In case u_1, u_2, v_1, v_2 are all from $\mathbf{a}_1$ or all from $\mathbf{a}_2$, then all contractions are zero so the brackets of two components from (3.22) or two from (3.23) are zero. If $u_1, v_1 \in \mathbf{a}_1$ and $u_2, v_2 \in \mathbf{a}_2$, then

$$\left[\int_C :u_1(w)\, v_1(-w):\, w^k \frac{dw}{w}, \int_{C_1} :u_2(w_0)\, v_2(-w_0):\, w_0^{\iota} \frac{dw_0}{w_0}\right]$$
$$= \int_C (\mp(-1)^{k+\iota}\{u_1, u_2\}_\pm :v_1(w_0)\, v_2(w_0):$$
$$+ \imath(-1)^k \{u_1, v_2\}_\pm :v_1(w_0)\, u_2(w_0):$$
$$+ \imath(-1)^{\iota} \{v_1, u_2\}_\pm :u_1(w_0)\, v_2(w_0):$$
$$\mp \{v_1, v_2\}_\pm :u_1(w_0)\, u_2(w_0):)\, w_0^{k+\iota} \frac{dw_0}{w_0}$$
$$+ k\delta_{k,-\iota}(\mp\{u_1, u_2\}_\pm \{v_1, v_2\}_\pm + \imath(-1)^k \{u_1, v_2\}_\pm \{v_1, u_2\}_\pm). \quad (3.57)$$

These calculations show that the components (all integral) of (3.21)–(3.23) with the identity operator form a closed Lie algebra acting on $V_\pm(Z')$.

To get the three extensions of this algebra we adjoin the components of (3.24)–(3.27) which include integral, semi-integral, or both kinds of components when $S = Z, Z'$, or $\frac{1}{2}\mathbb{Z}$, respectively.

Let us define

$$\zeta = \begin{cases} (w - w_0)/2 & \text{if } S = \mathbb{Z}, \\ (-ww_0)^{1/2} & \text{if } S = \mathbb{Z} + \frac{1}{2} \end{cases}$$

and

$$\zeta' = \begin{cases} (w + w_0)/2 & \text{if } S = \mathbb{Z}, \\ (ww_0)^{1/2} & \text{if } S = \mathbb{Z} + \frac{1}{2}. \end{cases}$$

Also, let $\operatorname{sgn}(\mathbb{Z}) = 1$ and $\operatorname{sgn}(\mathbb{Z} + \frac{1}{2}) = -1$. From Lemma 8(a) we have

LEMMA 9. (a) *For* $|w| > |w_0|$, *we have*

$$\underline{e_S(w)\, e_S(w_0)} = \underline{e_S(-w)\, e_S(-w_0)} = \mp 2 \frac{\zeta}{w + w_0},$$

$$\underline{e_S(-w)\, e_S(w_0)} = \mp 2 \frac{\zeta'}{w - w_0},$$

$$\underline{e_S(w)\, e_S(-w_0)} = \mp 2 \operatorname{sgn}(S) \frac{\zeta'}{w - w_0} \quad \textit{for} \quad S = \mathbb{Z}, \quad \mathbb{Z} + \tfrac{1}{2},$$

$$\underline{e_{1/2\mathbb{Z}}(w)\, e_{1/2\mathbb{Z}}(-w_0)} = \underline{e_{\mathbb{Z}}(-w)\, e_{\mathbb{Z}}(w_0)} - \underline{e_{\mathbb{Z}+1/2}(-w)\, e_{\mathbb{Z}+1/2}(w_0)}.$$

(b) $S = \mathbb{Z}$ *or* $\mathbb{Z} + \frac{1}{2}$, $u \in \mathbf{a}$, $|w| > |w_0|$, *we have*

$$\lim_{w \to -w_0} \zeta = -w_0, \qquad \lim_{w \to w_0} \zeta' = w_0,$$

$$\lim_{w \to -w_0} :u(-w)\, e_S(-w_0): = \imath :u(w_0)\, e_S(-w_0):,$$

$$\lim_{w \to -w_0} :e_S(-w)\, e_S(-w_0): = \operatorname{sgn}(S) :e_S(w_0)\, e_S(-w_0):,$$

$$\lim_{w \to -w_0} :e_S(w)\, e_S(w_0): = :e_S(-w_0)\, e_S(w_0):,$$

$$\lim_{w \to -w_0} :e_S(-w)\, e_{S'}(-w_0): = \operatorname{sgn}(S) :e_S(w_0)\, e_{S'}(-w_0):.$$

(c) *For* $S = \mathbb{Z}$ *or* $\mathbb{Z} + \frac{1}{2}$, $k \in \mathbb{Z}$, *we have*

$$:e_S(w)\, e_S(-w):_k = (-1)^{-k} \operatorname{sgn}(S) :e_S(-w)\, e_S(w):_k.$$

If $k \in \mathbb{Z} + \frac{1}{2}$, *we have*

$$:e_{\mathbb{Z}}(-w)\, e_{\mathbb{Z}'}(w):_k = \imath(-1)^k :e_{\mathbb{Z}}(w)\, e_{\mathbb{Z}'}(-w):_k,$$

while for $k \in \mathbb{Z}$ *this is zero. For* $S = \frac{1}{2}\mathbb{Z}$, *if* $k \in \mathbb{Z}$ *we have*

$$:e_S(-w)\, e_S(w):_k = :e_{\mathbb{Z}}(-w)\, e_{\mathbb{Z}}(w):_k + :e_{\mathbb{Z}'}(-w)\, e_{\mathbb{Z}'}(w):_k$$

and

$$:e_S(w)\, e_S(-w):_k = \imath(-1)^{-k}(:e_{\mathbb{Z}}(-w)\, e_{\mathbb{Z}}(w):_k - :e_{\mathbb{Z}'}(-w)\, e_{\mathbb{Z}'}(w):_k),$$

while if $k \in \mathbb{Z} + \frac{1}{2}$, *we have*

$$:e_S(-w)\, e_S(w):_k = (1 - \imath(-1)^k) :e_{\mathbb{Z}}(-w)\, e_{\mathbb{Z}'}(w):_k.$$

Let us begin with the case when $u_1, v \in \mathbf{a}_1$ and $u_2 \in \mathbf{a}_2$. We have

$$\left[\int_C :u_1(w)\, u_2(w):\, w^k \frac{dw}{w}, \int_{C_1} :v(w_0)\, e_S(w_0):\, w_0^{\iota} \frac{dw_0}{w_0}\right]$$
$$= \int_C \{u_2, v\}_\pm :u_1(w_0)\, e_S(w_0):\, w_0^{k+\iota} \frac{dw_0}{w_0}, \qquad (3.58)$$

while for $v \in \mathbf{a}_2$, we have

$$\left[\int_C :u_1(w)\, u_2(w):\, w^k \frac{dw}{w}, \int_{C_1} :v(-w_0)\, e_S(w_0):\, w_0^t \frac{dw_0}{w_0}\right]$$

$$= \int_C \mp\{u_1, v\}_\pm (-1)^k :u_2(-w_0)\, e_S(w_0):\, w_0^{k+t} \frac{dw_0}{w_0}. \tag{3.59}$$

It is also clear from Lemma 8 that the brackets of components from (3.21)–(3.23) with components from (3.26)–(3.27) are all zero. The same is true for components of (3.22) with (3.24), and (3.23) with (3.25). If $u, v \in \mathbf{a}_1$ and $u_2 \in \mathbf{a}_2$, then

$$\left[\int_C :u(w)\, v(-w):\, w^k \frac{dw}{w}, \int_{C_1} :u_2(-w_0)\, e_S(w_0):\, w_0^t \frac{dw_0}{w_0}\right]$$

$$= \int_C (\{v, u_2\}_\pm :u(w_0)\, e_S(w_0):$$

$$\mp \{u, u_2\}_\pm (-1)^k\, \iota :v(w_0)\, e_S(w_0):)\, w_0^{k+t} \frac{dw_0}{w_0}; \tag{3.60}$$

and if $u_1 \in \mathbf{a}_1$, $u, v \in \mathbf{a}_2$, then

$$\left[\int_C :u(w)\, v(-w):\, w^k \frac{dw}{w}, \int_{C_1} :u_1(w_0)\, e_S(w_0):\, w_0^t \frac{dw_0}{w_0}\right]$$

$$= \int_C (\mp\{u, u_1\}_\pm :v(-w_0)\, e_S(w_0):$$

$$+ \{v, u_1\}_\pm (-1)^k\, \iota :u(-w_0)\, e_S(w_0):)\, w_0^{k+t} \frac{dw_0}{w_0}. \tag{3.61}$$

That completes the brackets of components from (3.21)–(3.23) with those from (3.24)–(3.27).

Let $u, v \in \mathbf{a}$; then Corollary 6 says that as expressions

$:u(w)\, e_S(w)\, v(w_0)\, e_S(w_0):$

$$= \pm\, \mathrm{sgn}(S) :v(w_0)\, e_S(w_0)\, u(w)\, e_S(w): \quad \text{for} \quad S = \mathbb{Z} \quad \text{or} \quad S = \mathbb{Z} + \tfrac{1}{2}. \tag{3.62}$$

This determines whether we should commute (if $\pm\,\mathrm{sgn}(S) = 1$) or anticommute (if $\pm\,\mathrm{sgn}(S) = -1$) the components of $:u(w)\, e_S(w):$ and $:v(w_0)\, e_S(w_0):$. For $S = \frac{1}{2}\mathbb{Z}$, this depends on the components, that is, on k and t. But in all cases, we have

$$:u(w)\, e_Z(w)\, v(w_0)\, e_{Z'}(w_0): = :v(w_0)\, e_{Z'}(w_0)\, u(w)\, e_Z(w):. \tag{3.63}$$

So between components of $:u(w)\, e_Z(w):$ and $:v(w_0)\, e_{Z'}(w_0):$, we always use the Lie bracket.

For $u, v \in \mathbf{a}_1$ and $S = \mathbb{Z}$ or $\mathbb{Z} + \frac{1}{2}$, we have

$$\left[\int_C :u(w)\, e_S(w):\, w^k \frac{dw}{w}, \int_{C_1} :v(w_0)\, e_S(w_0):\, w_0^t \frac{dw_0}{w_0}\right]_{\mp \mathrm{sgn}(S)}$$

$$= \int_C \mp^S (\mp 2)(-1)^k :u(-w_0)\, v(w_0):\, w_0^{k+t} \frac{dw_0}{w_0}, \tag{3.64}$$

where

$$\mp^S = \begin{cases} \mp 1 & \text{if} \quad S = \mathbb{Z}, \\ +1 & \text{if} \quad S = \mathbb{Z} + \tfrac{1}{2}. \end{cases}$$

For $u, v \in \mathbf{a}_2$ and $S = \mathbb{Z}$ or $\mathbb{Z} + \frac{1}{2}$, we have

$$\left[\int_C :u(-w)\, e_S(w):\, w^k \frac{dw}{w}, \int_{C_1} :v(-w_0)\, e_S(w_0):\, w_0^t \frac{dw_0}{w_0}\right]_{\mp \mathrm{sgn}(S)}$$

$$= \int_C \mp^S (\mp 2)(-1)^k\, \iota :u(w_0)\, v(-w_0):\, w_0^{k+t} \frac{dw_0}{w_0}. \tag{3.65}$$

If $u \in \mathbf{a}_1$, $v \in \mathbf{a}_2$, $S = \mathbb{Z}$ or $\mathbb{Z} + \frac{1}{2}$, then

$$\left[\int_C :u(w)\, e_S(w):\, w^k \frac{dw}{w}, \int_{C_1} :v(-w_0)\, e_S(w_0):\, w_0^t \frac{dw_0}{w_0}\right]_{\mp \mathrm{sgn}(S)}$$

$$= \int_C (\mp^S \{u, v\}_\pm (-1)^k :e_S(-w_0)\, e_S(w_0):$$

$$\mp^S (\mp 2)(-1)^k :u(-w_0)\, v(-w_0):)\, w_0^{k+t} \frac{dw_0}{w_0}$$

$$\mp^S (\mp 2) \{u, v\}_\pm\, k(-1)^k\, \delta_{k,-t}. \tag{3.66}$$

If $u_1 \in \mathbf{a}_1$ and $S = \mathbb{Z}$ or $\mathbb{Z} + \frac{1}{2}$, we have the Lie bracket

$$\left[\int_C :u_1(w)\, e_S(w):\, w^k \frac{dw}{w}, \int_{C_1} :e_S(-w_0)\, e_S(w_0):\, w_0^t \frac{dw_0}{w_0}\right]$$

$$= \mp 2\, \mathrm{sgn}(S)(1 - (-1)^{-t}) \int_C :u_1(w_0)\, e_S(w_0):\, w_0^{k+t} \frac{dw_0}{w_0}; \tag{3.67}$$

while for $u_2 \in \mathbf{a}_2$, we can see that

$$\left[\int_C :u_2(-w)\, e_S(w):\, w^k \frac{dw}{w}, \int_{C_1} :e_S(-w_0)\, e_S(w_0):\, w_0^t \frac{dw_0}{w_0}\right]$$

$$= \mp 2\, \mathrm{sgn}(S)(1 - (-1)^{-t}) \int_C :u_2(-w_0)\, e_S(w_0):\, w_0^{k+t} \frac{dw_0}{w_0}. \tag{3.68}$$

We must also compute for $S=\mathbb{Z}$ or $\mathbb{Z}+\frac{1}{2}$ the Lie bracket

$$\left[\int_C :e_S(-w)\,e_S(w):\,w^k\,\frac{dw}{w},\int_{C_1} :e_S(-w_0)\,e_S(w_0):\,w_0^t\,\frac{dw_0}{w_0}\right]$$
$$=\mp 2\,\mathrm{sgn}(S)(1-(-1)^k(1+(-1)^{-k-t})$$
$$\times\int_C :e_S(-w_0)\,e_S(w_0):\,w_0^{k+t}\,\frac{dw_0}{w_0}$$
$$+4\,\mathrm{sgn}(S)\,k(1-(-1)^k)\,\delta_{k,-t}. \tag{3.69}$$

The above calculations show that the extensions defined by (3.24)–(3.27) for $S=\mathbb{Z}$ and for $S=\mathbb{Z}+\frac{1}{2}$ give closed algebras. The case of $S=\frac{1}{2}\mathbb{Z}$ requires more work. Since $e_S(w)=e_{\mathbb{Z}}(w)+e_{\mathbb{Z}+1/2}(w)$, the calculations of brackets of components from (3.21)–(3.23) with those from (3.24)–(3.27) for $S=\mathbb{Z}$ and $S=\mathbb{Z}+\frac{1}{2}$ combine to give the same results for $S=\frac{1}{2}\mathbb{Z}$. In (3.64)–(3.66), if $k,t\in\mathbb{Z}$ or $k,t\in\mathbb{Z}+\frac{1}{2}$, the calculation is the same as for $S=Z$ or $S=Z'$, respectively. But if $k\in\mathbb{Z}$ and $t\in\mathbb{Z}+\frac{1}{2}$, the Lie bracket is used and gives zero for (3.64) and (3.65), because all contractions are zero.

Let $u\in\mathbf{a}_1$, $v\in\mathbf{a}_2$, $k\in\mathbb{Z}$, $t\in\mathbb{Z}+\frac{1}{2}$, then instead of (3.66) we have

$$\left[\int_C :u(w)\,e_Z(w):\,w^k\,\frac{dw}{w},\int_{C_1} :v(-w_0)\,e_{Z'}(w_0):\,w_0^t\,\frac{dw_0}{w_0}\right]$$
$$=\int_C \mp^{Z}\{u,v\}_\pm(-1)^k :e_Z(-w_0)\,e_{Z'}(w_0):\,w_0^{k+t}\,\frac{dw_0}{w_0}. \tag{3.70}$$

From Lemma 9(c) this is a linear combination of components from (3.26) and (3.27). If $k\in\mathbb{Z}+\frac{1}{2}$ and $t\in\mathbb{Z}$, we would have instead

$$\left[\int_C :u(w)\,e_{Z'}(w):\,w^k\,\frac{dw}{w},\int_{C_1} :v(-w_0)\,e_Z(w_0):\,w_0^t\,\frac{dw_0}{w_0}\right]$$
$$=\int_C \mp^{Z'}\{u,v\}_\pm(-1)^{-t}\,\iota{:}e_Z(-w_0)\,e_{Z'}(w_0):\,w_0^{k+t}\,\frac{dw_0}{w_0}. \tag{3.71}$$

Concerning the calculation (3.67), if $t\in\mathbb{Z}$, then we still use the Lie bracket whether $k\in\mathbb{Z}$ or $k\in\mathbb{Z}+\frac{1}{2}$. But the case when $k,t\in\mathbb{Z}+\frac{1}{2}$ reduces to

$$\left[\int_C :u_1(w)\,e_{Z'}(w):\,w^k\,\frac{dw}{w},\int_{C_1} :e_Z(-w_0)\,e_{Z'}(w_0):\,w_0^t\,\frac{dw_0}{w_0}\right]_{\pm\iota}$$
$$=\int_C \mp 2(-1)^{-t} :u_1(w_0)\,e_Z(w_0):w_0^{k+t}\,\frac{dw_0}{w_0}, \tag{3.72}$$

because as expressions, $:u_1(w)\,e_{Z'}(w)\,e_Z(-w_0)\,e_{Z'}(w_0):$ equals $\mp\iota{:}e_Z(-w_0)\,e_{Z'}(w_0)\,u_1(w)\,e_{Z'}(w):$. If $k\in\mathbb{Z}$ and $t\in\mathbb{Z}+\frac{1}{2}$, then (3.67) reduces to

$$\left[\int_C :u_1(w)\,e_Z(w):\,w^k\,\frac{dw}{w},\int_{C_1} :e_Z(-w_0)\,e_{Z'}(w_0):w_0^t\,\frac{dw_0}{w_0}\right]_{\mp\iota}$$
$$=\int_C \mp 2\iota{:}u_1(w_0)\,e_{Z'}(w_0):\,w_0^{k+t}\,\frac{dw_0}{w_0}, \tag{3.73}$$

because as expressions, $:u_1(w)\,e_Z(w)\,e_Z(-w_0)\,e_{Z'}(w_0):$ equals $\pm\iota{:}e_Z(-w_0)\,e_{Z'}(w_0)\,u_1(w)\,e_Z(w):$.

The case of calculation (3.68) is quite similar to that of (3.67). If $t\in\mathbb{Z}$, the Lie bracket is used whether $k\in\mathbb{Z}$ or $k\in\mathbb{Z}+\frac{1}{2}$, but the case when $k,t\in\mathbb{Z}+\frac{1}{2}$ reduces to

$$\left[\int_C :u_2(-w)\,e_{Z'}(w):\,w^k\,\frac{dw}{w},\int_{C_1} :e_Z(-w_0)\,e_{Z'}(w_0):\,w_0^t\,\frac{dw_0}{w_0}\right]_{\pm\iota}$$
$$=\int_C \mp 2(-1)^{-t} :u_2(-w_0)\,e_Z(w_0):\,w_0^{k+t}\,\frac{dw_0}{w_0}, \tag{3.74}$$

and the case when $k\in\mathbb{Z}$, $t\in\mathbb{Z}+\frac{1}{2}$ reduces to

$$\left[\int_C :u_2(-w)\,e_Z(w):\,w^k\,\frac{dw}{w},\int_{C_1} :e_Z(-w_0)\,e_{Z'}(w_0):\,w_0^t\,\frac{dw_0}{w_0}\right]_{\mp\iota}$$
$$=\int_C \mp 2\iota{:}u_2(-w_0)\,e_{Z'}(w_0):\,w_0^{k+t}\,\frac{dw_0}{w_0}. \tag{3.75}$$

The last calculation to be considered is (3.69). Lemma 9(c) shows that when $S=\frac{1}{2}\mathbb{Z}$ we are dealing with the components of $:e_Z(-w)\,e_Z(w):$, $:e_{Z'}(-w)\,e_{Z'}(w):$, and $:e_Z(-w)\,e_{Z'}(w):$. The components of these first two are only integral, while the last are only semi-integral. The Lie brackets of the first two types with themselves have already been done in (3.69), and

$$\left[\int_C :e_Z(-w)\,e_Z(w):\,w^k\,\frac{dw}{w},\int_{C_1} :e_{Z'}(-w_0)\,e_{Z'}(w_0):\,w_0^t\,\frac{dw_0}{w_0}\right]=0, \tag{3.76}$$

because all contractions are zero. For $k\in\mathbb{Z}$, $t\in\mathbb{Z}+\frac{1}{2}$ we have Lie brackets

$$\left[\int_C :e_Z(-w)\,e_Z(w):\,w^k\,\frac{dw}{w},\int_{C_1} :e_Z(-w_0)\,e_{Z'}(w_0):\,w_0^t\,\frac{dw_0}{w_0}\right]$$
$$=\mp 2\iota(1-(-1)^k)\int_C :e_Z(-w_0)\,e_{Z'}(w_0):\,w_0^{k+t}\,\frac{dw_0}{w_0} \tag{3.77}$$

and

$$\left[\int_C :e_{Z'}(-w)\, e_{Z'}(w)\colon w^k \frac{dw}{w}, \int_{C_1} :e_Z(-w_0)\, e_{Z'}(w_0)\colon w_0^t \frac{dw_0}{w_0}\right]$$

$$= \mp 2\iota(1-(-1)^k)\int_C :e_Z(-w_0)\, e_{Z'}(w_0)\colon w_0^{k+t}\frac{dw_0}{w_0}. \tag{3.78}$$

If $k, t \in \mathbb{Z}+\frac{1}{2}$ we have the anticommutator

$$\left[\int_C :e_Z(-w)\, e_{Z'}(w)\colon w^k \frac{dw}{w}, \int_{C_1} :e_Z(-w_0)\, e_{Z'}(w_0)\colon w_0^t \frac{dw_0}{w_0}\right]_+$$

$$= \mp 2\int_C ((-1)^{-t} :e_Z(-w_0)\, e_Z(w_0)\colon$$

$$+ (-1)^k :e_{Z'}(-w_0)\, e_{Z'}(w_0)\colon)\, w_0^{k+t}\frac{dw_0}{w_0} + 4k(-1)^k \delta_{k,-t}. \tag{3.79}$$

We have established our first main theorem.

THEOREM A. *Let $Z=\mathbb{Z}$ or $\mathbb{Z}+\frac{1}{2}$, and let $A_\pm(Z^{2l})$ be the associative algebra with generators (3.2) and relations (3.3), and let $V_\pm(Z^l)$ be the representation space of $A_\pm(Z^{2l})$ generated by a vacuum vector v_0 (see paragraph containing (3.14)).*

(a) *For $S=Z$, Z' or $\frac{1}{2}\mathbb{Z}$ we have the extension $A^S_\pm(Z^l)$ of $A_\pm(Z^l)$ obtained by adjoining the Clifford algebra generators (3.5), and we have the extension $V^S_\pm(Z^l)$ of $V_\pm(Z^l)$ obtained by tensoring with Clifford module $V(S)$.*

Then the identity operator and the homogeneous components of generating functions $:u(w)\, v(w):$, $u, v \in \mathbf{a}$, represent a closed Lie algebra of operators on $V_\pm(Z^l)$. Each of the three extensions obtained by also including the components of $:u(w)\, e_S(w):$, and $:e_Z(w)\, e_{Z'}(w):$ for $S=\frac{1}{2}\mathbb{Z}$, represents a closed algebra of operators on $V^S_\pm(Z^l)$.

(b) *For $S=Z$, Z', or $\frac{1}{2}\mathbb{Z}$ we have the extension $A^S_\pm(Z^l)$ of $A_\pm(Z^l)$ obtained by adjoining the Clifford, Weyl, or Clifford–Weyl algebra (3.52), and we have the extension $V^S_\pm(Z^l)$ of $V_\pm(Z^l)$ obtained by tensoring with appropriate module $V(S)$.*

Then the identity operator and the homogeneous components of generating functions $:u_1(w)\, v_2(w):$, $:u_1(w)\, v_1(-w):$, $:u_2(w)\, v_2(-w):$, $u_1, v_1 \in \mathbf{a}_1$, $u_2, v_2 \in \mathbf{a}_2$, represent a closed Lie algebra of operators on $V_\pm(Z^l)$. Each of the three extensions obtained by also including the components of $:u_1(w)\, e_S(w):$, $:u_2(-w)\, e_S(w):$, $:e_S(-w)\, e_S(w):$, $:e_S(w)\, e_S(-w):$ represents a closed algebra of operators on $V^S_\pm(Z^l)$.

4. Identification of Affine Algebras

In this section we will review the root systems of the classical affine Kac–Moody algebras [6] and use the calculations of Section 3 to identify the algebras we have constructed.

The classical affine algebras will be organized into three series, the orthogonal series $D_l^{(1)}$, $B_l^{(1)}$, $D_{l+1}^{(2)}$, the symplectic series $C_l^{(1)}$, $B^{(1)}(0,l)$, $C^{(2)}(l+1)$, and the general linear series $A_{2l-1}^{(2)}$, $A_{2l}^{(2)}$, $A^{(2)}(0,2l-1)$, $A^{(4)}(0,2l)$. There are for each of these algebras two natural gradations according to which endpoint of the Dynkin diagram corresponds to the affine simple root α_0. This choice determines the finite-dimensional algebra $\mathbf{g}$ whose Dynkin diagram is that which remains when the affine point is removed. One way to obtain this grading and the corresponding root-space decomposition is to adjoin a derivation d to the affine algebra. This derivation acts as zero on the finite-dimensional algebra $\mathbf{g}$, and acts as 1 or $\frac{1}{2}$ on the α_0 root-space determined by convenience.

TABLE VII

The Orthogonal Series

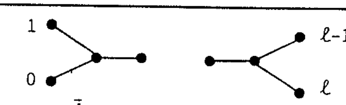
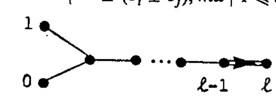
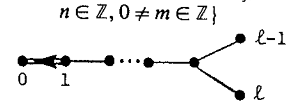
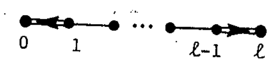
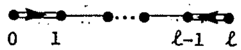
$\mathbf{o}^{(1)}(2l)$ $D_l^{(1)}$	 $\alpha_0=\bar{d}-\varepsilon_1-\varepsilon_2, \alpha_1=\varepsilon_1-\varepsilon_2,\ldots,\alpha_{l-1}=\varepsilon_{l-1}-\varepsilon_l, \alpha_l=\varepsilon_{l-1}+\varepsilon_l$ $\Delta=\{n\bar{d}\pm(\varepsilon_i\pm\varepsilon_j), m\bar{d} \mid 1\leqslant i,j\leqslant l, i<j, n\in\mathbb{Z}, 0\neq m\in\mathbb{Z}\}$
$\mathbf{o}^{(1)}(2l+1)$ $B_l^{(1)}$	 $\alpha_0=\bar{d}-\varepsilon_1-\varepsilon_2, \alpha_1=\varepsilon_1-\varepsilon_2,\ldots,\alpha_{l-1}=\varepsilon_{l-1}-\varepsilon_l, \alpha_l=\varepsilon_l$ $\Delta=\{n\bar{d}\pm\varepsilon_i, n\bar{d}\pm(\varepsilon_i\pm\varepsilon_j), m\bar{d} \mid 1\leqslant i,j\leqslant l, i<j, n\in\mathbb{Z}, 0\neq m\in\mathbb{Z}\}$
$\mathbf{o}^{(2)}(2l+1)$ $B_l^{(2)}\approx B_l^{(1)}$	 $\alpha_0=\frac{1}{2}\bar{d}-\varepsilon_1, \alpha_1=\varepsilon_1-\varepsilon_2,\ldots,\alpha_{l-1}=\varepsilon_{l-1}-\varepsilon_l, \alpha_l=\varepsilon_{l-1}+\varepsilon_l$ $\Delta=\{n\bar{d}\pm(\varepsilon_i\pm\varepsilon_j), (n+\frac{1}{2})\bar{d}\pm\varepsilon_i, m\bar{d} \mid 1\leqslant i,j\leqslant l, i<j, n\in\mathbb{Z}, 0\neq m\in\mathbb{Z}\}$
$\mathbf{o}^{(2)}(2l+2)$ $D_{l+1}^{(2)}$	 $\alpha_0=\frac{1}{2}\bar{d}-\varepsilon_1, \alpha_1=\varepsilon_1-\varepsilon_2,\ldots,\alpha_{l-1}=\varepsilon_{l-1}-\varepsilon_l, \alpha_l=\varepsilon_l$ $\Delta=\{n\bar{d}\pm(\varepsilon_i\pm\varepsilon_j), \frac{1}{2}n\bar{d}\pm\varepsilon_i, \frac{1}{2}m\bar{d} \mid 1\leqslant i,j\leqslant l, i<j, n\in\mathbb{Z}, 0\neq m\in\mathbb{Z}\}$

TABLE VIII

The Symplectic Series

$\mathbf{sp}^{(1)}(2l)$



$C_l^{(1)}$ $\quad \alpha_0 = \bar{d} - 2\varepsilon_1, \alpha_1 = \varepsilon_1 - \varepsilon_2, \ldots, \alpha_{l-1} = \varepsilon_{l-1} - \varepsilon_l, \alpha_l = 2\varepsilon_l$
$\Delta = \{n\bar{d} \pm (\varepsilon_i \pm \varepsilon_j), n\bar{d} \pm 2\varepsilon_i, m\bar{d} \mid 1 \leqslant i,j \leqslant l, i<j, n \in \mathbb{Z}, 0 \neq m \in \mathbb{Z}\}$

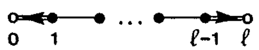
$\mathbf{osp}^{(1)}(1, 2l)$

$B^{(1)}(0,l)$ $\quad \alpha_0 = \bar{d} - 2\varepsilon_1, \alpha_1 = \varepsilon_1 - \varepsilon_2, \ldots, \alpha_{l-1} = \varepsilon_{l-1} - \varepsilon_l, \alpha_l = \varepsilon_l$
$\Delta = \{n\bar{d} \pm \varepsilon_i, n\bar{d} \pm 2\varepsilon_i, n\bar{d} \pm (\varepsilon_i \pm \varepsilon_j), m\bar{d} \mid 1 \leqslant i,j \leqslant l, i<j, n \in \mathbb{Z}, 0 \neq m \in \mathbb{Z}\}$

$\mathbf{osp}^{(2)}(1, 2l)$

$B^{(2)}(0,l) \approx B^{(1)}(0,l)$ $\quad \alpha_0 = \frac{1}{2}\bar{d} - \varepsilon_1, \alpha_1 = \varepsilon_1 - \varepsilon_2, \ldots, \alpha_{l-1} = \varepsilon_{l-1} - \varepsilon_l, \alpha_l = 2\varepsilon_l$
$\Delta = \{n\bar{d} \pm (\varepsilon_i \pm \varepsilon_j), n\bar{d} \pm 2\varepsilon_i, (n+\frac{1}{2})\bar{d} \pm \varepsilon_i, m\bar{d} \mid 1 \leqslant i,j \leqslant l, i<j, n \in \mathbb{Z}, 0 \neq m \in \mathbb{Z}\}$

$\mathbf{osp}^{(2)}(2, 2l)$



$C^{(2)}(l+1)$ $\quad \alpha_0 = \frac{1}{2}\bar{d} - \varepsilon_1, \alpha_1 = \varepsilon_1 - \varepsilon_2, \ldots, \alpha_{l-1} = \varepsilon_{l-1} - \varepsilon_l, \alpha_l = \varepsilon_l$
$\Delta = \{n\bar{d} \pm (\varepsilon_i \pm \varepsilon_j), n\bar{d} \pm 2\varepsilon_i, \frac{1}{2}n\bar{d} \pm \varepsilon_i, \frac{1}{2}m\bar{d} \mid 1 \leqslant i,j \leqslant l, i<j, n \in \mathbb{Z}, 0 \neq m \in \mathbb{Z}\}$

Let $\{h_1, \ldots, h_l\}$ be an orthonormal basis of the Cartan subalgebra $\mathbf{h}$ of $\mathbf{g}$ with respect to the invariant form as chosen in Section 2. The dual basis of $\mathbf{h}^*$ is denoted $\{\varepsilon_1, \ldots, \varepsilon_l\}$. The Cartan subalgebra $\hat{\mathbf{h}}$ of the extended affine algebra $\hat{\mathbf{g}}$ will have basis $\{h_1, \ldots, h_l, c, d\}$ such that

$$0 = \langle h_i, c\rangle = \langle h_i, d\rangle = \langle c, c\rangle = \langle d, d\rangle, \qquad 1 = \langle c, d\rangle \tag{4.1}$$

with respect to the extended invariant form. We denote the dual basis of $\hat{\mathbf{h}}^*$ by

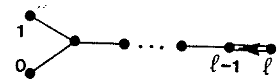
$$\{\varepsilon_1, \ldots, \varepsilon_l, \bar{c}, \bar{d}\}. \tag{4.2}$$

In Tables VII–IX, we give the simple roots, root system, and Dynkin diagram for the distinguishable gradations of the three series of classical affine algebras. An open circle in the Dynkin diagram indicates a super simple root. To assist in visualizing these root systems, we also give in Figs. 1–3 corresponding schematic root diagrams in which the super roots are indicated by an open circle.

The dimension of each root-space is one, except for the roots $m\bar{d}$, $0 \neq m \in \mathbb{Z}$, whose multiplicity is l. Note that the special linear series differs

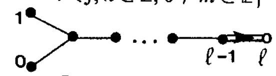
TABLE IX. The General Linear Series

(a) $\mathbf{gl}^{(2)}(2l)$



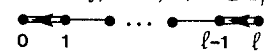
$A_{2l-1}^{(2)}$ $\quad \alpha_0 = \bar{d} - \varepsilon_1 - \varepsilon_2, \alpha_1 = \varepsilon_1 - \varepsilon_2, \ldots, \alpha_{l-1} = \varepsilon_{l-1} - \varepsilon_l, \alpha_l = 2\varepsilon_l$
$\Delta = \{n\bar{d} \pm (\varepsilon_i \pm \varepsilon_j), 2n\bar{d} \pm 2\varepsilon_i, m\bar{d} \mid 1 \leqslant i,j \leqslant l, i<j, n \in \mathbb{Z}, 0 \neq m \in \mathbb{Z}\}$

$\mathbf{gl}^{(2)}(1, 2l)$



$A^{(2)}(0, 2l-1)$ $\quad \alpha_0 = \bar{d} - \varepsilon_1 - \varepsilon_2, \alpha_1 = \varepsilon_1 - \varepsilon_2, \ldots, \alpha_{l-1} = \varepsilon_{l-1} - \varepsilon_l, \alpha_l = \varepsilon_l$
$\Delta = \{n\bar{d} \pm \varepsilon_i, n\bar{d} \pm (\varepsilon_i \pm \varepsilon_j), 2n\bar{d} \pm 2\varepsilon_i, m\bar{d} \mid 1 \leqslant i,j \leqslant l, i<j, n \in \mathbb{Z}, 0 \neq m \in \mathbb{Z}\}$

$\mathbf{gl}^{(4)}(2l+1)$

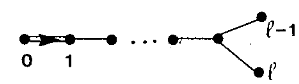


$A_{2l}^{(4)}$ $\quad \alpha_0 = \frac{1}{2}\bar{d} - \varepsilon_1, \alpha_1 = \varepsilon_1 - \varepsilon_2, \ldots, \alpha_{l-1} = \varepsilon_{l-1} - \varepsilon_l, \alpha_l = 2\varepsilon_l$
$\Delta = \{n\bar{d} \pm (\varepsilon_i \pm \varepsilon_j), 2n\bar{d} \pm 2\varepsilon_i, (n+\frac{1}{2})\bar{d} \pm \varepsilon_i, m\bar{d} \mid 1 \leqslant i,j \leqslant l, i<j, n \in \mathbb{Z}, 0 \neq m \in \mathbb{Z}\}$

$\mathbf{gl}^{(4)}(1, 2l+1)$

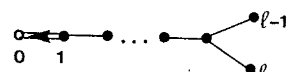
$A^{(4)}(0, 2l)$ $\quad \alpha_0 = \frac{1}{2}\bar{d} - \varepsilon_1, \alpha_1 = \varepsilon_1 - \varepsilon_2, \ldots, \alpha_{l-1} = \varepsilon_{l-1} - \varepsilon_l, \alpha_l = \varepsilon_l$
$\Delta = \{n\bar{d} \pm (\varepsilon_i \pm \varepsilon_j), 2n\bar{d} \pm 2\varepsilon_i, \frac{1}{2}n\bar{d} \pm \varepsilon_i, \frac{1}{2}m\bar{d} \mid 1 \leqslant i,j \leqslant l, i<j, n \in \mathbb{Z}, 0 \neq m \in \mathbb{Z}\}$

(b) $\mathbf{gl}^{(2)}(2l)$



$A_{2l-1}^{(2)}$ $\quad \alpha_1 = \bar{d} - 2\varepsilon_1, \alpha_1 = \varepsilon_1 - \varepsilon_2, \ldots, \alpha_{l-1} = \varepsilon_{l-1} - \varepsilon_l, \alpha_l = \varepsilon_{l-1} + \varepsilon_l$
$\Delta = \{n\bar{d} \pm (\varepsilon_i \pm \varepsilon_j), (2n+1)\bar{d} \pm 2\varepsilon_i, m\bar{d} \mid 1 \leqslant i,j \leqslant l, i<j, n \in \mathbb{Z}, 0 \neq m \in \mathbb{Z}\}$

$\mathbf{gl}^{(4)}(1, 2l)$



$A^{(4)}(0, 2l-1)$ $\quad \alpha_0 = \frac{1}{2}\bar{d} - \varepsilon_1, \alpha_1 = \varepsilon_1 - \varepsilon_2, \ldots, \alpha_{l-1} = \varepsilon_{l-1} - \varepsilon_l, \alpha_l = \varepsilon_{l-1} + \varepsilon_l$
$\Delta = \{n\bar{d} \pm (\varepsilon_i \pm \varepsilon_j), (2n+1)\bar{d} \pm 2\varepsilon_i, (n+\frac{1}{2})\bar{d} + \varepsilon_i, m\bar{d} \mid 1 \leqslant i,j \leqslant l, i<j, n \in \mathbb{Z}, 0 \neq m \in \mathbb{Z}\}$

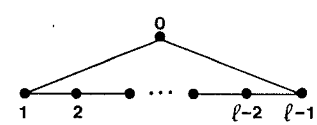
$\mathbf{gl}^{(2)}(2l+1)$

$A_{2l}^{(2)}$ $\quad \alpha_0 = \bar{d} - 2\varepsilon_1, \alpha_1 = \varepsilon_1 - \varepsilon_2, \ldots, \alpha_{l-1} = \varepsilon_{l-1} - \varepsilon_l, \alpha_l = \varepsilon_l$
$\Delta = \{n\bar{d} \pm \varepsilon_i, n\bar{d} \pm (\varepsilon_i \pm \varepsilon_j), (2n+1)\bar{d} \pm 2\varepsilon_i, m\bar{d} \mid 1 \leqslant i,j \leqslant l, i<j, n \in \mathbb{Z}, 0 \neq m \in \mathbb{Z}\}$

$\mathbf{gl}^{(4)}(1, 2l+1)$

$A^{(4)}(0, 2l)$ $\quad \alpha_0 = \frac{1}{2}\bar{d} - \varepsilon_1; \alpha_1 = \varepsilon_1 - \varepsilon_2, \ldots, \alpha_{l-1} = \varepsilon_{l-1} - \varepsilon_l, \alpha_l = \varepsilon_l$
$\Delta = \{\frac{1}{2}n\bar{d} \pm \varepsilon_i, n\bar{d} \pm (\varepsilon_i \pm \varepsilon_j), (2n+1)\bar{d} \pm 2\varepsilon_i, \frac{1}{2}m\bar{d} \mid 1 \leqslant i,j \leqslant l, i<j, n \in \mathbb{Z}, 0 \neq m \in \mathbb{Z}\}$

(c) $\mathbf{gl}^{(1)}(l)$



$A_{l-1}^{(1)}$ $\quad \alpha_0 = \bar{d} - \varepsilon_1 + \varepsilon_l, \alpha_1 = \varepsilon_1 - \varepsilon_2, \ldots, \alpha_{l-1} = \varepsilon_{l-1} - \varepsilon_l$
$\Delta = \{n\bar{d} \pm (\varepsilon_i - \varepsilon_j), m\bar{d} \mid 1 \leqslant i,j \leqslant l, i<j, n \in \mathbb{Z}, 0 \neq m \in \mathbb{Z}\}$

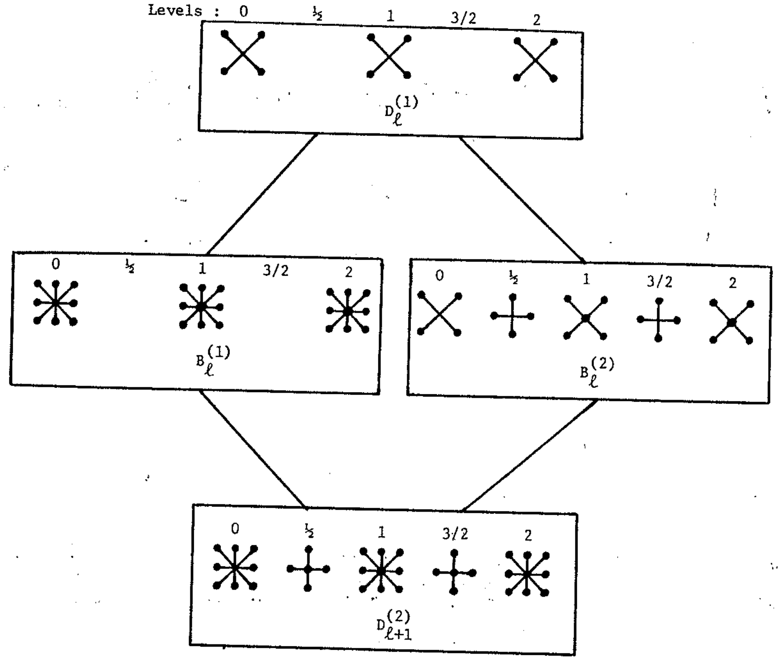


FIG. 1. The orthogonal series.

from our general linear series by having multiplicity $l-1$ in each $(2n+1)\bar{d}$ root-space, $n \in \mathbb{Z}$. Representations of the special linear series can be obtained from our general linear representations by taking the space of vacuum (highest weight) vectors for the infinite-dimensional Heisenberg algebra coming from the extra dimension in each $(2n+1)\bar{d}$ root-space.

Recall the bases which were chosen for the finite-dimensional algebras and the notation given at the end of Section 2 in formulas (2.5)–(2.10). The first type of construction we gave in Section 3 was based on the generating functions $:u(w)\,v(w):$, where $:uv: \in \mathbf{g}$ with $\mathbf{g} = \mathbf{o}(2l)$ in the fermionic case and $\mathbf{g} = \mathbf{sp}(2l)$ in the bosonic case. We then identify $h_i = \pm :a_i a_i^*:$ in the Cartan subalgebra $\mathbf{h}$ of $\mathbf{g}$ with

$$h_i(0) = \pm \int_C :a_i(w)\,a_i^*(w): \frac{dw}{w} \tag{4.3}$$

in the Cartan subalgebra $\hat{\mathbf{h}}$ of the affine algebra $\hat{\mathbf{g}}$. Then (3.34) shows that we have constructed a representation of $\mathbf{o}^{(1)}(2l)$ in which central element c acts as $+1$ in the fermionic case, and a representation of $\mathbf{sp}^{(1)}(2l)$ in which c acts as -1 in the bosonic case.

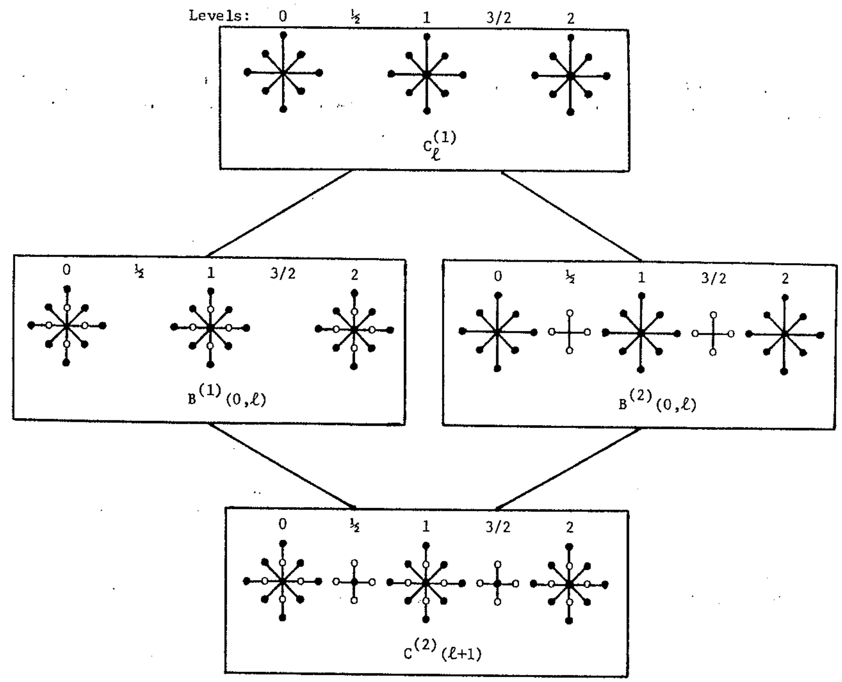


FIG. 2. The symplectic series.

The next construction we gave extended these algebras by allowing $:uv: \in \mathbf{g}$, where $\mathbf{g} = \mathbf{o}(2l+1)$ in the fermionic case and $\mathbf{g} = B(0,l)$ in the bosonic case. For the new short root vectors $:ue:$ the generating functions $:u(w)\,e_Z(w):$ gave only integral components. Formulas (3.39) and (3.40) show that the resulting algebra is of type $B_l^{(1)}$ in the fermionic case and $B^{(1)}(0,l)$ in the bosonic case. If, instead, we use the functions $:u(w)\,e_{Z'}(w):$ so that short roots are only on the semi-integral levels, then (3.43) and (3.44) show that we obtained $B_l^{(2)}$ and $B^{(2)}(0,l)$. Finally, the largest extension of the series obtained by using $:u(w)\,e_{1/2Z}(w):$ and $:e_Z(w)\,e_{Z'}(w):$ has the root system of $D_{l+1}^{(2)}$ in the fermionic case and that of $C^{(2)}(l+1)$ in the bosonic case. We have obtained purely fermionic constructions of the orthogonal series and purely bosonic constructions of the symplectic series. Note that since $\mathbf{gl}^{(1)}(l)$ is contained in $\mathbf{o}^{(1)}(2l)$, and $\mathbf{sp}^{(1)}(2l)$ we have obtained both fermionic and bosonic constructions for type $A_{l-1}^{(1)}$. We will show in Section 5 that the restriction of these representations to $\mathbf{gl}^{(1)}(l)$ has a simple decomposition into irreducible components.

The second type of construction we gave in Section 3 was based on the generating functions $:u_1(w)\,u_2(w):$, $:u_1(w)\,v_1(-w):$, $:u_2(w)\,v_2(-w):$ for