so that $\iota = (-1)^{2n}$ for $n \in Z$. Let $\mathbf{a} = \mathbf{a}_1 \oplus \mathbf{a}_2$ and $\{\cdot, \cdot\}_\pm$ be as defined in Section 2. Let the associative algebra $A_\pm(Z^{2l})$ be generated by

$$\{u(m) \mid u \in \mathbf{a}, m \in Z\} \tag{3.2}$$

with the relations

$$u(m)\, v(n) \pm v(n)\, u(m) = \{u, v\}_\pm\, \delta_{m,-n}. \tag{3.3}$$

The algebra $A_\pm(Z^{2l})$ is $\mathbb{Z}_2$-graded according to the parity of the number of generators in a monomial expression. We define the normal ordering of a quadratic expression to be

$$:u(m)\, v(n): = \begin{cases} u(m)\, v(n) & \text{if } n > m, \\ \frac{1}{2}(u(m)\, v(n) \mp v(n)\, u(m)) & \text{if } n = m, \\ \mp v(n)\, u(m) & \text{if } n < m \end{cases} \tag{3.4}$$
$$= \mp :v(n)\, u(m):.$$

We will require three extensions of the algebra $A_\pm(Z^{2l})$. For $S = Z, Z'$, or $\frac{1}{2}\mathbb{Z}$ the generators

$$\{e(m) \mid m \in S\} \tag{3.5}$$

span an infinite-dimensional Clifford algebra with relations

$$e(m)\, e(n) + e(n)\, e(m) = \mp 2\delta_{m,-n}. \tag{3.6}$$

Let $A^S_\pm(Z^{2l})$ denote the algebra obtained by adjoining to $A_\pm(Z^{2l})$ the generators (3.5) with relations (3.6) and

$$u(m)\, e(n) \pm e(n)\, u(m) = 0 \tag{3.7}$$

for $u \in \mathbf{a}$, $m \in Z$.

The following generating function formalism is very useful. For $u \in \mathbf{a}$ and formal complex variable w, we define

$$u(w) = \sum_{m \in Z} u(m)\, w^{-m}, \tag{3.8}$$

$$u^+(w) = \begin{cases} \sum_{0 < m \in Z} u(m)\, w^{-m} & \text{if } Z = \mathbb{Z} + \frac{1}{2}, \\ \frac{1}{2}u(0) + \sum_{0 < m \in Z} u(m)\, w^{-m} & \text{if } Z = \mathbb{Z}, \end{cases} \tag{3.9}$$

and $u^-(w) = u(w) - u^+(w)$. We also have

$$e_S(w) = \sum_{m \in S} e(m)\, w^{-m}, \tag{3.10}$$

where the sum runs over the appropriate set $S = Z$, Z' or $\frac{1}{2}\mathbb{Z}$ depending on the algebra being considered. For two generating functions $u(w)$ and $v(w_0)$ we define the normally ordered product

$$:u(w)\, v(w_0): = \sum_m \sum_n :u(m)\, v(n): w^{-m} w_0^{-n}. \tag{3.11}$$

If $w = w_0$ and $m, n \in Z$, then we have

$$x(w) = :u(w)\, v(w): = \sum_{k \in Z} \sum_{m \in Z} :u(m)\, v(k-m): w^{-k}, \tag{3.12}$$

which defines the kth homogeneous component

$$x(k) = :u(w)\, v(w):_k$$
$$= \sum_{m \in Z} :u(m)\, v(k-m): \quad \text{for} \quad k \in \mathbb{Z}. \tag{3.13}$$

If $v(w) = e_S(w)$, then the outer summation in (3.12) will be over $k \in \mathbb{Z}$, $\mathbb{Z} + \frac{1}{2}$ or $\frac{1}{2}\mathbb{Z}$ in case $S = Z$, Z', or $\frac{1}{2}\mathbb{Z}$, and $x(w)$ will have the appropriate components. Also note that for $u \in \mathbf{a}$, $:u(m)e(n):$ is as given in (3.4) and equals $u(m)\, e(n)$ because of (3.7). But in $:e(w)\, e(w_0):$ the normal ordering is fermionic.

The components of various normally ordered quadratic expressions $:u(w)\, v(w):$ are well-defined operators on appropriate representation spaces which we will now describe. Let $V_\pm(Z^l)$ be a simple Clifford (+) or Weyl (−) module as follows. We call the generators

$$\{u(m) \mid u \in \mathbf{a}, 0 < m \in Z\} \cup \{u(0) \mid u \in \mathbf{a}_1 \text{ if } Z = \mathbb{Z}\} \tag{3.14}$$

annihilation operators, and

$$\{u(m) \mid u \in \mathbf{a}, 0 > m \in Z\} \cup \{u(0) \mid u \in \mathbf{a}_2 \text{ if } Z = \mathbb{Z}\} \tag{3.15}$$

creation oerators. Then $V_\pm(Z^l)$ is a simple module containing an element v_0, called a "vacuum vector," which is killed by all annihilation operators. The space

$$V_\pm(Z^l) = A_\pm(Z^{2l}) \cdot v_0 \tag{3.16}$$

decomposes into even and odd subspaces

$$V_\pm(Z^l) = V^0_\pm(Z^l) \oplus V^1_\pm(Z^l) \tag{3.17}$$

according to the parity of the number of creation operators applied to v_0 in a monomial expression. Note that for any particular vector $v \in V_\pm(Z^l)$, only finitely many terms from (3.13) can make a nonzero contribution to $x(k) \cdot v$, so $x(k)$ is a well-defined operator on $V_\pm(Z^l)$. Also note that because $x(k)$ is composed of quadratic expressions, the even and odd subspaces of $V_\pm(Z^l)$ are preserved by these operators.

Let us now define representation spaces for each of the algebras $A^S_\pm(Z^{2l})$. Let $V(S)$ be a simple Clifford module for the Clifford algebra generated by (3.5) with relations (3.6). We call $e(m)$ an annihilation operator if $m > 0$, or a creation operator if $m < 0$. When $e(0)$ is present its action is determined by (3.6),

$$e(0)\,e(0) = \mp 1, \tag{3.18}$$

so we choose $e(0) = \sqrt{-1}$ in the fermionic case and $e(0) = 1$ in the bosonic case. $V(S)$ is the simple Clifford module containing "vacuum vector" v_0, which is killed by annihilation operators. Because of (3.7), we see that the $A^S_\pm(Z^{2l})$-module

$$V^S_\pm(Z^l) = V_\pm(Z^l) \otimes V(S) = A^S_\pm(Z^{2l}) \cdot v_0 \tag{3.19}$$

is simple. It is clear that the components of generating functions $:u(w)\,v(w):$, $:u(w)\,e_S(w):$, and $:e_Z(w)\,e_{Z'}(w):$ (when $S = \frac{1}{2}\mathbb{Z}$) are well-defined operators on $V^S_\pm(Z^l)$. We have the decomposition into even and odd subspaces

$$V^S_\pm(Z^l) = V^{S,0}_\pm(Z^l) \oplus V^{S,1}_\pm(Z^l), \tag{3.20}$$

which are preserved by the components of $:u(w)\,v(w):$ for $u, v \in \mathbf{a}$, and by the components of $:u(w)\,e_S(w):$, unless $e(0)$ is present. So the decomposition of $V^S_\pm(Z^l)$ is preserved for $S = Z$ if $Z = \mathbb{Z} + \frac{1}{2}$, but is not if $Z = \mathbb{Z}$. When $S = Z'$ it is preserved if $Z = \mathbb{Z}$, but is not if $Z = \mathbb{Z} + \frac{1}{2}$, and when $S = \frac{1}{2}\mathbb{Z}$ it is not preserved in either case. Later we will see that these preserved subspaces are irreducible highest weight representations of various affine algebras. Their highest weight vectors and weights will be determined by the choice of Z.

There are two ways of using generating functions to define operators which represent affine algebras. One way is to consider the homogeneous components of functions $:u(w)\,v(w):$ for $u, v \in \mathbf{a}$ and the identity operator 1. These may be extended three ways by adding on components of functions $:u(w)\,e_S(w):$ and $:e_Z(w)\,e_{Z'}(w):$ as shown above. There is another construction which yields a closed algebra, but it requires a "twist" of the first one. To obtain it, we consider the identity operator and the components of the functions

$$:u(w)\,v(w): \quad \text{for} \quad u \in \mathbf{a}_1, \quad v \in \mathbf{a}_2, \tag{3.21}$$

$$:u(w)\,v(-w): \quad \text{for} \quad u, v \in \mathbf{a}_1, \tag{3.22}$$

and

$$:u(w)\,v(-w): \quad \text{for} \quad u, v \in \mathbf{a}_2. \tag{3.23}$$

These may be extended three ways by adding on the components of the functions

$$:u(w)\,e_S(w): \quad \text{for} \quad u \in \mathbf{a}_1, \tag{3.24}$$

$$:u(-w)\,e_S(w): \quad \text{for} \quad u \in \mathbf{a}_2; \tag{3.25}$$

$$:e_S(-w)\,e_S(w): \tag{3.26}$$

and

$$:e_S(w)\,e_S(-w):, \tag{3.27}$$

where the generators $\{e(m) \mid m \in S\}$ form a Clifford, Weyl, or mixed algebra. Actually, (3.27) is only needed when $S = \frac{1}{2}\mathbb{Z}$. We will refer to these two types of constructions as "nontwisted" and "twisted," respectively. To show that these collections of operators span closed algebras it will be necessary to develop further the generating function formalism.

Let $[A, B]_\pm = AB \pm BA$ for operators A, B. It is straightforward to verify

LEMMA 1. *Let $u, v \in \mathbf{a}$, $S = \mathbb{Z}$, $\mathbb{Z} + \frac{1}{2}$ or $\frac{1}{2}\mathbb{Z}$, and let $\{e(n) \mid n \in S\}$ be fermionic generators satisfying* (3.6), (3.7).

(a) *Suppose $Z = \mathbb{Z}$ and $|w| > |w_0|$ for $w, w_0 \in \mathbb{C}^*$, then*

$$[u^+(w), v(w_0)]_\pm = [u(w), v^-(w_0)]_\pm$$

$$= \{u, v\}_\pm \left(\frac{1}{2} + \sum_{1 \leqslant n \in \mathbb{Z}} w_0^n w^{-n}\right) = \{u, v\}_\pm \frac{1}{2}\left(\frac{w + w_0}{w - w_0}\right).$$

(b) *Suppose $Z = \mathbb{Z} + \frac{1}{2}$ and $|w| > |w_0|$ for $w, w_0 \in \mathbb{C}^*$, then*

$$[u^+(w), v(w_0)]_\pm = [u(w), v^-(w_0)]_\pm$$

$$= \{u, v\}_\pm \left(\sum_{0 \leqslant n \in \mathbb{Z}} w_0^{n+1/2} w^{-(n+1/2)}\right) = \{u, v\}_\pm \frac{(ww_0)^{1/2}}{w - w_0}.$$

(c) *For $|w| > |w_0|$, $w, w_0 \in \mathbb{C}^*$ we have*

$$[e_S^+(w), e_S(w_0)]_+ = [e_S(w), e_S^-(w_0)]_+$$

$$= \mp 2\frac{\xi_S}{w - w_0} = \mp 2 \begin{cases} \dfrac{1}{2}\left(\dfrac{w + w_0}{w - w_0}\right) & \text{if} \quad S = \mathbb{Z}, \\[2ex] \dfrac{(ww_0)^{1/2}}{w - w_0} & \text{if} \quad S = \mathbb{Z} + \dfrac{1}{2}, \\[2ex] \dfrac{1}{2}\left(\dfrac{w + w_0}{w - w_0}\right) + \dfrac{(ww_0)^{1/2}}{w - w_0} & \text{if} \quad S = \dfrac{1}{2}\mathbb{Z}. \end{cases}$$

LEMMA 2. *For $u \in \mathbf{a}$, $v \in \mathbf{a}$ or $v = e_S$, either Z, any $w, w_0 \in \mathbb{C}^*$ we have*

$$:u(w)\,v(w_0): = u(w)\,v^+(w_0) \mp v^-(w_0)\,u(w) = u^-(w)\,v(w_0) \mp v(w_0)\,u^+(w).$$

For any S we have

$$:e_S(w)\,e_S(w_0): = e_S(w)\,e_S^+(w_0) - e_S^-(w_0)\,e_S(w) = e_S^-(w)\,e_S(w_0) - e_S(w_0)\,e_S^+(w)$$

in both the bosonic and fermionic cases.

DEFINITION. For any generating functions $u(w)$, $v(w_0)$, we define the contraction

$$\underline{u(w)\,v(w_0)} = u(w)\,v(w_0) - :u(w)\,v(w_0):. \tag{3.28}$$

Then for $u, v \in \mathbf{a}$ and $|w| > |w_0|$, we have from Lemmas 1 and 2 that

$$\underline{u(w)\,v(w_0)} = [u(w), v^-(w_0)]_\pm = \{u, v\}_\pm \frac{\xi}{w - w_0}, \tag{3.29}$$

where

$$\xi = \xi_Z = \begin{cases} \frac{1}{2}(w + w_0) & \text{if } Z = \mathbb{Z}, \\ (ww_0)^{1/2} & \text{if } Z = \mathbb{Z} + \frac{1}{2}. \end{cases} \tag{3.30}$$

For $u \in \mathbf{a}$ and any $w, w_0 \in \mathbb{C}^*$, we also have

$$\underline{u(w)\,e_S(w_0)} = 0, \tag{3.31}$$

while for $|w| > |w_0|$ we see that

$$\underline{e_S(w)\,e_S(w_0)} = [e_S(w), e_S^-(w_0)]_+ = \mp 2 \frac{\xi_S}{w - w_0} \tag{3.32}$$

is given in Lemma 1(c).

LEMMA 3. *For $u, v \in \mathbf{a}$, if $|w| > |w_0|$, we have*

(a) $\underline{u(w)\,v(-w_0)} = [u(w), v^-(-w_0)]_\pm = \{u, v\}_\pm\, \xi'/(w + w_0)$ *where*

$$\frac{\xi'}{w + w_0} = \begin{cases} \dfrac{1}{2} + \sum\limits_{1 \leq n \in Z} (-w_0)^n (w)^{-n} = \dfrac{1}{2}\left(\dfrac{w - w_0}{w + w_0}\right) & \text{if } \iota = 1, \\ \sum\limits_{0 \leq n \in Z} (-w_0)^{n+1/2} (w)^{-(n+1/2)} = \left(\dfrac{-w_0}{w}\right)^{1/2} \dfrac{w}{w + w_0} & \text{if } \iota = -1, \end{cases}$$

(b) $\underline{u(-w)\,v(w_0)} = [u(-w), v^-(w_0)]_\pm = \{u, v\}_\pm\, \iota\xi'/(w + w_0)$,

(c) $\underline{u(-w)\,v(-w_0)} = [u(-w), v^-(-w_0)]_\pm = \{u, v\}_\pm\, \xi/(w - w_0)$. *Thus, we have*

$$\underline{u(w)\,v(w_0)} = \underline{u(-w)\,v(-w_0)} \quad \textit{and} \quad \underline{u(w)\,v(-w_0)} = \iota\,\underline{u(-w)\,v(w_0)}.$$

LEMMA 4. *For $u, v \in \mathbf{a}$ and $|w| > |w_0|$, we have*

$$\lim_{w \to w_0} \xi = w_0 \quad \textit{and} \quad \lim_{w \to -w_0} \xi' = -w_0,$$

$$\lim_{w \to -w_0} :u(-w)\,v(w_0): = \iota :u(w_0)\,v(w_0):,$$

$$\lim_{w \to -w_0} :u(w)\,v(-w_0) = :u(-w_0)\,v(-w_0):.$$

DEFINITION. Let $x_1, \ldots, x_n$ be generators in an algebra of operators having fermionic (+) or bosonic (−) relations, and having a notion of normal ordering. For fermionic generators define

$$:x_1 \cdots \underline{x_i \cdots x_j} \cdots x_n: = \operatorname{sgn}(\sigma)\, \underline{x_i x_j}(:x_1 \cdots x_{i-1} x_{i+1} \cdots x_{j-1} x_{j+1} \cdots x_n:),$$

where $\operatorname{sgn}(\sigma)$ is the sign of the permutation

$$\sigma = \begin{pmatrix} 1 & 2 & 3 & \cdots & & & & & n \\ i & j & 1 & \cdots & \hat{\imath} & \cdots & \hat{\jmath} & \cdots & n \end{pmatrix}.$$

For bosonic generators, we give the same definition, but without the $\operatorname{sgn}(\sigma)$ factor. In the physics literature, the following Wick theorem is well known [7].

THEOREM 5. *We have $x_1 \cdots x_n = :x_1 \cdots x_n: + \sum :x_1 \cdots \underline{x_i \cdots x_j} \cdots x_n:$, where the summation is taken over all possible combinations of sets of contractions. Also, $(:x_1 \cdots x_n:)(:y_1 \cdots y_m:) = :x_1 \cdots x_n y_1 \cdots y_m: + \sum :x_1 \cdots \underline{x_n y_1} \cdots y_m:$, where the summation is taken over all possible combinations of contractions of some x's and some y's.*

COROLLARY 6. *In an algebra with both fermionic and bosonic generators, we have for any permutation σ*

$$:x_1 \cdots x_n: = (-1)^N :x_{\sigma(1)} \cdots x_{\sigma(n)}:,$$

where N is the number of fermionic–fermionic transpositions in a decomposition of σ.

Let V be any of the representation spaces we have defined, and let V^* be its algebraic dual space (functionals nonzero only on finite-dimensional subspaces of V). We denote by $\langle \cdot \mid \cdot \rangle$ the pairing between V and V^*. Let $:u(w)\,v(w):$ be a generating function having nonzero components $:u(w)\,v(w):_k$ only for $k \in \mathbb{Z}$ or only for $k \in \mathbb{Z}+\frac{1}{2}$. For any $v_1 \in V^*$, $v_2 \in V$, we have

$$\langle v_1 \mid :u(w)\,v(w):_k v_2 \rangle = \frac{1}{2\pi i}\int_C \langle v_1 \mid :u(w)\,v(w):\, v_2 \rangle\, w^k \frac{dw}{w}$$

for any circle C around the origin of the complex w-plane. In the rest of the paper we will omit v_1, v_2 and $1/2\pi i$ from our calculations and only write $:u(w)\,v(w):_k = \int_C :u(w)\,v(w):\, w^k\, dw/w$. We will use the residue theorem to compute commutators or anticommutators, as appropriate, of such components. The residue theorem may be avoided by treating w as a formal variable, but then the calculations can be rather lengthy. In case $:u(w)\,v(w):$ has nonzero components $:u(w)\,v(w):_k$ for $k \in \mathbb{Z}$ and $\mathbb{Z}+\frac{1}{2}$, we will always divide $:u(w)\,v(w):$ into two separate generating functions: one having only integral components and the other having only semi-integral components. Then $:u(w)\,v(w):_k$ can be expressed as an integral of the appropriate part of $:u(w)\,v(w):$ depending on k. For example, if $u \in \mathbf{a}$, then $:u(w)\,e_{1/2\mathbb{Z}}(w):_k$ equals

$$\int_C :u(w)\,e_{\mathbb{Z}}(w):\, w^k \frac{dw}{w} \qquad \text{if} \quad k \in \mathbb{Z},$$

$$\int_C :u(w)\,e_{\mathbb{Z}'}(w):\, w^k \frac{dw}{w} \qquad \text{if} \quad k \in \mathbb{Z}+\frac{1}{2}.$$

In the case of (3.26), $:e_{1/2\mathbb{Z}}(-w)\,e_{1/2\mathbb{Z}}(w):_k$ equals

$$\int_C (:e_{\mathbb{Z}}(-w)\,e_{\mathbb{Z}}(w) + :e_{\mathbb{Z}'}(-w)\,e_{\mathbb{Z}'}(w):)\, w^k \frac{dw}{w} \qquad \text{if} \quad k \in \mathbb{Z},$$

$$\int_C (:e_{\mathbb{Z}}(-w)\,e_{\mathbb{Z}'}(w): + :e_{\mathbb{Z}'}(-w)\,e_{\mathbb{Z}}(w):)\, w^k \frac{dw}{w} \qquad \text{if} \quad k \in \mathbb{Z}+\frac{1}{2}.$$

Now we proceed to the proof that the components of the generating functions in the nontwisted case form a closed algebra. The Wick theorem and (3.29) give

LEMMA 7. *For $r_1, r_2, s_1, s_2 \in \mathbf{a}$ and $|w| > |w_0|$, we have*

$$(:r_1(w)\,r_2(w):)(:s_1(w_0)\,s_2(w_0):) = :r_1(w)\,r_2(w)\,s_1(w_0)\,s_2(w_0):$$
$$+\frac{\xi}{w-w_0}(\{r_1,s_2\}_\pm :r_2(w)\,s_1(w_0): + \{r_2,s_1\}_\pm :r_1(w)\,s_2(w_0):$$

$$\mp \{r_1,s_1\}_\pm :r_2(w)\,s_2(w_0): \mp \{r_2,s_2\}_\pm :r_1(w)\,s_1(w_0):)$$
$$+\frac{\xi^2}{(w-w_0)^2}(\{r_1,s_2\}_\pm \{r_2,s_1\}_\pm \mp \{r_1,s_1\}_\pm \{r_2,s_2\}_\pm).$$

Note that if $|w_0| > |w|$, then $(:s_1(w_0)\,s_2(w_0):)(:r_1(w)\,r_2(w):)$ gives exactly the same expression as above.

For $x_1 = :r_1 r_2:$, $x_2 = :s_1 s_2:$ quadratic expressions from $A_\pm(2l)$, identity (2.1) gives us that

$$x_3 = [x_1, x_2] = \{r_1,s_2\}_\pm :r_2 s_1: + \{r_2,s_1\}_\pm :r_1 s_2:$$
$$\mp \{r_1,s_1\}_\pm :r_2 s_2: \mp \{r_2,s_2\} :r_1 s_1: \qquad (3.33)$$

in a finite-dimensional algebra, $\mathbf{o}(2l)$ or $\mathbf{sp}(2l)$. For $m, n \in \mathbb{Z}$ we can now use the generating function formalism (recall (3.8)–(3.13)) to prove that

$$[x_1(m), x_2(n)] = [x_1, x_2](m+n) \pm m\langle x_1, x_2\rangle\, \delta_{m,-n}. \qquad (3.34)$$

The left side of (3.34) is

$$\left[\int_C x_1(w)\,w^m \frac{dw}{w}, \int_{C_1} x_2(w_0)\,w_0^n \frac{dw_0}{w_0}\right], \qquad (3.35)$$

where C and C_1 are any circles around the origin of the complex plane. We may write this as

$$\int_C \left(\int_{C_R} x_1(w)\,x_2(w_0)\,w^m \frac{dw}{w} - \int_{C_r} x_2(w_0)\,x_1(w)\,w^m \frac{dw}{w}\right) w_0^n \frac{dw_0}{w_0}, \qquad (3.36)$$

where C_R and C_r have radii R and r, respectively, and $r < |w_0| < R$ for w_0 on C. From Lemma 7 and (2.3), we then get

$$\int_C \left(\int_{C_R \setminus C_r} x_1(w)\,x_2(w_0)\,w^m \frac{dw}{w}\right) w_0^n \frac{dw_0}{w_0} \qquad (3.37)$$
$$= \int_C x_3(w_0)\,w_0^{m+n} \frac{dw_0}{w_0} \pm \langle x_1, x_2\rangle \int_C m w_0^{m+n} \frac{dw_0}{w_0}$$
$$= x_3(m+n) \pm m\langle x_1, x_2\rangle\, \delta_{m,-n}.$$

Thus, the identity operator and the components of $:u(w)\,v(w):$, for $u, v \in \mathbf{a}$ as operators on $V_\pm(Z^l)$, form a closed Lie algebra.

Let us now extend this algebra by adding on the components of $:u(w)\,e_{\mathbb{Z}}(w):$ for $u \in \mathbf{a}$, where

$$e_{\mathbb{Z}}(w) = \sum_{m \in \mathbb{Z}} e(m)\,w^{-m}. \qquad (3.38)$$

If $x = {:}uv{:}$, $x_1 = {:}u_1 e{:}$, $x_2 = {:}u_2 e{:}$ for $u, v, u_1, u_2 \in \mathbf{a}$ are elements of the finite-dimensional algebra $\mathbf{o}(2l+1)$ or $B(0, l)$, then $x(w) = {:}u(w)\,v(w){:}$, $x_1(w) = {:}u_1(w)\,e_Z(w){:}$, and $x_2(w) = {:}u_2(w)\,e_Z(w){:}$ have only integral components. Lemma 7 still holds for $x(w)$ and $x_1(w_0)$ if we understand e to be orthogonal to $\mathbf{a}$ with respect to $\{\cdot,\cdot\}_\pm$. This gives, as above,

$$[x(m), x_1(n)] = [x, x_1](m+n) \tag{3.39}$$

for $m, n \in \mathbb{Z}$, and $\langle x, x_1\rangle = 0$ from (2.3). To obtain the expression (3.37) it is necessary to use $[x_1(m), x_2(n)]_\mp$, because ${:}e_Z(w)\,e_Z(w_0){:} = -{:}e_Z(w_0)\,e_Z(w){:}$ in both the fermionic and bosonic cases. The same is true of $[x_1, x_2]_\mp$ in the finite-dimensional algebra, and this technique gives

$$[x_1(m), x_2(n)]_\mp = [x_1, x_2]_\mp(m+n) \pm m\delta_{m,-n}\langle x_1, x_2\rangle. \tag{3.40}$$

Again we have obtained a closed algebra, this time consisting of operators acting on $V_\pm^Z(Z^l)$.

If we extend the algebra of operators on $V_\pm(Z^l)$ by adding on the components of ${:}u(w)\,e_{Z'}(w){:}$ for $u \in \mathbf{a}$, where

$$e_{Z'}(w) = \sum_{m \in Z'} e(m)\,w^{-m}, \tag{3.41}$$

then $x_1(w)$ and $x_2(w)$ will have only semi-integral $(\mathbb{Z} + \frac{1}{2})$ components. The contractions needed are

$$\underbrace{u(w)\,e_{Z'}(w_0)} = 0$$

and

$$\underbrace{e_{Z'}(w)\,e_{Z'}(w_0)} = \mp 2\,\frac{\xi_{Z'}}{w - w_0},$$

where

$$\xi_{Z'} = \begin{cases} \frac{1}{2}(w + w_0) & \text{if } \iota = -1, \\ (ww_0)^{1/2} & \text{if } \iota = 1. \end{cases} \tag{3.42}$$

Although some of the details of the calculation are slightly modified, we easily obtain as before

$$[x(m), x_1(n+\tfrac{1}{2})] = [x, x_1](m+n+\tfrac{1}{2}), \tag{3.43}$$

and

$$[x_1(m+\tfrac{1}{2}), x_2(n+\tfrac{1}{2})]_\mp$$
$$= [x_1, x_2]_\mp(m+n+1) \pm (m+\tfrac{1}{2})\,\delta_{(m+1/2),-(n+1/2)}\langle x_1, x_2\rangle \tag{3.44}$$

for $m, n \in \mathbb{Z}$. This closed algebra of operators acts on $V_\pm^{Z'}(Z^l)$.

The largest algebra extension, containing all of the previous three, is obtained by including the components of ${:}u(w)\,e_{1/2Z}(w){:}$ for $u \in \mathbf{a}$, where

$$e_{1/2Z}(w) = e_Z(w) + e_{Z'}(w) = \sum_{m \in 1/2Z} e(m)\,w^{-m}, \tag{3.45}$$

and the components of ${:}e_Z(w)\,e_{Z'}(w){:}$. For $x_1 = {:}u_1 e{:}$, we have

$$x_1(w) = {:}u_1(w)\,e_{1/2Z}(w){:} = \sum_{k \in 1/2Z} \sum_{m \in Z} {:}u_1(m)\,e(k-m){:}\,w^{-k} \tag{3.46}$$

whose integral and semi-integral components were considered separately above. We obtain again the formulas (3.39), (3.40), (3.43), and (3.44), as well as

$$[x_1(m), x_2(n+\tfrac{1}{2})]_\mp = \mp\{u_1, u_2\}_\pm \int_C {:}e_Z(w_0)\,e_{Z'}(w_0){:}\,w_0^{m+n+1/2}\,\frac{dw_0}{w_0}. \tag{3.47}$$

To show closure of this largest extension we must bracket the components of ${:}e_Z(w)\,e_{Z'}(w){:}$ with each other and with $x_1(n)$, $x_1(n+\frac{1}{2})$ and $x(n)$. We obtain

$$\left[x_1(n), \int_{C_1} {:}e_Z(w_0)\,e_{Z'}(w_0){:}\,w_0^{m+1/2}\,\frac{dw_0}{w_0}\right] = \mp 2x_1(n+m+\tfrac{1}{2}), \tag{3.48}$$

$$\left[x_1(n+\tfrac{1}{2}), \int_{C_1} {:}e_Z(w_0)\,e_{Z'}(w_0){:}\,w_0^{m+1/2}\,\frac{dw_0}{w_0}\right] = \pm 2x_1(n+m+1), \tag{3.49}$$

the bracket with $x(n)$ is zero, and

$$\left[\int_C {:}e_Z(w)\,e_{Z'}(w){:}\,w^{n+1/2}\,\frac{dw}{w}, \int_{C_1} {:}e_Z(w_0)\,e_{Z'}(w_0){:}\,w_0^{m+1/2}\,\frac{dw_0}{w_0}\right]$$
$$= -4(n+\tfrac{1}{2})\,\delta_{n+m+1,0}. \tag{3.50}$$

Now we repeat the four constructions given above in the twisted case, i.e., starting with the identity operator and the components of the functions (3.21)–(3.23), then adding on the components of (3.24)–(3.27) for $S = Z$, Z' or $\frac{1}{2}\mathbb{Z}$. But now several points are different. First, we apparently can no longer associate ${:}u(w)\,v(-w){:} = x(w)$ with some element x in a finite-dimensional algebra. This means that the bracket formulas will not look quite as simple as before. (In fact, there is a way of doing so based on the decomposition $\mathbf{gl}(2l) = \mathbf{sp}(2l) \oplus \mathbf{g}'$, but it would require introducing twice as many generators and a set of relations which reduce to our chosen approach.) We must be very careful using integrals to express the components $:u(w)\,v(-w):_k$ (see Lemmas 8(b) and 9(b), (c)). We note that there is a more straightforward alternative way of proceeding. We may use $u(w)=\sum_{m\in\mathbb{Z}} u(m)\,w^{-2m}$ instead of (3.8), $e_S(w)=\sum_{m\in S} e(m)\,w^{-2m}$ instead of (3.10), and $\sqrt{-1}\,w$ instead of $-w$ in (3.21)–(3.27). Using these functions, however, the bracket calculations become twice as long, each residue we do now splits into two parts, each giving half of the answer.

The second point of difference is in the algebra composed by the components of $e_S(w)$ for $S=\mathbb{Z}$, $\mathbb{Z}+\frac{1}{2}$ or $\frac{1}{2}\mathbb{Z}$. We now let

$$\{e(n)\mid n\in S\} \tag{3.51}$$

generate an algebra with the relations

$$\begin{aligned} &e(m)\,e(n)+e(n)\,e(m)=\mp 2(-1)^m\,\delta_{m,-n} \quad \text{if} \quad m,n\in\mathbb{Z},\\ &e(m)\,e(n)-e(n)\,e(m)=0\begin{cases}\text{if} & m\in\mathbb{Z},\quad n\in\mathbb{Z}+\frac{1}{2}\\ \text{if} & m\in\mathbb{Z}+\frac{1}{2},\quad n\in\mathbb{Z},\end{cases}\\ &e(m)\,e(n)-e(n)\,e(m)=\mp 2(-1)^m\,\delta_{m,-n} \quad \text{if} \quad m,n\in\mathbb{Z}+\tfrac{1}{2}.\end{aligned} \tag{3.52}$$

If $S=\mathbb{Z}$, we have a pure Clifford algebra as before. But if $S=\mathbb{Z}+\frac{1}{2}$, we get a pure Weyl algebra, and if $S=\frac{1}{2}\mathbb{Z}$, we get a mixed Clifford–Weyl algebra. This naturally requires the normal ordering $:e(m)\,e(n):$ to be fermionic for $m,n\in\mathbb{Z}$ and bosonic if either $m\in\mathbb{Z}+\frac{1}{2}$ or $n\in\mathbb{Z}+\frac{1}{2}$.

We must also replace the relation (3.7) by the relations

$$\begin{aligned} &u(m)\,e(n)+e(n)\,u(m)=0 \quad \text{if} \quad n\in\mathbb{Z},\quad u(m)\in A_+(Z^{2l}),\\ &u(m)\,e(n)-e(n)\,u(m)=0\begin{cases}\text{if} & n\in\mathbb{Z}+\frac{1}{2},\quad u(m)\in A_+(Z^{2l})\\ \text{if} & u(m)\in A_-(Z^{2l}).\end{cases}\end{aligned} \tag{3.53}$$

This means that the normal ordering $:u(m)\,e(n):$ is bosonic if $u(m)$ is bosonic. But if $u(m)$ is fermionic, then the normal ordering is fermionic for $n\in\mathbb{Z}$, bosonic for $n\in\mathbb{Z}+\frac{1}{2}$. Consequently, we have

LEMMA 8. *For $S=\mathbb{Z}$, $\mathbb{Z}+\frac{1}{2}$ or $\frac{1}{2}\mathbb{Z}$ let*

$$e_S(w)=\sum_{n\in S} e(n)\,w^{-n}$$

have components from (3.51) *with relations* (3.52) *and* (3.53).

(a) *For $|w|>|w_0|$ we have*

$$\underbrace{e_Z(w)\,e_Z(w_0)}=[e_Z(w),e_Z^-(w_0)]_+=\mp 2\left(\frac{1}{2}\,\frac{w-w_0}{w+w_0}\right),$$

$$\underbrace{e_{Z+1/2}(w)\,e_{Z+1/2}(w_0)}=[e_{Z+1/2}(w),e_{Z+1/2}^-(w_0)]_-=\mp 2(-1)^{1/2}\,\frac{(ww_0)^{1/2}}{w+w_0}.$$

(b) *For $u,v\in\mathbf{a}$, $k\in\mathbb{Z}$ we have*

$$:u(-w)\,v(-w):_k=(-1)^{-k}:u(w)\,v(w):_k,$$
$$:u(w)\,v(-w):_k=\imath(-1)^{-k}:u(-w)\,v(w):_k.$$

Also, we have

$$:u(-w)\,e_S(-w):_k=(-1)^{-k}:u(w)\,e_S(w):_k,$$
$$:u(w)\,e_S(-w):_k=\imath(-1)^{-k}:u(-w)\,e_S(w):_k$$

for $S=Z, Z', \frac{1}{2}\mathbb{Z}$ and $k\in\mathbb{Z}$, $\mathbb{Z}+\frac{1}{2}$, $\frac{1}{2}\mathbb{Z}$, respectively.

We begin as before with the associative algebra $A_\pm(Z^{2l})$ generated by (3.2) with relations (3.3). But now we get the three extensions $A_\pm^S(Z^{2l})$ for $S=Z$, Z', and $\frac{1}{2}\mathbb{Z}$ by adjoining generators (3.51) with relations (3.52) and (3.53). The representation space $V_\pm(Z^l)=A_\pm(Z^{2l})\cdot v_0$ is defined as before, but the space $V(S)$ will be a Clifford module if $S=\mathbb{Z}$, a Weyl module if $S=\mathbb{Z}+\frac{1}{2}$, and a mixed Clifford–Weyl module if $S=\frac{1}{2}\mathbb{Z}$. With these modifications the notation of (3.19) and (3.20) will still be used.

We are now ready to prove that the components of (3.21)–(3.23) and the identity operator acting on $V_\pm(Z^l)$ represent a closed algebra. Since the components of $:u(w)\,v(w):$ for $u\in\mathbf{a}_1$, $v\in\mathbf{a}_2$ are associated with the element $x=:uv:$ from the finite-dimensional algebra $\mathbf{gl}(l)$, we do have the usual bracket formula

$$[x(m),y(n)]=[x,y](m+n)\pm m\langle x,y\rangle\,\delta_{m,-n} \tag{3.54}$$

for $m,n\in\mathbb{Z}$, $x,y\in\mathbf{gl}(l)$.

Let $u_1\in\mathbf{a}_1$, $v_1\in\mathbf{a}_2$. If $u,v\in\mathbf{a}_1$ we find

$$\begin{aligned}&\left[\int_C :u_1(w)\,v_1(w):\,w^k\,\frac{dw}{w},\int_{C_1}:u(w_0)\,v(-w_0):\,w_0^t\,\frac{dw_0}{w_0}\right]\\ &=\int_C(\{v_1,u\}_\pm :u_1(w_0)\,v(-w_0):\\ &\quad+(-1)^k\{v_1,v\}_\pm :u_1(w_0)\,u(-w_0):)\,w_0^{k+t}\,\frac{dw_0}{w_0};\end{aligned} \tag{3.55}$$

while for $u,v\in\mathbf{a}_2$, we find this bracket equals

$$\begin{aligned}&\mp\int_C(\{u_1,u\}_\pm :v_1(w_0)\,v(-w_0):\\ &\quad+(-1)^k\{u_1,v\}_\pm :u(w_0)\,v_1(-w_0):)\,w_0^{k+t}\,\frac{dw_0}{w_0}.\end{aligned} \tag{3.56}$$