{B,C\}_\pm\mp\{A,C\}_\pm B, \tag{2.2}$$

we obtain $\mathbf{o}(2l+1)$, type B_l in the fermionic case (+), and the superalgebra $B(0,l)$ in the bosonic case (−). If $u,v\in\mathbf{a}$, then the commutator $[u,v]_-=2{:}uv{:}$ is in $\mathbf{o}(2l+1)$, while the anticommutator $[u,v]_+=2{:}uv{:}$ is in $B(0,l)$. Note that in either case the span of the elements $:uv:$, $u\in\mathbf{a}_1$, $v\in\mathbf{a}_2$ gives the Lie algebra $\mathbf{gl}(l)$ inside both $\mathbf{o}(2l)$ and $\mathbf{sp}(2l)$.

These constructions give the following representations. Let $V_\pm$ be a simple Clifford (+) or Weyl (−) module having a "vacuum vector" v_0 such that $\mathbf{a}_1\cdot v_0=0$. We call these operators which kill v_0 annihilation operators and those from $\mathbf{a}_2$ we call creation operators. The 2^l-dimensional fermionic representation space $V_+=A_+(2l)\cdot v_0$ is just the irreducible spinor representation of $\mathbf{o}(2l+1)$. The decomposition $V_+=V_+^0\oplus V_+^1$ into even and odd subspaces (the number of creation operators applied to v_0) is preserved by $\mathbf{o}(2l)$ giving the irreducible semispinor representations. The infinite-dimensional bosonic representation space $V_-=A_-(2l)\cdot v_0$ is irreducible under $B(0,l)$ but decomposes into two infinite-dimensional irreducible subspaces $V_-^0\oplus V_-^1$ under $\mathbf{sp}(2l)$. In this finite-dimensional theory the symmetry between the bosonic and fermionic constructions is grossly broken by the infinite dimensionality of V_-. This discrepancy does not exist in the affine theory where all representation spaces are infinite-dimensional.

To carry out the constructions of the affine algebras, it is useful to present all elements of $\mathbf{o}(2l+1)$ and $B(0,l)$ as normally ordered quadratic expressions. This is easily done by introducing a new generator, e, into the algebra $A_\pm(2l)$, such that $ue\pm eu=0$ for $u\in\mathbb{C}^{2l}$ and $e^2=\mp1$. The larger algebra $A_\pm(2l+1)$ acts on $V_\pm$ with $e\cdot v_0=\sqrt{-1}\,v_0$ in the fermionic case and $e\cdot v_0=v_0$ in the bosonic case. To obtain $\mathbf{o}(2l+1)$ from $\mathbf{o}(2l)$, and $B(0,l)$ from $\mathbf{sp}(2l)$, we add the quadratic expressions $:ue:=ue$ for $u\in\mathbf{a}$. Then we find as before $[ue, ve]_\mp=2{:}uv{:}$.

Any nondegenerate symmetric invariant bilinear form $\langle\cdot,\cdot\rangle$ on any of the above algebras is uniquely determined up to a scalar multiple. It allows the identification of the Cartan subalgebra $\mathbf{h}$ with its dual $\mathbf{h}^*$. If r_1, r_2, s_1, $s_2\in\mathbf{a}$, then our choice for the invariant form on $\mathbf{o}(2l)$ or $\mathbf{sp}(2l)$ is

$$\langle :r_1r_2:, :s_1s_2:\rangle=\pm(\{r_1,s_2\}_\pm\{r_2,s_1\}_\pm\mp\{r_1,s_1\}_\pm\{r_2,s_2\}_\pm). \tag{2.3}$$

The same expression gives an invariant form on $\mathbf{o}(2l+1)$ in the fermionic case, and a supersymmetric invariant form on $B(0,l)$ in the bosonic case. Recall that supersymmetry of a form on a superalgebra means that

$$\langle a,b\rangle=(-1)^{(\deg a)(\deg b)}\langle b,a\rangle. \tag{2.4}$$

Before giving the constructions of affine algebras in the next section, we would like to set up our notation for the root systems, root vectors, and fundamental weights of the finite-dimensional algebras $\mathbf{gl}(l)$, $\mathbf{o}(2l)$, $\mathbf{o}(2l+1)$, $\mathbf{sp}(2l)$, and $B(0,l)$ (see Table V).

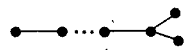
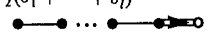
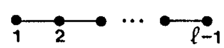
Let $\{a_i\mid 1\leqslant i\leqslant l\}$ be a basis of $\mathbf{a}_1$ and let $\{a_i^*\mid 1\leqslant i\leqslant l\}$ be the dual basis of $\mathbf{a}_2$, so that we have

$$\begin{aligned}\{a_i,a_j\}_\pm=\{a_i^*,a_j^*\}_\pm=0 \quad &\text{and}\\ \{a_i,a_j^*\}_\pm=\delta_{ij} \quad &\text{for}\quad 1\leqslant i,j\leqslant l.\end{aligned} \tag{2.5}$$

Then

$$\sum_{1\leqslant i,j\leqslant l}\mathbb{C}{:}a_ia_j^*{:}\oplus\sum_{1\leqslant i<j\leqslant l}\mathbb{C}{:}a_ia_j{:}\oplus\sum_{1\leqslant j<i\leqslant l}\mathbb{C}{:}a_i^*a_j^*{:} \tag{2.6}$$

TABLE V

$\mathbf{o}(2l)$ D_l	$\Delta = \{\pm(\varepsilon_i \pm \varepsilon_j) \mid 1 \leqslant i,j \leqslant l, i<j\}$ $\alpha_1 = \varepsilon_1 - \varepsilon_2, \ldots, \alpha_{l-1} = \varepsilon_{l-1} - \varepsilon_l, \alpha_l = \varepsilon_{l-1} + \varepsilon_l$ $\omega_1 = \varepsilon_1, \ldots, \omega_{l-2} = \varepsilon_1 + \cdots + \varepsilon_{l-2},$ $\omega_{l-1} = \frac{1}{2}(\varepsilon_1 + \cdots + \varepsilon_{l-1} - \varepsilon_l)$ $\omega_l = \frac{1}{2}(\varepsilon_1 + \cdots + \varepsilon_{l-1} + \varepsilon_l)$ $\tilde{\alpha} = \omega_2 = \tilde{\alpha}^s$ 
$\mathbf{o}(2l+1)$ B_l	$\Delta = \{\pm\varepsilon_i, \pm(\varepsilon_i \pm \varepsilon_j) \mid 1 \leqslant i,j \leqslant l, i<j\}$ $\alpha_1 = \varepsilon_1 - \varepsilon_2, \ldots, \alpha_{l-1} = \varepsilon_{l-1} - \varepsilon_l, \alpha_l = \varepsilon_l$ $\omega_1 = \varepsilon_1, \ldots, \omega_{l-1} = \varepsilon_1 + \cdots + \varepsilon_{l-1}, \omega_l = \frac{1}{2}(\varepsilon_1 + \cdots + \varepsilon_l)$ $\tilde{\alpha} = \omega_2, \tilde{\alpha}^s = \omega_1$ 
$\mathbf{sp}(2l)$ C_l	$\Delta = \{\pm 2\varepsilon_i, \pm(\varepsilon_i \pm \varepsilon_j) \mid 1 \leqslant i,j \leqslant l, i<j\}$ $\alpha_1 = \varepsilon_1 - \varepsilon_2, \ldots, \alpha_{l-1} = \varepsilon_{l-1} - \varepsilon_l, \alpha_l = 2\varepsilon_l$ $\omega_1 = \varepsilon_1, \ldots, \omega_{l-1} = \varepsilon_1 + \cdots + \varepsilon_{l-1}, \omega_l = \varepsilon_1 + \cdots + \varepsilon_l$ $\tilde{\alpha} = 2\omega_1, \tilde{\alpha}^s = \omega_1$ 
$\mathbf{osp}(1, 2l)$ $B(0, l)$	$\Delta = \{\pm\varepsilon_i, \pm 2\varepsilon_i, \pm(\varepsilon_i \pm \varepsilon_j) \mid 1 \leqslant i,j \leqslant l, i<j\}$ $\alpha_1 = \varepsilon_1 - \varepsilon_2, \ldots, \alpha_{l-1} = \varepsilon_{l-1} - \varepsilon_l, \alpha_l = \varepsilon_l$ $\omega_1 = \varepsilon_1, \ldots, \omega_{l-1} = \varepsilon_1 + \cdots + \varepsilon_{l-1}, \omega_l = \frac{1}{2}(\varepsilon_1 + \cdots + \varepsilon_l)$ $\tilde{\alpha} = 2\omega_1, \tilde{\alpha}^s = \omega_1$ 
$\mathbf{gl}(l)$ A_{l-1}	$\Delta = \{\pm(\varepsilon_i - \varepsilon_j) \mid 1 \leqslant i,j \leqslant l, i<j\}$ $\alpha_1 = \varepsilon_1 - \varepsilon_2, \ldots, \alpha_{l-1} = \varepsilon_{l-1} - \varepsilon_l$ $\omega_1 = \varepsilon_1, \ldots, \omega_{l-1} = \varepsilon_1 + \cdots + \varepsilon_{l-1}$ $\tilde{\alpha} = \varepsilon_1 - \varepsilon_l = \tilde{\alpha}^s$ 

is a root space decomposition of $\mathbf{o}(2l)$ in the fermionic case, $\mathbf{sp}(2l)$ in the bosonic case with respect to the Cartan subalgebra $\mathbf{h}$ spanned by

$$\{h_i = \pm{:}a_i a_i^*{:} = -{:}a_i^* a_i{:} \pm \tfrac{1}{2} \mid 1 \leqslant i \leqslant l\}. \tag{2.7}$$

Note that in the fermionic case ${:}a_i a_i{:} = 0 = {:}a_i^* a_i^*{:}$. Let $\{\varepsilon_1, \ldots, \varepsilon_l\}$ be the basis of $\mathbf{h}^*$ dual to $\{h_1, \ldots, h_l\}$, so $\varepsilon_i(h_j) = \delta_{ij}$; and from (2.3), we see that $\langle h_i, h_j \rangle = \delta_{ij} = \langle \varepsilon_i, \varepsilon_j \rangle$. If we extend (2.6) by adding on

$$\sum_{1 \leqslant i \leqslant l} \mathbb{C}{:}a_i e{:} \oplus \sum_{1 \leqslant i \leqslant l} \mathbb{C}{:}a_i^* e{:}, \tag{2.8}$$

the result is a root space decomposition of $\mathbf{o}(2l+1)$ in the fermionic case, $B(0, l)$ in the bosonic case with respect to the same Cartan subalgebra $\mathbf{h}$.

The root system is denoted by Δ, with simple roots $\alpha_1, \ldots, \alpha_l$, highest root $\tilde{\alpha}$, highest short root $\tilde{\alpha}^s$ and fundamental weights $\omega_1, \ldots, \omega_l$. These weights are determined by the condition

$$\frac{2\langle \omega_i, \alpha_j \rangle}{\langle \alpha_j, \alpha_j \rangle} = \delta_{ij}, \qquad 1 \leqslant i, j \leqslant l, \tag{2.9}$$

TABLE VI

$\mathbf{o}(2l)$	$v_0 \in V_+^0$ of weight $a_l^* v_0 \in V_+^1$ of weight	$\omega_l = \frac{1}{2}(\varepsilon_1 + \cdots + \varepsilon_l)$ $\omega_{l-1} = \frac{1}{2}(\varepsilon_1 + \cdots + \varepsilon_{l-1} - \varepsilon_l)$
$\mathbf{o}(2l+1)$	$v_0 \in V_+$ of weight	$\omega_l = \frac{1}{2}(\varepsilon_1 + \cdots + \varepsilon_l)$
$\mathbf{sp}(2l)$	$v_0 \in V_-^0$ of weight $a_l^* v_0 \in V_-^1$ of weight	$-\frac{1}{2}\omega_l = -\frac{1}{2}(\varepsilon_1 + \cdots + \varepsilon_l)$ $-\frac{3}{2}\omega_l + \omega_{l-1} = -\frac{1}{2}(\varepsilon_1 + \cdots + \varepsilon_l) - \varepsilon_l$
$B(0, l)$	$v_0 \in V_-$ of weight	$-\omega_l = -\frac{1}{2}(\varepsilon_1 + \cdots + \varepsilon_l)$
$\mathbf{gl}(l)$ in $\mathbf{o}(2l)$	V_+ decomposes into $l+1$ irreducible components having the highest weight vectors with weights as follows:	
	v_0 of weight $a_l^* v_0$ of weight $a_{l-1}^* a_l^* v_0$ of weight $\vdots$ $a_1^* \cdots a_l^* v_0$ of weight	$\frac{1}{2}(\varepsilon_1 + \cdots + \varepsilon_l)$ $\frac{1}{2}(\varepsilon_1 + \cdots + \varepsilon_{l-1} - \varepsilon_l)$ $\frac{1}{2}(\varepsilon_1 + \cdots + \varepsilon_{l-2} - \varepsilon_{l-1} - \varepsilon_l)$ $-\frac{1}{2}(\varepsilon_1 + \cdots + \varepsilon_l)$
$\mathbf{gl}(l)$ in $\mathbf{sp}(2l)$	V_- decomposes into infinitely many irreducible components having highest weight vectors with weights as follows:	
	v_0 of weight $(a_l^*)^k v_0$ of weight for each $k \geqslant 1$	$-\frac{1}{2}(\varepsilon_1 + \cdots + \varepsilon_l)$ $-\frac{1}{2}(\varepsilon_1 + \cdots + \varepsilon_l) - k\varepsilon_l$

once simple roots are chosen. We give in Table V the relevant information for these five algebras, including their Dynkin diagrams.

The correspondence of root vectors with roots is as follows:

$$\begin{aligned} &{:}a_i a_j^*{:} \leftrightarrow (\varepsilon_i - \varepsilon_j), \\ &{:}a_i a_j{:} \leftrightarrow (\varepsilon_i + \varepsilon_j), \\ &{:}a_i^* a_j^*{:} \leftrightarrow -(\varepsilon_i + \varepsilon_j), \\ &{:}a_i e{:} \leftrightarrow \varepsilon_i, \\ &{:}a_i^* e{:} \leftrightarrow -\varepsilon_i. \end{aligned} \tag{2.10}$$

In Table VI we give the highest weight vectors in the irreducible components of $V_\pm$ and their weights.

3. Constructions of Affine Algebras

Let $Z = \mathbb{Z}$ or $\mathbb{Z} + \frac{1}{2}$, and let Z' denote the complement of Z in $\frac{1}{2}\mathbb{Z}$. Define

$$\iota = \operatorname{sgn}(Z) = \begin{cases} +1 & \text{if } Z = \mathbb{Z}, \\ -1 & \text{if } Z = \mathbb{Z} + \frac{1}{2}, \end{cases} \tag{3.1}$$