and

\[
\left[ h_\ell(0), \int_c :a_j(w_0) a^*_k(w_0); w_0^\prime \frac{dw_0}{w_0} \right] = (\epsilon_j - \epsilon_k)(h_\ell) \int_c :a_j(w_0) a^*_k(w_0); w_0^\prime \frac{dw_0}{w_0},
\]

for \(1 \leq i, j, k \leq l\). Later we will define the operator which represents derivation \(d\) and show that its bracket with these integrals gives them back with eigenvalue \(t\). Before we compare this information with the root systems of \(A_{2l-1}^{(1)}\) in Tables IX (a), (b) and Fig. 3, we note that some components of our generating functions are zero. We have by the use of Lemmas 2 and 8(b),

\[
\int_c :a_j(w) a_j(-w); w^\prime \frac{dw}{w} = \mp \int_c :a_j(-w) a_j(w); w^\prime \frac{dw}{w},
\]

and the same for \(a^*_k\) in place of \(a_j\). This says that these components are 0 when \(\pm t = 1\), that is, the long roots \(\pm \epsilon_l\) only occur on even levels when \(\pm t = -1\) and on odd levels when \(\pm t = 1\). We have shown that the algebra \(\mathfrak{gl}^{(1)}(2l)\) of type \(A_{2l-1}^{(1)}\) has both fermionic and bosonic constructions.

We next considered the extensions of this algebra obtained by adjoining the components of the functions \(u_i(w) e_i(w); u_i(-w) e_i(w); e_i(w) e_i(-w)\); for \(u_i \in \mathfrak{a}_1, u_i \in \mathfrak{a}_2\), and \(S = Z\) or \(Z'\). If \(S = Z\), the first two have only integral components, but if \(S = Z'\), they have only semi-integral components. Also note that Lemma 9(c) shows the individual components of \(e_i(-w) e_i(w)\) are multiples of those of \(e_i(w) e_i(-w)\); so we really needed only one of these. (For the largest extension, however, when \(S = \frac{1}{2}Z\), we will need both.) In fact, using Corollary 6 and Lemma 9(c), we find

\[
\int_c :e_i(w) e_i(-w); w^\prime \frac{dw}{w} = -(-1)^t \int_c :e_i(w) e_i(-w); w^\prime \frac{dw}{w}
\]

for \(t \in Z\); so this function has only odd integral components.

Using (3.58), (3.59), we find

\[
\left[ h_\ell(0), \int_c :a_j(w_0) e_j(w_0); w_0^\prime \frac{dw_0}{w_0} \right] = \epsilon_j(h_\ell) \int_c :a_j(w_0) e_j(w_0); w_0^\prime \frac{dw_0}{w_0},
\]

FIG. 3. The general linear series.
The algebras constructed in Theorem A(b) are those of the general linear series. In case \( \pm t = -1 \), we get the series of Table IX(a) shown on the upper halves of Fig. 3, while for \( \pm t = 1 \) we get the series of Table IX(b) shown on the lower halves of Fig. 3.

5. THE VIRA SOBO ALGEBRAS AND IRREDUCIBILITY

Every highest weight representation of each affine algebra can be extended to a semidirect product with the Virasoro algebra by means of the Segal operators [2]. These provide projective representations of the extensions of the affine algebras by the derivations \( d(n) = i^{n + 1/d(n)} \), so that \( d = d(0) \). The Virasoro algebra plays an important role in the dual resonance models of physics (see references in [3]). For the bosonic and fermionic constructions we have given of the classical affine algebras, the Segal operators considerably simplify. In the first part of this section, we will prove identities for the Segal operators. In the second part of this section, we will use those identities to find the irreducible components of each representation we have constructed.

Let us consider first the operators which will act on \( V_{\pm}(Z') \) in the \( D_{\pm} \) and \( C_{\pm} \) cases. Recall that

\[
\hat{h}(w) = \pm \hat{a}(w) \hat{a}(w): \quad \text{for } 1 \leq i \leq l
\]

(5.1)
defines the components \( \hat{h}(m), m \in \mathbb{Z} \), which generate a Heisenberg algebra,

\[
[h_{i}(m), h_{j}(n)] = \pm m \delta_{ij} \delta_{m-n}.
\]

(5.2)

Let us introduce the notation

\[
\hat{x}_{i}(w) = \hat{a}(w) \hat{a}(w), \quad \hat{x}_{ij}(w) = \hat{a}(w) \hat{a}(w),
\]

(5.3)

and the new bosonic normal ordering

\[
\hat{x}(m) \hat{y}(n) = \begin{cases} x(m) y(n) & \text{if } n > m, \\ \frac{1}{2} (x(m) y(n) + y(n) x(m)) & \text{if } n = m, \\ y(n) x(m) & \text{if } n < m, \end{cases}
\]

(5.4)

for \( x(m), y(n) \) any components from (5.3). In the fermionic case \( x_{ij}(w) \) and \( x_{ij}^{*}(w) \) are identically zero. We will also use the notation

\[
\hat{D}_{i}^{*}(w) = \hat{x}_{ij}^{*}(w) \hat{x}_{ij}(w), \quad \hat{D}_{ij}(w) = \hat{x}_{ij}(w) \hat{x}_{ij}^{*}(w),
\]

(5.5)
The Segal operators \([2]\) are defined to be the components of the generating function

\[
L(w) = \sum x_i(w) x_i^*(w),
\]

where the summation runs through basis \(\{x_i\}\) of the finite-dimensional subalgebra, \(o(2l)\) or \(sp(2l)\), of the affine algebra, \(o^{(1)}(2l)\) or \(sp^{(1)}(2l)\), and where \(\{x_i^*\}\) is a dual basis, so that \(\langle x_i, x_j^* \rangle = 1\). If we choose \(\{x_i\}\) from

\[
\langle x_i, x_j^* \rangle = \delta_{ij}, \quad x_i = \delta_{ij}, \quad x_i^* = \delta_{ij}; \quad 1 \leq i, j \leq l,
\]

then the invariant form \((2.3)\) determines the dual basis. Note that

\[
\langle x_i^*, x_j \rangle = +1 \quad \text{and} \quad \langle x_i, x_j^* \rangle = -\delta_{ij}.
\]

so in the fermionic case

\[
L_+(w) = \sum_{i=1}^{l} D_{ii}^\alpha + \sum_{1 \leq i < j \leq l} (D_{ii}^\alpha + D_{jj}^\alpha) - \sum_{1 \leq i < j \leq l} (D_{ij} + D_{ji}^\alpha)
\]

and in the bosonic case

\[
L_-(w) = \sum_{i=1}^{l} D_{ii}^\alpha + \sum_{1 \leq i < j \leq l} (D_{ii}^\alpha + D_{jj}^\alpha) - \sum_{1 \leq i < j \leq l} (D_{ij} + D_{ji}^\alpha) - \frac{1}{2} \sum_{i=1}^{l} (D_{ii}^\alpha + D_{ii}^\alpha)
\]

It is known \([2]\) that the components \(L(m), m \in \mathbb{Z}\), form a Virasoro algebra on highest weight representations. We will show that in our representations \(L_+(w)\) has a considerable simplification and is a multiple of the Segal operator for a much smaller subalgebra.

For any generating functions \(u(w), v(w)\), we define the operator \(\tilde{\sigma}_w\) by

\[
\tilde{\sigma}_w v(w) = u(w) \left( w \frac{d}{dw} v(w) - \left( w \frac{d}{dw} u(w) \right) v(w) \right)
\]

so that for fixed \(w_0\), we have

\[
\lim_{w \to w_0} \left( u(w_0) v(w) - u(w) v(w_0) \right) = \tilde{\sigma}_w v(w) \big|_{w = w_0}.
\]

We also wish to denote by \(x_i^*(w) x_j^*(w_0)\) the contraction such that for \(\|w\| > |w_0|\)

\[
x_j^*(w) x_i^*(w_0) = :a_i(w) a_j^*(w_0) a_i^*(w): + x_i^*(w) x_j^*(w_0).
\]

Then we have the following special cases of Lemma 7.

**Lemma 10.** For \(\|w\| > |w_0|\), \(1 \leq i, j \leq l\), we have

\[
\begin{align*}
(a) & \quad x_j^*(w) x_i^*(w_0) = \left( \frac{\delta_{ij}}{w - w_0} \right) (a_j(w_0) a_i^*(w) - :a_i(w) a_j^*(w_0):) \\
& \quad + \frac{\xi^2}{(w - w_0)^3}, \\
(b) & \quad x_j^*(w) x_i^*(w_0) = \left( \frac{\delta_{ij}}{w - w_0} \right) (a_j(w_0) a_i^*(w) - :a_i(w) a_j^*(w_0):) \\
& \quad + \frac{\xi^2}{(w - w_0)^3}, \\
(c) & \quad x_j^*(w) x_i^*(w_0) = \left( \frac{\delta_{ij}}{w - w_0} \right) (a_j(w_0) a_i^*(w) - :a_i(w) a_j^*(w_0):) \\
& \quad + \frac{\xi^2}{(w - w_0)^3},
\end{align*}
\]

where \(\xi\) is given in \((3.30)\).

If \(x(w)\) and \(y(w_0)\) are generating functions from \((5.3)\) we will denote by \(x(w) y(w_0)\) the bosonic contraction with respect to \((5.4)\),

\[
\tilde{\sigma}_w x(w) y(w_0) = x(w) (\frac{d}{dw} y(w_0) - \left( \frac{d}{dw} x(w) \right) y(w_0)).
\]

Note that generally the analog of Lemma 2 does not hold for this bosonic normal ordering. That is, this bosonic contraction may not equal \([x(w), y(w_0)]\). The contraction may be computed directly from \((5.4)\), but for our purposes we only need the following lemma.

**Lemma 11.** For \(\|w\| > |w_0|\), \(1 \leq i, j \leq l\) we have

\[
\begin{align*}
(a) & \quad x_i^*(w) x_j^*(w_0) + x_j^*(w) x_i^*(w_0) = \frac{1}{(w - w_0)^3}, \\
(b) & \quad x_i^*(w) x_j^*(w_0) + x_j^*(w) x_i^*(w_0) = \frac{x_j^*(w) x_i^*(w_0)}{(w - w_0)^3}.
\end{align*}
\]

**Proof.** From \((3.34)\) or Lemma 10 we have for \(m, k \in \mathbb{Z}\)

\[
[x_i^*(m), x_j^*(m, k)] = \pm (x_i^*(m) x_j^*(m, k) - x_j^*(m, k) x_i^*(m)) + m \delta_{mk}.
\]

\[
[x_i^*(m), x_j^*(m, k)] = (\delta_{ij} + 1)(x_i^*(m, k) x_j^*(m) - x_j^*(m, k) x_i^*(m)) + m \delta_{mk}.
\]

Using definitions \((5.4)\) and \((5.14)\) these give

\[
\sum_{m, k \in \mathbb{Z}} 2 \pm m \delta_{mk} w^{-m} w_0^k.
\]
and

\[ x_{ij}(w) x_{j\ell}^*(w_0) + x_{j\ell}^*(w) x_{ij}(w_0) = \sum_{m, k \in \mathbb{Z}} 2(\delta_{ij} + 1) m \delta_{mk} w^{-m} w_0^k. \]

These give the result.

**Lemma 12.** For \( 1 \leq i, j \leq l \) we have

(a) \( D_{ij}^\dagger(w) + D_{ji}^\dagger(w) = 2 :a_i(w) a_j(w) a_j^*(w) a_i^*(w): + (:a_i(w) \delta_{ii} a_j^*(w): - \frac{1}{2}(1 + i)), \)

(b) \( D_j(w) + D_j^\dagger(w) = 2 :a_i(w) a_j(w) a_j^*(w) a_i^*(w): - (\delta_{ij} + 1)(:a_i(w) \delta_{ii} a_j^*(w): + :a_j(w) \delta_{ii} a_i^*(w):) + (\delta_{ij} + 1) \frac{1}{2}(1 + i). \)

**Proof.** The first equation follows from Lemma 11(a) where for \( |w| > |w_0| \) we use

\[ D_{ij}^\dagger(w_0) + D_{ji}^\dagger(w_0) = \lim_{w \to w_0} \left( :x_{ij}^*(w) x_{ij}^*(w_0): + :x_{ji}^*(w) x_{ji}^*(w_0): \right) = \lim_{w \to w_0} \left( :x_{ij}^*(w) x_{ij}^*(w_0): + :x_{ji}^*(w) x_{ji}^*(w_0): + 2 \frac{ww_0}{(w - w_0)^2} \right), \]

and then use (5.13) and Lemma 10(a). The second equation is similarly done using Lemmas 11(b) and 10(b), (c).

**Proposition 13.** We have

\[ L_\pm(w) = \mp(2l + 1) \sum_{i=1}^l (:a_i(w) \delta_{ii} a_i^*(w): - \frac{1}{2}(1 + i)). \]

In the fermionic case, we have

\[ L_+(w) = (2l - 1) \sum_{i=1}^l D_{ii}^\dagger(w), \tag{5.15} \]

while in the bosonic case we have

\[ L_-(w) = \frac{1}{2}(2l + 1) \left( \sum_{i=1}^l D_{ii}^\dagger(w) - \frac{1}{2} \sum_{i=1}^l (D_{ii}(w) + D_{ii}^\dagger(w)) \right). \tag{5.16} \]

**Proof.** These follow directly from (5.9), (5.10), and Lemma 12.

**Remark.** Note that (5.15) says \( L_+(w) \) is a multiple of the Segal generator of the Heisenberg subalgebra \( \mathfrak{h} \), while (5.16) says \( L_-(w) \) is a multiple of the Segal generator of the subalgebra generated by the Heisenberg and the long root vectors. In the first case the subalgebra is just \( \mathfrak{o}^{(1)}(2) \), and in the second case it is \( \mathfrak{sp}^{(1)}(2) \). Since we know that the vertex construction for \( D^{(1)} \) is based on the Fock space representation of the classical affine algebras, it seems reasonable to hope for an analogous construction of \( C^{(1)} \) based on \( \mathfrak{sp}^{(1)}(2) \). The operators representing short root vectors analogous to vertex operators should be quite interesting.

Let us denote for \( 1 \leq j \leq l \),

\[ D_j(w) = \pm \frac{i}{l} (a_j(w) \delta_{ii} a_i^*(w): - \frac{1}{2}(1 + i)) \]

\[ = \mp \frac{1}{l} \left( 1 + i \right) \pm \sum_{k \in \mathbb{Z}} \sum_{n \in \mathbb{Z}} \left( n + \frac{k}{2} \right) :a_j(n) a_i^*(k - n): w^{-k} \tag{5.17} \]

and

\[ D(w) = \sum_{j=1}^l D_j(w), \tag{5.18} \]

so that

\[ L_\pm(w) = -2(2l + 1) D(w). \tag{5.19} \]

**Proposition 14.** For \( 1 \leq i, j \leq l, k, n \in \mathbb{Z}, m \in \mathbb{Z} \), we have

(a) \[ [D_j(k), a_j(m)] = \delta_{ij}(m + (k/2)) a_i(m + k), \]

(b) \[ [D_j(k), a_j^*(m)] = \delta_{ij}(m + (k/2)) a_i^*(m + k), \]

(c) \[ [D_j(k), x(n)] = nx(n + k) \text{ for } x(w) = :u(w) v(w): \text{ when } u, v \in a, \]

(d) \[ [D_j(k), D(n)] = (n - k) D(k + n) \pm (l/12)(k^3 - k) \delta_{k, -n}. \]

**Proof.** The proof is based on the Wick theorem and the calculus of residues. For (a), we have

\[ [D_j(k), a_j(m)] = \left[ \int_c D_j(w) w^k \frac{dw}{w}, \int_c a_j(w_0) w_0^m \frac{dw_0}{w_0} \right] \]

\[ = - \frac{1}{2} \delta_{j,k} \int_c \left( \frac{v}{(w - w_0)^2} a_j(w) \right) \frac{dw}{w} \frac{dw_0}{w_0} \]

\[ + \frac{1}{2} \delta_{j,k} \int_c a_j^{(1)}(w) \frac{dw}{w} \frac{dw_0}{w_0} \]

\[ = - \frac{1}{2} \delta_{j,k} \int_c \left( ma_j(w_0) + 2a_j^{(1)}(w_0) \right) w_0^m \frac{dw_0}{w_0} \]

\[ = \delta_{j,k} \left( m + \frac{k}{2} \right) a_j(m + k), \]

where

\[ v = \begin{cases} \frac{ww_0}{v}, & \text{if } Z = \mathbb{Z}, \\ \frac{1}{2}(w + w_0)(ww_0)^{1/2}, & \text{if } Z = \mathbb{Z} + \frac{1}{2}, \end{cases} \tag{5.20} \]
and we denote for \( u \in a \)
\[
u^{(i)}(w) = w \frac{d}{dw} u(w).
\] (5.21)

The calculation for (b) is identical, and (c) follows from (a) and (b). Up to a scalar term (d) follows from (c) and (5.19), but we must be careful to determine the scalar term. In fact, the term \( \pi(16)(1 + i) \) in \( D(0) \) is needed to get \( k^3 - k \) rather than \( k^3 + 2k \) when \( Z = Z' \).

To understand the situation for the extensions of \( D_l^{(1)} \) and \( C_l^{(1)} \) corresponding to \( S = Z \) and \( S = Z' \) we must return to the Segal operators. First we should extend the notations (5.3) by adding
\[
x_{ie}(w) = :a_i(w) e_\nu(w):; \quad x_\nu^a(w) = :a_\nu^a(w) e_\nu(w):
\] (5.22)
for \( 1 \leq i \leq l \). We should still use the bosonic normal ordering (5.4) if \( x(m) \) is a component from (5.3) and \( y(n) \) is a component from (5.22). But if \( x(m) \) and \( y(n) \) are both components from (5.22), then (3.40) shows that we should use
\[
x(m) y(n) = \begin{cases} x(m) y(n) & \text{if } n > m, \\ \frac{1}{2}(x(m) y(n) + y(n) x(m)) & \text{if } n = m, \\ \pm x(n) y(m) & \text{if } n < m. \end{cases}
\] (5.23)

Let \( D_{ie}(w) = :x_{ie}(w) x_\nu^a(w)\); \( D_{ie}^a(w) = :x_\nu^a(w) x_{ie}(w)\); be added to (5.5). To extend the set (5.7), add \( x_{ie} = :a_i e_\nu; \quad x_\nu^a = :a_\nu^a e_\nu\); for \( 1 \leq i \leq l \), and then note that the invariant form (2.3) gives
\[
\langle x_{ie}, x_\nu^a \rangle = \pm 2 \delta_{ij} \quad \text{and} \quad \langle x_{ie}, x_{ie} \rangle = 2 \delta_{ij}.
\] (5.24)

Remember that the invariant form is supersymmetric (2.4) in the bosonic case. This gives the generating function for the Segal operators
\[
L_{a}(w) = L_{a}(w) + \frac{1}{i} \sum_{i=1}^{l} (D_{ie}(w) \pm D_{ie}^a(w))
\] (5.25)
for \( S = Z \) or \( S = Z' \).

**Lemma 15.** For \( |w| > |w_0|, 1 \leq i, j \leq l, S = Z \) or \( S = Z' \) we have

(a) \( x_{ie}(w) x_\nu^a(w_0) = :\left(\xi/(w - w_0)\right) \delta_\nu_i e_\nu(w) e_\nu(w_0); + \left(\xi/(w - w_0)\right) 2 :a_i(w) a_\nu^a(w_0); + 2 \delta_{ij} \left(\xi/(w - w_0)\right)^3 \),

(b) \( x_\nu^a(w) x_{ie}(w_0) = :\left(\xi/(w - w_0)\right) \delta_\nu_i e_\nu(w) e_\nu(w_0); - \left(\xi/(w - w_0)\right) 2 :a_i(w) a_\nu^a(w_0); + 2 \delta_{ij} \left(\xi/(w - w_0)\right)^3 \),

where \( \xi \) and \( \xi_0 \) are given in (3.30) and Lemma 1(c).

We must define the following expressions to compactly express the next result. Let
\[
\eta = \begin{cases} w w_0 & \text{if } S = Z, \\ \frac{1}{2} (w + w_0) (w w_0)^{1/2} & \text{if } S = Z', \\ w w_0 + \frac{1}{2} (w + w_0) (w w_0)^{1/2} & \text{if } S = \frac{1}{2} Z.
\end{cases}
\] (5.26)

Let \( x(w), y(w_0) = x(w) y(w_0) - :x(w) y(w_0)\) be the appropriate type of contraction with respect to (5.23) for \( x(w) \) and \( y(w_0) \) from (5.22).

**Lemma 16.** For \( |w| > |w_0|, 1 \leq i \leq l \) we have
\[
x_{ie}(w) x_\nu^a(w_0) \pm x_\nu^a(w) x_{ie}(w_0) = 4 \eta/(w - w_0)^2.
\] (5.27)

**Proof.** From (3.40), (3.44), or Lemma 15 we have for \( m, k \in Z + S, S = Z \) or \( S = Z' \), \( 1 \leq i, j \leq l \),
\[
x_{ie}(m), x_\nu^a(m - k) = 2(x_{ie}(m - k) + m \delta_{ij} \delta_{mk}).
\]
Using definitions (5.23) and (5.27) with \( j = i \) one finds that
\[
x_{ie}(w) x_\nu^a(w_0) \pm x_\nu^a(w) x_{ie}(w_0) = \sum_{m, k \in Z + S \atop m > -k} 4m \delta_{mk} (w - w_0)^{-m - w_0}.
\]

Then the result follows from the formulas
\[
\sum_{1 \leq i, j \leq l} p \left( \frac{w_0}{w} \right)^p = \frac{w w_0}{(w - w_0)^2},
\] (5.28)
and
\[
\sum_{0 \leq l \leq Z} \left(p + \frac{1}{2}\right) \left( \frac{w_0}{w} \right)^{p + 1/2} = \frac{(w + w_0) (w w_0)^{1/2}}{2 (w - w_0)^2}.
\] (5.29)

**Lemma 17.** For \( 1 \leq i, j \leq l, S = Z, Z' \), we have
\[
D_{ie}(w) \pm D_{ie}^a(w) = :e_\nu(w) \tilde{e}_\nu(w); + \left(1 + \frac{1}{2}\right) \left(1 + \text{sgn}(S)\right).
\]

**Proof.** The method of Lemma 12 gives this result using Lemmas 15 and 16 and
\[
\lim_{w \to w_0} \left( \tilde{\xi}_s - \eta \right) = \left(1 + \frac{1}{2}\right) \left(1 + \text{sgn}(S)\right).
\] (5.30)
Proposition 18. For $S = Z$ or $Z'$, we have

$$L^S_{\pm}(w) = - 4lD(w) \pm \frac{l}{2} : e_S(w) \tilde{e}_w e_S(w) : + \frac{l}{2} \left( \frac{1 + i}{4} \right) \text{sgn}(S).$$

Let us denote for $S = Z$ or $Z'$,

$$D^S_{\pm}(w) = \frac{1}{8} \left( : e_S(w) \tilde{e}_w e_S(w) : \pm \text{sgn}(S) \left( \frac{1 + i}{4} \right) \right)$$

$$= - \text{sgn}(S) \left( \frac{1 + i}{32} \right) \pm \sum_{k \in \mathbb{Z}} \sum_{n \in \mathbb{Z}} \left( n - \frac{k}{2} \right) e(n) e(k - n) w^{-k},$$  \hspace{1cm} (5.31)

so that for $S = Z$ or $Z'$

$$L^S_{\pm}(w) = - 4l(D(w) \pm D^S_{\pm}(w)) = - 4lD^S(w).$$  \hspace{1cm} (5.32)

Proposition 19. For $1 \leq j \leq l$, $k \in \mathbb{Z}$, $S = Z, Z'$, we have

(a) $[D^S_{\pm}(k), a_j(m)] = 0 = [D^S_{\pm}(k), a^*_j(m)]$ for $m \in \mathbb{Z}$,

(b)

$$[D^S_{\pm}(k), e(m)] = \begin{cases} (m + \frac{k}{2}) e(m + k) & \text{for } m \in S, \\ 0 & \text{for } m \in S', \end{cases}$$

(c) $[D^S_{\pm}(k), x(n)] = nx(n + k)$ for $x(w) = :u(w)v(w):$, $u, v \in \mathbb{A}$, $n \in \mathbb{Z}$, or for $x(w) = :u(w) \tilde{e}_w :$, $n \in \mathbb{Z} + S$,

(d) $[D^S_{\pm}(n), D^S_{\pm}(m)] = (n - k) D^S_{\pm}(n + k) + (l/12)(k^3 - k) \delta_{k, -n}$ for $n \in \mathbb{Z}$.

The generating function for the largest extension corresponding to $S = \frac{1}{2}Z$ is larger than (5.25) because of the root vectors coming from

$$x_{\pm}(w) = : e(S) e_{\pm} (w) :.$$  \hspace{1cm} (5.33)

The normal ordering $: e_{\pm}(m) e_{\pm}(n) ;$, $m, n \in \mathbb{Z} + \frac{1}{2}$, is the bosonic one (5.4) because of (3.50). Letting

$$D_{\pm}(w) = : x_{\pm}(w) x_{\pm}(w) ;,$$  \hspace{1cm} (5.34)

we find the generating function for $S = \frac{1}{2}Z$ to be

$$L^{1/2}_{\pm}(w) = L_{\pm}(w) + \frac{i}{3} \sum_{\ell = 1}^l (D_{\ell}(w) \pm D^S_{\ell}(w)) + \frac{i}{3} D_{\pm}(w).$$  \hspace{1cm} (5.35)

Lemma 20. For $|w| > |w_0|$, we have

$$x_{\pm}(w) x_{\pm}(w_0) = \pm 2 \frac{\xi_{\pm}}{w - w_0} : e_{\pm}(w) e_{\pm}(w_0) :$$

$$\pm \frac{\xi_{\pm}}{w - w_0} : e_{\pm}(w) e_{\pm}(w_0) : = \frac{4 \xi_{\pm}^2 \xi_{\mp}^2}{(w - w_0)^2}.$$  \hspace{1cm} (5.36)

From (3.50) or Lemma 20, we have

$$[x_{\pm}(m), x_{\pm}(-k)] = - 4m \delta_{mk},$$  \hspace{1cm} (5.36)

for $m, k \in \mathbb{Z} + \frac{1}{2}$, which gives

Lemma 21. For $|w| > |w_0|$, we have the bosonic contraction with respect to (5.4)

$$x_{\pm}(w) x_{\pm}(w_0) = - 4 \frac{(w + w_0)(w_0 w_0)}{2(w - w_0)^2} = - 4 \frac{\xi_{\pm}^2 \xi_{\mp}^2}{(w - w_0)^2}.$$  \hspace{1cm} (5.36)

Lemma 22. We have

$$D_{\pm}(w) = \pm (e_{\pm}(w) \tilde{e}_w e_{\pm}(w) : + e_{\pm}(w) \tilde{e}_w e_{\pm}(w) :).$$  \hspace{1cm} (5.36)

Proposition 23. We have

$$L^{1/2}_{\pm}(w) = (4l \pm 2)(D(w) + D^S_{\pm}(w) + D^{S'}_{\pm}(w)).$$

If we define $D^{1/2}_{\pm}(w) = D(w) + D^S_{\pm}(w) + D^{S'}_{\pm}(w)$ and $D_{\pm}^{1/2}(w) = D_{\pm}(w) + D^{S}_{\pm}(w)$, then we have an analog of Proposition 19 for $S = \frac{1}{2}Z$.

It is straightforward technical work to modify what we have done above for the general linear series. For example, in (5.3) we should use

$$x_{\pm}(w) = : a_{\pm}(w) a_{\pm}(w) ;,$$  \hspace{1cm} (5.37)

according to (3.21)–(3.23), and the normal orderings we use must be appropriate. The final results, however, are quite analogous to those we have obtained above.

We can use the above results to find the irreducible components of the representation spaces $V_{\pm}(Z')$, $V^{S'}_{\pm}(Z')$. We will follow the idea of the proof in [2]. From Lemma 13 we see that each $D_{\pm}(w)$ is a linear combination of the identity operator, $D_{\pm}^0(w)$, $D_{\pm}^0(w)$, and $D_{\pm}^0(w)$. But the components of these generating functions consist of operators from the affine algebra, so the components of $D_{\pm}(w)$ preserve irreducible subspaces of $V_{\pm}(Z')$.

The operators $D_{\pm}(w)$ provide a projective representation of the derivatives $d(n) = i^{n+1}(d/dt)$ according to Proposition 14. If we only adjoin the