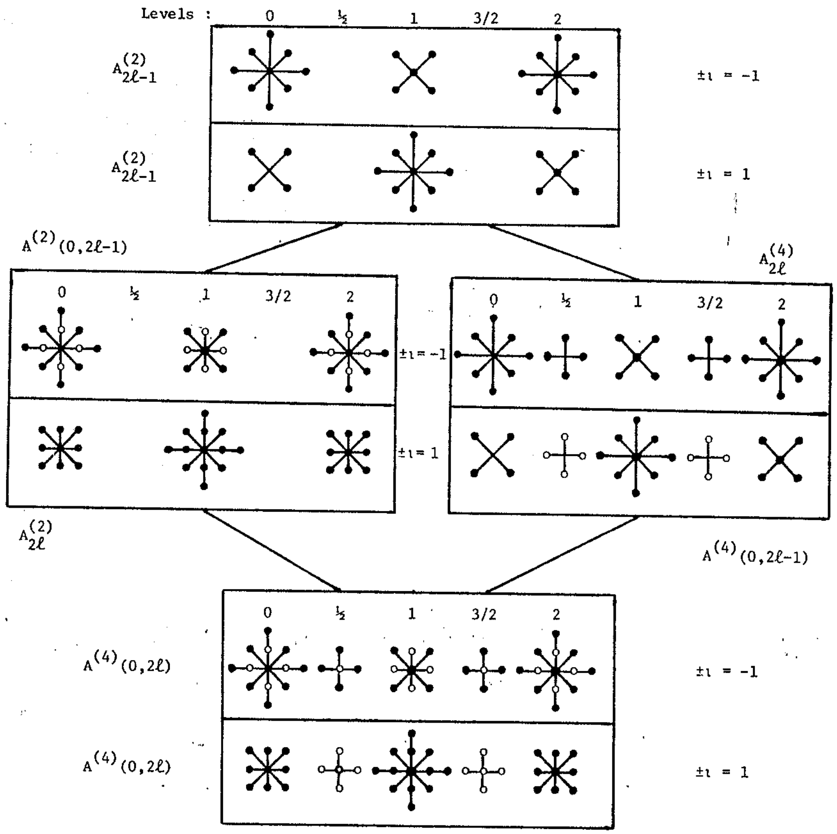

FIG. 3. The general linear series.

$u_1, v_1 \in \mathbf{a}_1$, $u_2, v_2 \in \mathbf{a}_2$. From (3.54) we see that the components of functions $:a_i(w)\, a_j^*(w):$ for $1 \leqslant i,j \leqslant l$ and ± 1 represent $\mathbf{gl}^{(1)}(l)$. Using (3.55) and (3.56) and we find the roots corresponding to the components of $:a_j(w)\, a_k(-w):$ and $:a_j^*(w)\, a_k^*(-w):$. We get

$$\left[h_l(0), \int_C :a_j(w_0)\, a_k(-w_0):\, w_0^t \frac{dw_0}{w_0}\right] = (\varepsilon_j + \varepsilon_k)(h_l) \int_C :a_j(w_0)\, a_k(-w_0):\, w_0^t \frac{dw_0}{w_0} \tag{4.4}$$

and

$$\left[h_l(0), \int_C :a_j^*(w_0)\, a_k^*(-w_0):\, w_0^t \frac{dw_0}{w_0}\right] = -(\varepsilon_j + \varepsilon_k)(h_l) \int_C :a_j^*(w_0)\, a_k^*(-w_0):\, w_0^t \frac{dw_0}{w_0} \tag{4.5}$$

and

$$\left[h_l(0), \int_C :a_j(w_0)\, a_k^*(w_0):\, w_0^t \frac{dw_0}{w_0}\right] = (\varepsilon_j - \varepsilon_k)(h_l) \int_C :a_j(w_0)\, a_k^*(w_0):\, w_0^t \frac{dw_0}{w_0} \tag{4.6}$$

for $1 \leqslant i, j, k \leqslant l$. Later we will define the operator which represents derivation d and show that its bracket with these integrals gives them back with eigenvalue t. Before we compare this information with the root systems of $A^{(2)}_{2l-1}$ in Tables IX (a), (b) and Fig. 3, we note that some components of our generating functions are zero. We have by the use of Lemmas 2 and 8(b),

$$\int_C :a_j(w)\, a_j(-w):\, w^t \frac{dw}{w} = \mp \int_C :a_j(-w)\, a_j(w):\, w^t \frac{dw}{w} = \mp\iota(-1)^t \int_C :a_j(w)\, a_j(-w):\, w^t \frac{dw}{w} \tag{4.7}$$

and the same for a_j^* in place of a_j. This says that these components are 0 when $\pm\iota(-1)^t = 1$, that is, the long roots $t\bar{d} \pm 2\varepsilon_j$ only occur on even levels when $\pm\iota = -1$ and on odd levels when $\pm\iota = 1$. We have shown that the algebra $\mathbf{gl}^{(2)}(2l)$ of type $A^{(2)}_{2l-1}$ has both fermionic and bosonic constructions.

We next considered the extensions of this algebra obtained by adjoining the components of the functions $:u_1(w)\, e_S(w):$, $:u_2(-w)\, e_S(w):$, $:e_S(-w)\, e_S(w):$, $:e_S(w)\, e_S(-w):$ for $u_1 \in \mathbf{a}_1$, $u_2 \in \mathbf{a}_2$ and $S = Z$ or Z'. If $S = Z$, the first two have only integral components, but if $S = Z'$, they have only semi-integral components. Also note that Lemma 9(c) shows the individual components of $:e_S(-w)\, e_S(w):$ are multiples of those of $:e_S(w)\, e_S(-w):$, so we really needed only one of these. (For the largest extension, however, when $S = \frac{1}{2}\mathbb{Z}$, we will need both.) In fact, using Corollary 6 and Lemma 9(c), we find

$$\int_C :e_S(w)\, e_S(-w):\, w^t \frac{dw}{w} = -(-1)^{-t} \int_C :e_S(w)\, e_S(-w):\, w^t \frac{dw}{w} \tag{4.8}$$

for $t \in \mathbb{Z}$; so this function has only odd integral components.

Using (3.58), (3.59), we find

$$\left[h_l(0), \int_C :a_j(w_0)\, e_S(w_0):\, w_0^t \frac{dw_0}{w_0}\right] = \varepsilon_j(h_l) \int_C :a_j(w_0)\, e_S(w_0):\, w_0^t \frac{dw_0}{w_0}, \tag{4.9}$$

$$\left[h_i(0), \int_C :a_j^*(-w_0)\, e_S(w_0): w_0^{\iota} \frac{dw_0}{w_0}\right]$$
$$= -\varepsilon_j(h_i) \int_C :a_j^*(-w_0)\, e_S(w_0): w_0^{\iota} \frac{dw_0}{w_0}, \tag{4.10}$$

and clearly we have

$$\left[h_i(0), \int_C :e_S(-w_0)\, e_S(w_0): w_0^{\iota} \frac{dw_0}{w_0}\right] = 0. \tag{4.11}$$

We have seen in (3.64)–(3.66) that we should use $[\cdot,\cdot]_{\mp \mathrm{sgn}(S)}$ as the operation between these new short root vectors. If $S = Z$, $\mp \mathrm{sgn}(S) = \mp\iota$ so they are super roots if $\pm\iota = -1$ and Lie roots if $\pm\iota = 1$ as shown in Fig. 3. If $S = Z'$, $\mp \mathrm{sgn}(S) = \pm\iota$, and they are super roots when $\pm\iota = 1$, Lie roots when $\pm\iota = -1$. When $\pm\iota = -1$ the algebra we have constructed has root system of type $A^{(2)}(0, 2l-1)$ if $S = Z$, $A_{2l}^{(4)}$ if $S = Z'$. When $\pm\iota = 1$, we get the root system of type $A_{2l}^{(2)}$ if $S = Z$, $A^{(4)}(0, 2l-1)$ if $S = Z'$. Of course, because of the isomorphisms $\mathbf{gl}^{(2)}(2l+1) \approx \mathbf{gl}^{(4)}(2l+1)$ and $\mathbf{gl}^{(2)}(1, 2l) \approx \mathbf{gl}^{(4)}(1, 2l)$, we have only constructed two algebras, but each has both fermionic and bosonic constructions.

The largest extension was obtained by using $S = \frac{1}{2}\mathbb{Z}$. For the functions $:u_1(w)\, e_S(w):$ and $:u_2(-w)\, e_S(w):$, we have short roots on both integral and semi-integral levels, but they retain the Lie or super status they had in the last construction. As expected, we found in (3.70) and (3.71) that the Lie bracket is used between a Lie and a super root vector. From Lemma 9(c), we see that this algebra contains the components of $:e_Z(-w)\, e_Z(w):$, $:e_{Z'}(-w)\, e_{Z'}(w):$, and $:e_Z(-w)\, e_{Z'}(w):$. The first two give Lie root vectors on integral levels, and the third gives super root vectors on semi-integral levels. This is consistent with the use of $[\cdot,\cdot]_{\pm\iota}$ in (3.72) and (3.74) and with the use of $[\cdot,\cdot]_{\mp\iota}$ in (3.73) and (3.75). We have obtained fermionic and bosonic constructions of $\mathbf{gl}^{(4)}(1, 2l+1)$, the algebra of type $A^{(4)}(0, 2l)$.

Thus, we have in our second main theorem the identifications of the algebras constructed in Theorem A.

THEOREM B. *The algebras constructed in Theorem* A(a) *are those of the orthogonal series in the fermionic case and the symplectic series in the bosonic case. Figures* 1 *and* 2 *correspond to the diagram of representation spaces*

$$\begin{array}{ccccc} & & V_{\pm}(Z^l) & & \\ & \diagup & & \diagdown & \\ V_{\pm}^{\mathbf{Z}}(Z^l) & & & & V_{\pm}^{\mathbf{Z}'}(Z^l) \\ & \diagdown & & \diagup & \\ & & V_{\pm}^{1/2\mathbf{Z}}(Z^l) & & \end{array} \tag{4.12}$$

The algebras constructed in Theorem A(b) *are those of the general linear series. In case* $\pm\iota = -1$, *we get the series of Table* IX(a) *shown on the upper halves of Fig.* 3, *while for* $\pm\iota = 1$ *we get the series of Table* IX(b) *shown on the lower halves of Fig.* 3.

5. THE VIRASORO ALGEBRAS AND IRREDUCIBILITY

Every highest weight representation of each affine algebra can be extended to a semidirect product with the Virasoro algebra by means of the Segal operators [2]. These provide projective representations of the extensions of the affine algebras by the derivations $d(n) = t^{n+1}(d/dt)$, so that $d = d(0)$. The Virasoro algebra plays an important role in the dual resonance models of physics (see references in [3]). For the bosonic and fermionic constructions we have given of the classical affine algebras, the Segal operators considerably simplify. In the first part of this section, we will prove identities for the Segal operators. In the second part of this section, we will use those identities to find the irreducible components of each representation we have constructed.

Let us consider first the operators which will act on $V_{\pm}(Z^l)$ in the $D_l^{(1)}$ and $C_l^{(1)}$ cases. Recall that

$$h_i(w) = \pm :a_i(w)\, a_i^*(w): \qquad \text{for} \quad 1 \leqslant i \leqslant l \tag{5.1}$$

defines the components $h_i(m)$, $m \in \mathbb{Z}$, which generate a Heisenberg algebra,

$$[h_i(m), h_j(n)] = \pm m \delta_{ij} \delta_{m,-n}. \tag{5.2}$$

Let us introduce the notation

$$\begin{aligned} x_{ij}^*(w) &= :a_i(w)\, a_j^*(w):, \qquad x_{ij}(w) = :a_i(w)\, a_j(w):, \\ x_{i\,j}^{**}(w) &= :a_i^*(w)\, a_j^*(w): \qquad \text{for} \quad 1 \leqslant i, j \leqslant l, \end{aligned} \tag{5.3}$$

and the new bosonic normal ordering

$$\vdots x(m)\, y(n) \vdots = \begin{cases} x(m)\, y(n) & \text{if} \quad n > m, \\ \frac{1}{2}(x(m)\, y(n) + y(n)\, x(m)) & \text{if} \quad n = m, \\ y(n)\, x(m) & \text{if} \quad n < m, \end{cases} \tag{5.4}$$

for $x(m)$, $y(n)$ any components from (5.3). In the fermionic case $x_{ii}(w)$ and $x_{i\,i}^{**}(w)$ are identically zero. We will also use the notation

$$\begin{aligned} D_{ij}^*(w) &= \vdots x_{ij}^*(w)\, x_{ji}^*(w) \vdots, \qquad D_{ij}(w) = \vdots x_{ij}(w)\, x_{i\,j}^{**}(w) \vdots, \\ D_{i\,j}^{**}(w) &= \vdots x_{i\,j}^{**}(w)\, x_{ij}(w) \vdots. \end{aligned} \tag{5.5}$$

The Segal operators [2] are defined to be the components of the generating function

$$L(w) = \sum :x_\gamma(w)\, x'_\gamma(w):, \tag{5.6}$$

where the summation runs through basis $\{x_\gamma\}$ of the finite-dimensional subalgebra, $\mathbf{o}(2l)$ or $\mathbf{sp}(2l)$, of the affine algebra, $\mathbf{o}^{(1)}(2l)$ or $\mathbf{sp}^{(1)}(2l)$, and where $\{x'_\gamma\}$ is a dual basis, so that $\langle x_\gamma, x'_\gamma\rangle = 1$. If we choose $\{x_\gamma\}$ from

$$\{x_{ij}^* = :a_i a_j^*:, x_{ij} = :a_i a_j:, x_{i\,j}^{**} = :a_i^* a_j^*: \mid 1 \leqslant i,j \leqslant l\}, \tag{5.7}$$

then the invariant form (2.3) determines the dual basis. Note that

$$\langle x_{ij}^*, x_{ji}^*\rangle = +1 \qquad \text{and} \qquad \langle x_{ij}, x_{i\,j}^{**}\rangle = -1 \pm \delta_{ij} \tag{5.8}$$

so in the fermionic case

$$L_+(w) = \sum_{i=1}^{l} D_{ii}^* + \sum_{1\leqslant i<j\leqslant l} (D_{ij}^* + D_{ji}^*) - \sum_{1\leqslant i<j\leqslant l} (D_{ij} + D_{i\,j}^{**}) \tag{5.9}$$

and in the bosonic case

$$L_-(w) = \sum_{i=1}^{l} D_{ii}^* + \sum_{1\leqslant i<j\leqslant l} (D_{ij}^* + D_{ji}^*) - \sum_{1\leqslant i<j\leqslant l} (D_{ij} + D_{i\,j}^{**})$$
$$-\tfrac{1}{2} \sum_{i=1}^{l} (D_{ii} + D_{i\,i}^{**}). \tag{5.10}$$

It is known [2] that the components $L(m)$, $m \in \mathbb{Z}$, form a Virasoro algebra on highest weight representations. We will show that in our representations $L_\pm(w)$ has a considerable simplification and is a multiple of the Segal operator for a much smaller subalgebra.

For any generating functions $u(w), v(w)$, we define the operator $\vec{\partial}_w$ by

$$u(w)\,\vec{\partial}_w v(w) = u(w)\left(w\frac{d}{dw}v(w)\right) - \left(w\frac{d}{dw}u(w)\right)v(w), \tag{5.11}$$

so that for fixed w_0, we have

$$\lim_{w\to w_0} w\left(\frac{u(w_0)\,v(w) - u(w)\,v(w_0)}{w - w_0}\right) = u(w)\,\vec{\partial}_w v(w)\,|_{w=w_0}. \tag{5.12}$$

We also wish to denote by $\underline{x_{ij}^*(w)\, x_{ji}^*(w_0)}$ the contraction such that for $|w| > |w_0|$

$$x_{ij}^*(w)\, x_{ji}^*(w_0) = :a_i(w)\, a_j^*(w)\, a_j(w_0)\, a_i^*(w_0):$$
$$+ \underline{x_{ij}^*(w)\, x_{ji}^*(w_0)}. \tag{5.13}$$

Then we have the following special cases of Lemma 7.

LEMMA 10. *For $|w| > |w_0|$, $1 \leqslant i,j \leqslant l$, we have*

(a) $\underline{x_{ij}^*(w)\, x_{ji}^*(w_0)} = (\mp\xi/(w - w_0))(:a_j(w_0)\, a_j^*(w): - :a_i(w)\, a_i^*(w_0):)$ $\pm\ \xi^2/(w - w_0)^2$,

(b) $\underline{x_{ij}(w)\, x_{i\,j}^{**}(w_0)} = (\delta_{ij} \mp 1)(\xi/(w - w_0))(:a_j(w)\, a_j^*(w_0):$ $+ :a_i(w)\, a_i^*(w_0):) + (\delta_{ij} \mp 1)(\xi^2/(w - w_0)^2)$,

(c) $\underline{x_{i\,j}^{**}(w)\, x_{ij}(w_0)} = -(\delta_{ij} \mp 1)(\xi/(w - w_0))(:a_j(w_0)\, a_j^*(w):$ $+ :a_i(w_0)\, a_i^*(w):) + (\delta_{ij} \mp 1)(\xi^2/(w - w_0)^2)$,

where ξ is given in (3.30).

If $x(w)$ and $y(w_0)$ are generating functions from (5.3) we will denote by $\underline{x(w)\, y(w_0)}$ the bosonic contraction with respect to (5.4),

$$\underline{x(w)\, y(w_0)} = x(w)\, y(w_0) - \,\vdots x(w)\, y(w_0)\vdots\,. \tag{5.14}$$

Note that generally the analog of Lemma 2 does not hold for this bosonic normal ordering. That is, this bosonic contraction may not equal $[x(w), y^-(w_0)]$. The contraction may be computed directly from (5.4), but for our purposes we only need the following lemma.

LEMMA 11. *For $|w| > |w_0|$, $1 \leqslant i,j \leqslant l$ we have*

(a) $\underline{x_{ij}^*(w) x_{ji}^*(w_0)} + \underline{x_{ji}^*(w) x_{ij}^*(w_0)} = \pm 2\,\dfrac{ww_0}{(w - w_0)^2}$,

(b) $\underline{x_{ij}(w)\, x_{i\,j}^{**}(w_0)} + \underline{x_{i\,j}^{**}(w)\, x_{ij}(w_0)} = 2(\delta_{ij} \mp 1)\,\dfrac{ww_0}{(w - w_0)^2}$.

Proof. From (3.34) or Lemma 10 we have for $m, k \in \mathbb{Z}$

$$[x_{ij}^*(m), x_{ji}^*(-k)] = \pm(x_{ii}^*(m-k) - x_{jj}^*(m-k) + m\delta_{mk}),$$

$$[x_{ij}(m), x_{i\,j}^{**}(-k)] = (\delta_{ij} \mp 1)(x_{ii}^*(m-k) + x_{jj}^*(m-k) + m\delta_{mk}).$$

Using definitions (5.4) and (5.14) these give

$$\underline{x_{ij}^*(w)\, x_{ji}^*(w_0)} + \underline{x_{ji}^*(w)\, x_{ij}^*(w_0)} = \sum_{\substack{m,k\in\mathbb{Z}\\ m>-k}} \pm 2m\delta_{mk} w^{-m} w_0^k$$

and

$$\underbrace{x_{ij}(w)\, x_{i\,j}^{**}}(w_0) + \underbrace{x_{i\,j}^{**}(w)\, x_{ij}}(w_0) = \sum_{\substack{m,k\in\mathbb{Z}\\ m>-k}} 2(\delta_{ij} \mp 1)\, m\delta_{mk} w^{-m} w_0^k.$$

These give the result. ∎

LEMMA 12. *For $1 \leqslant i,j \leqslant l$ we have*

(a) $D_{ij}^*(w) + D_{ji}^*(w) = 2{:}a_i(w)a_j(w)a_i^*(w)a_j^*(w){:} \mp ({:}a_i(w)\,\vec{\partial}_w a_i^*(w){:} + {:}a_j(w)\,\vec{\partial}_w a_j^*(w){:}) \pm \frac{1}{4}(1+\imath)$,

(b) $D_{ij}(w) + D_{i\,j}^{**}(w) = 2{:}a_i(w)\, a_j(w)\, a_i^*(w)\, a_j^*(w){:} - (\delta_{ij} \mp 1)({:}a_i(w)\,\vec{\partial}_w a_i^*(w){:} + {:}a_j(w)\,\vec{\partial}_w a_j^*(w){:}) + (\delta_{ij} \mp 1)\,\frac{1}{4}(1+\imath)$.

Proof. The first equation follows from Lemma 11(a) where for $|w| > |w_0|$ we use

$$D_{ij}^*(w_0) + D_{ji}^*(w_0) = \lim_{w\to w_0} ({:}x_{ij}^*(w)\, x_{ji}^*(w_0){:} + {:}x_{ji}^*(w)\, x_{ij}^*(w_0){:})$$

$$= \lim_{w\to w_0} \left((x_{ij}^*(w)\, x_{ji}^*(w_0) + x_{ji}^*(w)\, x_{ij}^*(w_0) \mp 2\frac{ww_0}{(w-w_0)^2} \right),$$

and then use (5.13) and Lemma 10(a). The second equation is similarly done using Lemmas 11(b) and 10(b), (c). ∎

PROPOSITION 13. *We have*

$$L_\pm(w) = \mp(2l \mp 1) \sum_{i=1}^{l} ({:}a_i(w)\,\vec{\partial}_w a_i^*(w){:} - \tfrac{1}{8}(1+\imath)).$$

In the fermionic case, we have

$$L_+(w) = (2l-1) \sum_{i=1}^{l} D_{ii}^*(w), \tag{5.15}$$

while in the bosonic case we have

$$L_-(w) = \tfrac{1}{3}(2l+1) \left(\sum_{i=1}^{l} D_{ii}^*(w) - \tfrac{1}{2} \sum_{i=1}^{l} (D_{ii}(w) + D_{i\,i}^{**}(w)) \right). \tag{5.16}$$

Proof. These follow directly from (5.9), (5.10), and Lemma 12. ∎

Remark. Note that (5.15) says $L_+(w)$ is a multiple of the Segal generator of the Heisenberg subalgebra $\hat{\mathbf{h}}$, while (5.16) says $L_-(w)$ is a multiple of the Segal generator of the subalgebra generated by the Heisenberg and the long root vectors. In the first case the subalgebra is just $\mathbf{o}^{(1)}(2)^l$, and in the second case it is $\mathbf{sp}^{(1)}(2)^l$. Since we know that the vertex construction for $D_l^{(1)}$ is based on the Fock space representation of the Heisenberg, it seems reasonable to hope for an analogous construction of $C_l^{(1)}$ based on $\mathbf{sp}^{(1)}(2)^l$. The operators representing short root vectors analogous to vertex operators should be quite interesting.

Let us denote for $1 \leqslant i \leqslant l$,

$$D_i(w) = \pm\tfrac{1}{2}({:}a_i(w)\,\vec{\partial}_w a_i^*(w){:} - \tfrac{1}{8}(1+\imath))$$

$$= \mp\tfrac{1}{16}(1+\imath) \pm \sum_{k\in\mathbb{Z}} \sum_{n\in Z} \left(n - \frac{k}{2}\right) {:}a_i(n)\, a_i^*(k-n){:}\, w^{-k} \tag{5.17}$$

and

$$D(w) = \sum_{i=1}^{l} D_i(w), \tag{5.18}$$

so that

$$L_\pm(w) = -2(2l \mp 1)\, D(w). \tag{5.19}$$

PROPOSITION 14. *For $1 \leqslant i,j \leqslant l$, $k, n \in \mathbb{Z}$, $m \in Z$, we have*

(a) $[D_i(k), a_j(m)] = \delta_{ij}(m + (k/2))\, a_j(m+k)$,

(b) $[D_i(k), a_j^*(m)] = \delta_{ij}(m + (k/2))\, a_j^*(m+k)$,

(c) $[D(k), x(n)] = nx(n+k)$ *for* $x(w) = {:}u(w)\, v(w){:}$ *with* $u, v \in \mathbf{a}$.

(d) $[D(k), D(n)] = (n-k)\, D(k+n) \pm (l/12)(k^3 - k)\,\delta_{k,-n}$.

Proof. The proof is based on the Wick theorem and the calculus of residues. For (a), we have

$$[D_i(k), a_j(m)] = \left[\int_C D_i(w)\, w^k \frac{dw}{w}, \int_{C_1} a_j(w_0)\, w_0^m \frac{dw_0}{w_0} \right]$$

$$= -\frac{\delta_{ij}}{2} \int_C \left(\int_{C_R\setminus C_r} \left(\frac{v}{(w-w_0)^2} a_j(w) + \frac{\xi}{w-w_0} a_j^{(1)}(w) \right) w^k \frac{dw}{w} \right) w_0^m \frac{dw_0}{w_0}$$

$$= -\frac{\delta_{ij}}{2} \int_C \left(m a_j(w_0) + 2a_j^{(1)}(w_0) \right) w_0^{k+m} \frac{dw_0}{w_0}$$

$$= \delta_{ij} \left(m + \frac{k}{2} \right) a_j(m+k),$$

where

$$v = \begin{cases} ww_0 & \text{if } Z = \mathbb{Z}, \\ \frac{1}{2}(w+w_0)(ww_0)^{1/2} & \text{if } Z = \mathbb{Z} + \frac{1}{2}, \end{cases} \tag{5.20}$$

and we denote for $u \in \mathbf{a}$

$$u^{(1)}(w) = w \frac{d}{dw} u(w). \tag{5.21}$$

The calculation for (b) is identical, and (c) follows from (a) and (b). Up to a scalar term (d) follows from (c) and (5.19), but we must be careful to determine the scalar term. In fact, the term $\mp(l/16)(1+\iota)$ in $D(0)$ is needed to get $k^3 - k$ rather than $k^3 + 2k$ when $Z = \mathbb{Z}$. ∎

To understand the situation for the extensions of $D_l^{(1)}$ and $C_l^{(1)}$ corresponding to $S = Z$ and $S = Z'$ we must return to the Segal operators. First we should extend the notations (5.3) by adding

$$x_{ie}(w) = :a_i(w)\, e_S(w):, \qquad x_{ie}^*(w) = :a_i^*(w)\, e_S(w): \tag{5.22}$$

for $1 \leqslant i \leqslant l$. We should still use the bosonic normal ordering (5.4) if $x(m)$ is a component from (5.3) and $y(n)$ is a component from (5.22). But if $x(m)$ and $y(n)$ are both components from (5.22), then (3.40) shows that we should use

$$:x(m)\,y(n): = \begin{cases} x(m)\,y(n) & \text{if } \ n > m, \\ \tfrac{1}{2}(x(m)\,y(n) \pm y(n)\,x(m)) & \text{if } \ n = m, \\ \pm y(n)\,x(m) & \text{if } \ n < m. \end{cases} \tag{5.23}$$

Let $D_{ie}(w) = :x_{ie}(w)\, x_{ie}^*(w):$, $D_{ie}^*(w) = :x_{ie}^*(w)\, x_{ie}(w):$ be added to (5.5). To extend the set (5.7), add $x_{ie} = :a_i e:$, $x_{ie}^* = :a_i^* e:$ for $1 \leqslant i \leqslant l$, and then note that the invariant form (2.3) gives

$$\langle x_{ie}, x_{je}^* \rangle = \pm 2\delta_{ij} \qquad \text{and} \qquad \langle x_{je}^*, x_{ie} \rangle = 2\delta_{ij}. \tag{5.24}$$

Remember that the invariant form is supersymmetric (2.4) in the bosonic case. This gives the generating function for the Segal operators

$$L_{\pm}^S(w) = L_{\pm}(w) + \tfrac{1}{2} \sum_{i=1}^{l} (D_{ie}(w) \pm D_{ie}^*(w)) \tag{5.25}$$

for $S = Z$ or $S = Z'$.

LEMMA 15. *For $|w| > |w_0|$, $1 \leqslant i, j \leqslant l$, $S = Z$ or Z' we have*

(a) $\underline{x_{ie}(w)\, x_{je}^*(w_0)} = \mp(\xi/(w - w_0))\, \delta_{ij} :e_S(w)\, e_S(w_0):$ $+ (\xi_S/(w - w_0))\, 2 :a_i(w)\, a_j^*(w_0): + 2\delta_{ij}(\xi\xi_S/(w - w_0)^2)$,

(b) $\underline{x_{ie}^*(w)\, x_{je}(w_0)} = -(\xi/(w - w_0))\, \delta_{ij} :e_S(w)\, e_S(w_0):$ $+ (\xi_S/(w - w_0))\, 2 :a_i^*(w)\, a_j(w_0): \pm 2\delta_{ij}(\xi\xi_S/(w - w_0)^2)$,

where ξ and ξ_S are given in (3.30) *and Lemma* 1(c).

We must define the following expressions to compactly express the next result. Let

$$\eta = \begin{cases} w w_0 & \text{if } \ S = Z, \\ \tfrac{1}{2}(w + w_0)(w w_0)^{1/2} & \text{if } \ S = Z', \\ w w_0 + \tfrac{1}{2}(w + w_0)(w w_0)^{1/2} & \text{if } \ S = \tfrac{1}{2}\mathbb{Z}. \end{cases} \tag{5.26}$$

Let

$$\underline{x(w)\, y(w_0)} = x(w)\, y(w_0) - :x(w)\, y(w_0): \tag{5.27}$$

be the appropriate type of contraction with respect to (5.23) for $x(w)$ and $y(w_0)$ from (5.22).

LEMMA 16. *For $|w| > |w_0|$, $1 \leqslant i \leqslant l$ we have*

$$\underline{x_{ie}(w)\, x_{ie}^*(w_0)} \pm \underline{x_{ie}^*(w)\, x_{ie}(w_0)} = 4\eta/(w - w_0)^2.$$

Proof. From (3.40), (3.44), or Lemma 15 we have for $m, k \in Z + S$, $S = Z$ or $S = Z'$, $1 \leqslant i, j \leqslant l$,

$$[x_{ie}(m), x_{je}^*(-k)]_{\mp} = 2(x_{ij}^*(m - k) + m\delta_{ij}\delta_{mk}).$$

Using definitions (5.23) and (5.27) with $j = i$ one finds that

$$\underline{x_{ie}(w)\, x_{ie}^*(w_0)} \pm \underline{x_{ie}^*(w)\, x_{ie}(w_0)} = \sum_{\substack{m, k \in Z + S \\ m > -k}} 4m\delta_{mk} w^{-m} w_0^k.$$

Then the result follows from the formulas

$$\sum_{1 \leqslant p \in Z} p \left(\frac{w_0}{w}\right)^p = \frac{w w_0}{(w - w_0)^2}, \tag{5.28}$$

and

$$\sum_{0 \leqslant p \in Z} \left(p + \frac{1}{2}\right) \left(\frac{w_0}{w}\right)^{p + 1/2} = \frac{(w + w_0)(w w_0)^{1/2}}{2(w - w_0)^2}. \; ∎ \tag{5.29}$$

LEMMA 17. *For $1 \leqslant i, j \leqslant l$, $S = Z$, Z', we have*

$$D_{ie}(w) \pm D_{ie}^*(w) = \pm :e_S(w)\, \vec{\partial}_w e_S(w):$$
$$- 2 :a_i(w)\, \vec{\partial}_w a_i^*(w): + \left(\frac{1 + \iota}{2}\right)\left(\frac{1 + \operatorname{sgn}(S)}{2}\right).$$

Proof. The method of Lemma 12 gives this result using Lemmas 15 and 16 and

$$\lim_{w \to w_0} 4 \left(\frac{\xi\xi_S - \eta}{(w - w_0)^2}\right) = \left(\frac{1 + \iota}{2}\right)\left(\frac{1 + \operatorname{sgn}(S)}{2}\right). \; ∎ \tag{5.30}$$

Proposition 18. *For $S = Z$ or Z', we have*

$$L^S_\pm(w) = -4lD(w) \pm \frac{l}{2} :e_S(w)\,\vec{\partial}_w e_S(w): + \frac{l}{2}\left(\frac{1+\imath}{4}\right)\operatorname{sgn}(S).$$

Let us denote for $S = Z$ or Z',

$$D^S_0(w) = \mp\frac{1}{8}\left(:e_S(w)\,\vec{\partial}_w e_S(w): \pm \operatorname{sgn}(S)\left(\frac{1+\imath}{4}\right)\right)$$

$$= -\operatorname{sgn}(S)\left(\frac{1+\imath}{32}\right) \mp \sum_{k\in Z}\sum_{n\in S}\left(n-\frac{k}{2}\right):e(n)\,e(k-n): w^{-k}, \qquad (5.31)$$

so that for $S = Z$ or Z'

$$L^S_\pm(w) = -4l(D(w) + D^S_0(w)) = -4lD^S(w). \qquad (5.32)$$

Proposition 19. *For $1 \leqslant j \leqslant l$, $k \in \mathbb{Z}$, $S = Z, Z'$, we have*

(a) $[D^S_0(k), a_j(m)] = 0 = [D^S_0(k), a^*_j(m)]$ *for $m \in Z$,*

(b)

$$[D^S_0(k), e(m)] = \begin{cases} \left(m+\frac{k}{2}\right)e(m+k) & \text{for } m \in S, \\ 0 & \text{for } m \in S', \end{cases}$$

(c) $[D^S(k), x(n)] = nx(n+k)$ *for $x(w) = :u(w)\,v(w):$, $u, v \in \mathbf{a}$, $n \in \mathbb{Z}$, or for $x(w) = :u(w)\,e_S(w):$, $n \in Z + S$,*

(d) $[D^S(k), D^S(n)] = (n-k)\,D^S(k+n) \pm (l/12)(k^3-k)\,\delta_{k,-n}$ *for $n \in \mathbb{Z}$.*

The generating function for the largest extension corresponding to $S = \frac{1}{2}\mathbb{Z}$ is larger than (5.25) because of the root vectors coming from

$$x_{ee}(w) = :e_Z(w)\,e_{Z'}(w): \qquad (5.33)$$

The normal ordering $\vdots x_{ee}(m)\,x_{ee}(n)\vdots$, $m, n \in \mathbb{Z} + \frac{1}{2}$, is the bosonic one (5.4) because of (3.50). Letting

$$D_{ee}(w) = \vdots x_{ee}(w)\,x_{ee}(w)\vdots, \qquad (5.34)$$

we find the generating function for $S = \frac{1}{2}\mathbb{Z}$ to be

$$L^{1/2\,Z}_\pm(w) = L_\pm(w) + \tfrac{1}{2}\sum_{i=1}^{l}(D_{ie}(w) \pm D^*_{ie}(w)) \mp \tfrac{1}{4}D_{ee}(w). \qquad (5.35)$$

Lemma 20. *For $|w| > |w_0|$, we have*

$$\underline{x_{ee}(w)\,x_{ee}(w_0)} = \pm 2\frac{\xi_Z}{w-w_0}:e_{Z'}(w)\,e_{Z'}(w_0): \pm 2\frac{\xi_{Z'}}{w-w_0}:e_Z(w)\,e_Z(w_0): - \frac{4\xi_Z\xi_{Z'}}{(w-w_0)^2}.$$

From (3.50) or Lemma 20, we have

$$[x_{ee}(m), x_{ee}(-k)] = -4m\delta_{mk} \qquad (5.36)$$

for $m, k \in \mathbb{Z} + \frac{1}{2}$, which gives

Lemma 21. *For $|w| > |w_0|$, we have the bosonic contraction with respect to (5.4)*

$$\underline{x_{ee}(w)\,x_{ee}(w_0)} = -4\frac{(w+w_0)(ww_0)^{1/2}}{2(w-w_0)^2} = -4\frac{\xi_Z\xi_{Z'}}{(w-w_0)^2}.$$

Lemma 22. *We have*

$$D_{ee}(w) = \mp(:e_Z(w)\,\vec{\partial}_w e_Z(w): + :e_{Z'}(w)\,\vec{\partial}_w e_{Z'}(w):).$$

Proposition 23. *We have*

$$L^{1/2\,Z}_\pm(w) = (-4l \mp 2)(D(w) + D^Z_0(w) + D^{Z'}_0(w)).$$

If we define $D^{1/2\,Z}(w) = D(w) + D^Z_0(w) + D^{Z'}_0(w)$ and $D^{1/2\,Z}_0(w) = D^Z_0(w) + D^{Z'}_0(w)$, then we have an analog of Proposition 19 for $S = \frac{1}{2}\mathbb{Z}$.

It is straightforward technical work to modify what we have done above for the general linear series. For example, in (5.3) we should use

$$x_{ij}(w) = :a_i(w)\,a_j(-w):, \qquad x^{**}_{ij}(w) = :a^*_i(w)\,a^*_j(-w): \qquad (5.37)$$

according to (3.21)–(3.23), and the normal orderings we use must be appropriate. The final results, however, are quite analogous to those we have obtained above.

We can use the above results to find the irreducible components of the representation spaces $V_\pm(Z^l)$, $V^S_\pm(Z^l)$. We will follow the idea of the proof in [2]. From Lemma 13 we see that each $D_i(w)$ is a linear combination of the identity operator, $D_{ii}^*(w)$, $D_{ii}(w)$, and $D^{**}_{ii}(w)$. But the components of these generating functions consist of operators from the affine algebra, so the components of $D_i(w)$ preserve irreducible subspaces of $V_\pm(Z^l)$.

The operators $D(n)$ provide a projective representation of the derivations $d(n) = t^{n+1}(d/dt)$ according to Proposition 14. If we only adjoin the