operators $D_i(0)$, $1 \leqslant i \leqslant l$, then we get an abelian extension which gives us a group of operators on $V_\pm(Z^l)$ with generators

$$U_i(\tau) = \mathbf{e}(\tau D_i(0)), \qquad \tau \in \mathbb{R}, \tag{5.38}$$

where $\mathbf{e}(\tau) = \exp(2\pi i \tau)$. These still preserve each irreducible component of $V_\pm(Z^l)$. From Proposition 14(a), (b) we have

$$U_i(\tau)\, a_j(m)\, U_i(-\tau) = a_j(m)\, \mathbf{e}(\delta_{ij} m\tau); \tag{5.39}$$

and similarly for $a_j^*(m)$. It follows that for $x_{ij}(k)$, $i \neq j$, the operator

$$x_{ij}(k,\tau) = U_i(\tau)\, x_{ij}(k)\, U_i(-\tau) = \sum_{m \in Z} :a_i(m)\, a_j(k-m):\, \mathbf{e}(m\tau) \tag{5.40}$$

also preserves each irreducible component of $V_\pm(Z^l)$. The same is true of operators

$$x_{ij}^*(k,\tau) = \sum_{m \in Z} :a_i(m)\, a_j^*(k-m):\ \mathbf{e}(m\tau) \tag{5.41}$$

and

$$x_{i\,j}^{**}(k,\tau) = \sum_{m \in Z} :a_i^*(m)\, a_j^*(k-m):\, \mathbf{e}(m\tau) \tag{5.42}$$

for $i \neq j$. Let $x(k,\tau)$ be any one of these three types, and consider the operator

$$\int_0^1 x(k,\tau)\, \mathbf{e}(-m\tau)\, d\tau. \tag{5.43}$$

The representation space $V = V_\pm(Z^l)$ is graded into finite-dimensional subspaces under the action of d,

$$V = \sum_j V_j, \tag{5.44}$$

where V_j is the j-eigenspace of d. Then we see that

$$x(k,\tau) \cdot V_j \subset V_{j+k}; \tag{5.45}$$

so (5.43) does the same.

This shows that (5.43) is a well-defined operator on V, which preserves irreducible components of V. But (5.43) provides the Fourier coefficients of $x(k,\tau)$, that is, the individual terms $:a_i(m)\, a_j(k-m):$, $:a_i(m)\, a_j^*(k-m):$, and $:a_i^*(m)\, a_j^*(k-m):$ for $1 \leqslant i \neq j \leqslant l$, $k \in \mathbb{Z}$, $m \in Z$. This implies that any irreducible component of $V_\pm(Z^l)$ is preserved by the even part of $A_\pm(Z^{2l})$,

so that (3.17) is the irreducible decomposition of $V_\pm(Z^l)$. This argument also works in the general linear case with generating functions (3.21)–(3.23).

The situation for the extensions corresponding to $S = Z, Z'$ and $\frac{1}{2}\mathbb{Z}$ is slightly different. As before, we see that the operators

$$x_{ie}(k,\tau) = U_i(\tau)\, x_{ie}(k)\, U_i(-\tau), \tag{5.46}$$

$$x_{ie}^*(k,\tau) = U_i(\tau)\, x_{ie}^*(k)\, U_i(-\tau), \tag{5.47}$$

and

$$\int_0^1 x(k,\tau)\, \mathbf{e}(-m\tau)\, d\tau \tag{5.48}$$

preserve irreducible components of $V_\pm^S(Z^l)$. But (5.48) provides us with the terms $:a_i(m)\, e(k-m):$ and $:a_i^*(m)\, e(k-m):$ for $1 \leqslant i \leqslant l$, $m \in Z$, $k \in \mathbb{Z}$, $\mathbb{Z} + \frac{1}{2}$, or $\frac{1}{2}\mathbb{Z}$ depending on S. If $e(0)$ is present in any of these terms, then the even and odd subspaces of $V_\pm^S(Z^l)$ will not be irreducible. When $S = \frac{1}{2}\mathbb{Z}$, $e(0)$ is definitely present, but for $S = Z$ or Z' it is present when $0 \in S$. The same situation holds for the extensions in the general linear series.

THEOREM C. *Let $V_\pm^S(Z^l)$ be the appropriate representation space of an affine algebra from the orthogonal, symplectic, or general linear series, $S = \phi$, Z, Z', $\frac{1}{2}\mathbb{Z}$. Then $V_\pm^S(Z^l)$ is irreducible when $0 \in S$, but has two irreducible components (the even and odd subspaces) when $0 \notin S$. In each space the central element c acts as $+1$ in the fermionic constructions and -1 in the bosonic constructions.*

Finally, we wish to discuss the decomposition of the representation spaces $V_\pm(Z^l)$ under $\mathbf{gl}^{(1)}(l)$ which occurs inside each of $\mathbf{o}^{(1)}(2l)$, $\mathbf{sp}^{(1)}(2l)$, and $\mathbf{gl}^{(2)}(2l)$. It has as basis the components of $:u_1(w)\, v_2(w):$ for $u_1 \in \mathbf{a}_1$, $v_2 \in \mathbf{a}_2$ along with the identity operator. The irreducible components of $V_\pm(Z^l)$ under $\mathbf{gl}^{(1)}(l)$ must be preserved by the operators $a_i(n)\, a_j^*(m)$ for $1 \leqslant i, j \leqslant l$, $n, m \in Z$. From (4.3) we see that

$$[h_i(0), a_j(n)] = \delta_{ij} a_j(n)$$

and

$$[h_i(0), a_j^*(n)] = -\delta_{ij} a_j^*(n); \tag{5.49}$$

so that if we define

$$h(0) = \sum_{i=1}^{l} h_i(0), \tag{5.50}$$

then

$$[h(0), a_j(n)] = a_j(n) \qquad (5.51)$$

and

$$[h(0), a_j^*(n)] = -a_j^*(n). \qquad (5.52)$$

Also, we see that for each i, $1 \leqslant i \leqslant l$, we have

$$h_i(0) \cdot v_0 = \pm \left(\frac{1+\imath}{4}\right) v_0; \qquad (5.53)$$

so if $h(0)$ is applied to a monomial expression

$$v = a_{i_1}(n_1) \cdots a_{i_s}(n_s)\, a_{j_1}^*(m_1) \cdots a_{j_t}^*(m_t)\, v_0 \qquad (5.54)$$

from $V_\pm(Z^l)$, we obtain

$$h(0) \cdot v = \left(s - t \pm l\left(\frac{1+\imath}{4}\right)\right) v. \qquad (5.55)$$

Let us denote by $V_\pm^k(Z^l)$, $k \in \mathbb{Z}$, the eigenspace of vectors v satisfying (5.55) with $s - t = k$. Each of these is clearly preserved and irreducible under the operators $a_i(n)\, a_j^*(m)$.

THEOREM D. *The representation space $V_\pm(Z^l)$ has the irreducible decomposition*

$$V_\pm(Z^l) = \sum_{k \in \mathbf{Z}} V_\pm^k(Z^l) \qquad (5.56)$$

under $\mathbf{gl}^{(1)}(l)$, where $V_\pm^k(Z^l)$ is the $(k \pm l(1+\imath)/4)$-eigenspace of the operator $h(0) = \pm \sum_{i=1}^{l} \sum_{n \in \mathbf{Z}} :a_i(n)\, a_i^*(-n):$ from the Cartan subalgebra.

6. Identification of Irreducible Representations and Their Characters

To identify the irreducible representations we have constructed we must first discuss the fundamental weights and the simple root vectors for each algebra. The Cartan subalgebra $\hat{\mathbf{h}}$ is the same in each series, having basis $\{h_1(0), ..., h_l(0), c, d\}$. These elements of the affine algebra are represented by the operators

$$h_i(0) = \pm \sum_{n \in \mathbf{Z}} :a_i(n)\, a_i^*(-n):, \qquad 1 \leqslant i \leqslant l, \qquad (6.1)$$

$$c = \pm 1, \qquad (6.2)$$

$$D(0) = \mp l\left(\frac{1+\imath}{16}\right) \pm \sum_{i=1}^{l} \sum_{n \in \mathbf{Z}} n{:}a_i(n)\, a_i^*(-n){:} \qquad (6.3)$$

on $V_\pm(Z^l)$. In the extensions $V_\pm^S(Z^l)$ for $S = Z, Z', \text{ or } \frac{1}{2}\mathbb{Z}$, $D(0)$ is replaced by

$$D^S(0) = D(0) \mp \tfrac{1}{4} \sum_{n \in S} n{:}e(n)\, e(-n){:}. \qquad (6.4)$$

Note that we have for vacuum vector v_0,

$$D(0) \cdot v_0 = D^S(0) \cdot v_0 = \mp l\left(\frac{1+\imath}{16}\right) v_0, \qquad (6.5)$$

which together with (5.53) and (6.2) says that v_0 has weight $\pm\bar{c} \mp l((1+\imath)/16)\,\bar{d} \pm ((1+\imath)/4)(\varepsilon_1 + \cdots + \varepsilon_l)$ in terms of basis (4.2) of $\hat{\mathbf{h}}^*$.

The fundamental weights

$$\{\hat{\omega}_0, \hat{\omega}_1, ..., \hat{\omega}_l\} \qquad (6.6)$$

are elements of $\hat{\mathbf{h}}^*$ determined up to a multiple of $\bar{d}$ by the conditions

$$\frac{2\langle \hat{\omega}_i, \alpha_j \rangle}{\langle \alpha_j, \alpha_j \rangle} = \delta_{ij}, \qquad 0 \leqslant i, j \leqslant l. \qquad (6.7)$$

We give in Table X the fundamental weights for the algebras from Tables VII–IX.

TABLE X

Fundamental Weights

$A_{l-1}^{(1)}$	$(n_0, ..., n_{l-1}) = (1, 1, ..., 1)$ with $\omega_1, ..., \omega_{l-1}$ of type A_{l-1}
$D_l^{(1)}$	$(n_0, ..., n_l) = (1, 1, 2, 2, ..., 2, 1, 1)$ with $\omega_1, ..., \omega_l$ of type D_l
$B_l^{(1)}$	$(n_0, ..., n_l) = (1, 1, 2, 2, ..., 2, 2, 1)$ with $\omega_1, ..., \omega_l$ of type B_l
$B_l^{(2)}$	$(n_0, ..., n_l) = (1, 2, 2, 2, ..., 2, 1, 1)$ with $\omega_1, ..., \omega_l$ of type D_l
$D_{l+1}^{(2)}$	$(n_0, ..., n_l) = (1, 2, 2, 2, ..., 2, 2, 1)$ with $\omega_1, ..., \omega_l$ of type B_l
$C_l^{(1)}$	$(n_0, ..., n_l) = (2, 2, 2, 2, ..., 2, 2, 2)$ with $\omega_1, ..., \omega_l$ of type C_l
$B^{(1)}(0, l)$	$(n_0, ..., n_l) = (2, 2, 2, 2, ..., 2, 2, 1)$ with $\omega_1, ..., \omega_l$ of type $B(0, l)$
$B^{(2)}(0, l)$	$(n_0, ..., n_l) = (1, 2, 2, 2, ..., 2, 2, 2)$ with $\omega_1, ..., \omega_l$ of type C_l
$C^{(2)}(l+1)$	$(n_0, ..., n_l) = (1, 2, 2, 2, ..., 2, 2, 1)$ with $\omega_1, ..., \omega_l$ of type $B(0, l)$
$A_{2l-1}^{(2)}(a)$	$(n_0, ..., n_l) = (1, 1, 2, 2, ..., 2, 2, 2)$ with $\omega_1, ..., \omega_l$ of type C_l
$A_{2l-1}^{(2)}(b)$	$(n_0, ..., n_l) = (2, 2, 2, 2, ..., 2, 1, 1)$ with $\omega_1, ..., \omega_l$ of type D_l
$A^{(2)}(0, 2l-1)$	$(n_0, ..., n_l) = (1, 1, 2, 2, ..., 2, 2, 1)$ with $\omega_1, ..., \omega_l$ of type $B(0, l)$
$A^{(4)}(0, 2l-1)$	$(n_0, ..., n_l) = (1, 2, 2, 2, ..., 2, 1, 1)$ with $\omega_1, ..., \omega_l$ of type D_l
$A_{2l}^{(4)}$	$(n_0, ..., n_l) = (1, 2, 2, 2, ..., 2, 2, 2)$ with $\omega_1, ..., \omega_l$ of type C_l
$A_{2l}^{(2)}$	$(n_0, ..., n_l) = (2, 2, 2, 2, ..., 2, 2, 1)$ with $\omega_1, ..., \omega_l$ of type B_l
$A^{(4)}(0, 2l)(a)$	$(n_0, ..., n_l) = (1, 2, 2, 2, ..., 2, 2, 1)$ with $\omega_1, ..., \omega_l$ of type $B(0, l)$
$A^{(4)}(0, 2l)(b)$	$(n_0, ..., n_l) = (1, 2, 2, 2, ..., 2, 2, 1)$ with $\omega_1, ..., \omega_l$ of type B_l

TABLE XI

Identification of Irreducible Representation Spaces for the Orthogonal Series

$D_l^{(1)}$	$V_+^0(\mathbb{Z}^l), v_0, \hat\omega_l - (l/8)\bar d$
	$V_+^1(\mathbb{Z}^l), a_l^*(0)\, v_0, \hat\omega_{l-1} - (l/8)\bar d$
	$V_+^0((\mathbb{Z}+\frac12)^l), v_0, \hat\omega_0$
	$V_+^1((\mathbb{Z}+\frac12)^l), a_1(-\frac12)\, v_0, \hat\omega_1 - \frac12\bar d$
$B_l^{(1)}$	$V_+^{\mathbb{Z}}(\mathbb{Z}^l), v_0, \hat\omega_l - (l/8)\bar d$
$S=\mathbb{Z}$	$V_+^{\mathbb{Z}+1/2,0}((\mathbb{Z}+\frac12)^l), v_0, \hat\omega_0$
	$V_+^{\mathbb{Z}+1/2,1}((\mathbb{Z}+\frac12)^l), a_1(-\frac12)\, v_0, \hat\omega_1 - \frac12\bar d$
$B_l^{(2)}$	$V_+^{\mathbb{Z}+1/2,0}(\mathbb{Z}^l), v_0, \hat\omega_l - (l/8)\bar d$
$S=\mathbb{Z}'$	$V_+^{\mathbb{Z}+1/2,1}(\mathbb{Z}^l), a_l^*(0)\, v_0, \hat\omega_{l-1} - (l/8)\bar d$
	$V_+^{\mathbb{Z}}((\mathbb{Z}+\frac12)^l), v_0, \hat\omega_0$
$D_{l+1}^{(2)}$	$V_+^{1/2\mathbb{Z}}(\mathbb{Z}^l), v_0, \hat\omega_l - (l/8)\bar d$
$S=\frac12\mathbb{Z}$	$V_+^{1/2\mathbb{Z}}((\mathbb{Z}+\frac12)^l), v_0, \hat\omega_0$

We write

$$\hat\omega_0 = n_0\bar c, \qquad \hat\omega_i = n_i\bar c + \omega_i, \quad 1 \leqslant i \leqslant l, \tag{6.8}$$

where $\omega_1,\ldots,\omega_l$ are the fundamental weights of a finite-dimensional algebra from Table V.

From (2.10) and the choices of simple roots in Tables VII–IX, we get the simple root vectors for each algebra. These allow us to find the highest

TABLE XII

Identification of Irreducible Representation Spaces for the Symplectic Series

$C_l^{(1)}$	$V_-^0(\mathbb{Z}^l), v_0, -\frac12\hat\omega_l + (l/8)\bar d$
	$V_-^1(\mathbb{Z}^l), a_l^*(0)\, v_0, \hat\omega_{l-1} - \frac32\hat\omega_l + (l/8)\bar d$
	$V_-^0((\mathbb{Z}+\frac12)^l), v_0, -\frac12\hat\omega_0$
	$V_-^1((\mathbb{Z}+\frac12)^l), a_1(-\frac12)\, v_0, -\frac32\hat\omega_0 + \hat\omega_1 + \frac12\bar d$
$B^{(1)}(0,l)$	$V_-^{\mathbb{Z}}(\mathbb{Z}^l), v_0, -\hat\omega_l + (l/8)\bar d$
$S=\mathbb{Z}$	$V_-^{\mathbb{Z}+1/2,0}((\mathbb{Z}+\frac12)^l), v_0, -\frac12\hat\omega_0$
	$V_-^{\mathbb{Z}+1/2,1}((\mathbb{Z}+\frac12)^l), a_1(-\frac12)\, v_0, -\frac32\hat\omega_0 + \hat\omega_1 + \frac12\bar d$
$B^{(2)}(0,l)$	$V_-^{\mathbb{Z}+1/2,0}(\mathbb{Z}^l), v_0, -\frac12\hat\omega_l + (l/8)\bar d$
$S=\mathbb{Z}'$	$V_-^{\mathbb{Z}+1/2,1}(\mathbb{Z}^l), a_l^*(0)\, v_0, \hat\omega_{l-1} - \frac32\hat\omega_l + (l/8)\bar d$
	$V_-^{\mathbb{Z}}((\mathbb{Z}+\frac12)^l), v_0, -\hat\omega_0$
$C^{(2)}(l+1)$	$V_-^{1/2\mathbb{Z}}(\mathbb{Z}^l), v_0, -\hat\omega_l + (l/8)\bar d$
$S=\frac12\mathbb{Z}$	$V_-^{1/2\mathbb{Z}}((\mathbb{Z}+\frac12)^l), v_0, -\hat\omega_0$

TABLE XIII

Identification of Irreducible Representation Spaces for the General Linear Series

$A_{2l-1}^{(2)}$	$V_-^0(\mathbb{Z}^l), v_0, -\frac12\hat\omega_l + (l/8)\bar d$
$\pm 1 = -1$	$V_-^1(\mathbb{Z}^l), a_l^*(0)\, v_0, \hat\omega_{l-1} - \frac32\hat\omega_l + (l/8)\bar d$
	$V_+^0((\mathbb{Z}+\frac12)^l), v_0, \hat\omega_0$
	$V_+^1((\mathbb{Z}+\frac12)^l), a_1(-\frac12)\, v_0, \hat\omega_1 - \frac12\bar d$
$A_{2l-1}^{(2)}$	$V_+^0(\mathbb{Z}^l), v_0, \hat\omega_l - (l/8)\bar d$
$\pm 1 = 1$	$V_+^1(\mathbb{Z}^l), a_l^*(0)\, v_0, \hat\omega_{l-1} - (l/8)\bar d$
	$V_-^0((\mathbb{Z}+\frac12)^l), v_0, -\frac12\hat\omega_0$
	$V_-^1((\mathbb{Z}+\frac12)^l), a_1(-\frac12)\, v_0, -\frac32\hat\omega_0 + \hat\omega_1 + \frac12\bar d$
$A^{(2)}(0, 2l-1)$	$V_+^{\mathbb{Z}+1/2,0}((\mathbb{Z}+\frac12)^l), v_0, \hat\omega_0$
$S=\mathbb{Z}$	$V_+^{\mathbb{Z}+1/2,1}((\mathbb{Z}+\frac12)^l), a_1(-\frac12)\, v_0, \hat\omega_1 - \frac12\bar d$
	$V_-^{\mathbb{Z}}(\mathbb{Z}^l), v_0, -\hat\omega_l + (l/8)\bar d$
$A^{(4)}(0, 2l-1)$	$V_+^{\mathbb{Z}+1/2,0}(\mathbb{Z}^l), v_0, \hat\omega_l - (l/8)\bar d$
$S=\mathbb{Z}'$	$V_+^{\mathbb{Z}+1/2,1}(\mathbb{Z}^l), a_l^*(0)\, v_0, \hat\omega_{l-1} - (l/8)\bar d$
	$V_-^{\mathbb{Z}}((\mathbb{Z}+\frac12)^l), v_0, -\hat\omega_0$
$A_{2l}^{(2)}$	$V_+^{\mathbb{Z}}(\mathbb{Z}), v_0, \hat\omega_l - (l/8)\bar d$
$S=\mathbb{Z}$	$V_-^{\mathbb{Z}+1/2,0}((\mathbb{Z}+\frac12)^l), v_0, -\frac12\hat\omega_0$
	$V_-^{\mathbb{Z}+1/2,1}((\mathbb{Z}+\frac12)^l), a_1(-\frac12)\, v_0, -\frac32\hat\omega_0 + \hat\omega_1 + \frac12\bar d$
$A_{2l}^{(4)}$	$V_+^{\mathbb{Z}}((\mathbb{Z}+\frac12)^l), v_0, \hat\omega_0$
$S=\mathbb{Z}'$	$V_-^{\mathbb{Z}+1/2,0}(\mathbb{Z}^l), v_0, -\frac12\hat\omega_l + (l/8)\bar d$
	$V_-^{\mathbb{Z}+1/2,1}(\mathbb{Z}^l), a_l^*(0)\, v_0, \hat\omega_{l-1} - \frac32\hat\omega_l + (l/8)\bar d$
$A^{(4)}(0, 2l)$	$V_+^{1/2\mathbb{Z}}(\mathbb{Z}^l), v_0, \hat\omega_l - (l/8)\bar d$
$S=\frac12\mathbb{Z}$	$V_+^{1/2\mathbb{Z}}((\mathbb{Z}+\frac12)^l), v_0, \hat\omega_0$
	$V_-^{1/2\mathbb{Z}}(\mathbb{Z}^l), v_0, -\hat\omega_l + (l/8)\bar d$
	$V_-^{1/2\mathbb{Z}}((\mathbb{Z}+\frac12)^l), v_0, -\hat\omega_0$

weight vector in each irreducible representation space. The highest weight vectors and their weights expressed in terms of the fundamental weights (6.6) are listed in Tables XI–XIV. These uniquely identify the representations we have constructed.

Finally, we give the homogeneous characters of the representation spaces we have constructed. For each affine algebra $\hat{\mathbf{g}}$, let $\mathbf{g}$ be the finite-dimensional scalar subalgebra consisting of the zero-level elements $x(0) \in \hat{\mathbf{g}}$. If V is a $\hat{\mathbf{g}}$-module we decompose it into k-eigenspaces V_k under the action of derivation d, and then write

$$\operatorname{ch}(V) = \sum_k \operatorname{ch}(V_k)\, q^{-k}. \tag{6.9}$$

TABLE XIV. Identification of Irreducible Representation Spaces for $A_{l-1}^{(1)}$

$A_{l-1}^{(1)} \subset D_l^{(1)}$ or $A_{l-1}^{(1)} \subset A_{2l-1}^{(2)}$ Fermionic	$V_+^k(\mathbb{Z}^l)$ for $k \in \mathbb{Z}$;
	$k=0, v_0, \lambda_0 = \bar{c} + \frac{1}{2}(\varepsilon_1 + \cdots + \varepsilon_l) - (l/8)\,\bar{d}$
	$k=1, a_1(-1)\,v_0, \lambda_1 = \lambda_0 + \varepsilon_1 - \bar{d}$
	$k=2, a_2(-1)\,a_1(-1)\,v_0, \lambda_2 = \lambda_1 + \varepsilon_2 - \bar{d}$
	$\vdots$
	$k=l, a_l(-1) \cdots a_2(-1)\,a_1(-1)\,v_0, \lambda_l = \lambda_{l-1} + \varepsilon_l - \bar{d}$
	$k=l+1, a_1(-2)\,a_l(-1) \cdots a_1(-1)\,v_0, \lambda_{l+1} = \lambda_l + \varepsilon_1 - 2\bar{d}$
	$\vdots$
	$k=2l, a_l(-2) \cdots a_1(-2)\,a_l(-1) \cdots a_1(-1)\,v_0,$
	$\lambda_{2l} = \lambda_{2l-1} + \varepsilon_l - 2\bar{d}$ etc.
	$k=-1, a_l^*(0)\,v_0, \lambda_{-1} = \lambda_0 - \varepsilon_l$
	$k=-2, a_{l-1}^*(0)\,a_l^*(0)\,v_0, \lambda_{-2} = \lambda_{-1} - \varepsilon_{l-1}$
	$\vdots$
	$k=-l, a_1^*(0) \cdots a_{l-1}^*(0)\,a_l^*(0)\,v_0, \lambda_{-l} = \lambda_{-(l-1)} - \varepsilon_1$
	$k=-l-1, a_l^*(-1)\,a_1^*(0) \cdots a_l^*(0)\,v_0, \lambda_{-l-1} = \lambda_{-l} - \varepsilon_l - \bar{d}$
	$\vdots$
	$k=-2l, a_1^*(-1) \cdots a_l^*(-1)\,a_1^*(0) \cdots a_l^*(0)\,v_0,$
	$\lambda_{-2l} = \lambda_{-2l+1} - \varepsilon_1 - \bar{d}$ etc.
	$V_+^k((\mathbb{Z}+\frac{1}{2})^l)$ for $k \in \mathbb{Z}$;
	$k=0, v_0, \lambda_0 = \bar{c}$
	$k=1, a_1(-\frac{1}{2})\,v_0, \lambda_1 = \lambda_0 + \varepsilon_1 - \frac{1}{2}\bar{d}$
	$k=2, a_2(-\frac{1}{2})\,a_1(-\frac{1}{2})\,v_0, \lambda_2 = \lambda_1 + \varepsilon_2 - \frac{1}{2}\bar{d}$
	$\vdots$
	$k=l, a_l(-\frac{1}{2}) \cdots a_1(-\frac{1}{2})\,v_0, \lambda_l = \lambda_{l-1} + \varepsilon_l - \frac{1}{2}\bar{d}$
	$k=l+1, a_1(-\frac{3}{2})\,a_l(-\frac{1}{2}) \cdots a_1(-\frac{1}{2})\,v_0,$
	$\lambda_{l+1} = \lambda_l + \varepsilon_1 - \frac{3}{2}\bar{d}$
	$\vdots$
	$k=2l, a_l(-\frac{3}{2}) \cdots a_1(-\frac{3}{2})\,a_l(-\frac{1}{2}) \cdots a_1(-\frac{1}{2})\,v_0,$
	$\lambda_{2l} = \lambda_{2l-1} + \varepsilon_l - \frac{3}{2}\bar{d}$ etc.
	$k=-1, a_l^*(-\frac{1}{2})\,v_0, \lambda_{-1} = \lambda_0 - \varepsilon_l - \frac{1}{2}\bar{d}$
	$k=-2, a_{l-1}^*(-\frac{1}{2})\,a_l^*(-\frac{1}{2})\,v_0, \lambda_{-2} = \lambda_{-1} - \varepsilon_{l-1} - \frac{1}{2}\bar{d}$
	$\vdots$
	$k=-l, a_1^*(-\frac{1}{2}) \cdots a_l^*(-\frac{1}{2})\,v_0, \lambda_{-l} = \lambda_{-(l-1)} - \varepsilon_1 - \frac{1}{2}\bar{d}$
	$k=-l-1, a_l^*(-\frac{3}{2})\,a_1^*(-\frac{1}{2}) \cdots a_l^*(-\frac{1}{2})\,v_0,$
	$\lambda_{-l-1} = \lambda_{-l} - \varepsilon_l - \frac{3}{2}\bar{d}$
	$\vdots$
	$k=-2l, a_1^*(-\frac{3}{2}) \cdots a_l^*(-\frac{3}{2})\,a_1^*(-\frac{1}{2}) \cdots a_l^*(-\frac{1}{2})\,v_0,$
	$\lambda_{-2l} = \lambda_{-2l+1} - \varepsilon_1 - \frac{3}{2}\bar{d}$ etc.
$A_{l-1}^{(1)} \subset C_l^{(1)}$ or $A_{l-1}^{(1)} \subset A_{2l-1}^{(2)}$ Bosonic	$V_-^k(\mathbb{Z}^l)$ for $k \in \mathbb{Z}$;
	$k=0, v_0, \lambda_0 = -\bar{c} - \frac{1}{2}(\varepsilon_1 + \cdots + \varepsilon_l) + (l/8)\,\bar{d}$
	$k \geqslant 1, a_l^*(0)^k\,v_0, \lambda_k = \lambda_0 - k\varepsilon_l$
	$k \leqslant -1, a_1(-1)^{-k}\,v_0, \lambda_k = \lambda_0 - k(\varepsilon_1 - \bar{d})$
	$V_-^k((\mathbb{Z}+\frac{1}{2})^l)$ for $k \in \mathbb{Z}$;
	$k=0, v_0, \lambda_0 = -\bar{c}$
	$k \geqslant 1, a_l^*(-\frac{1}{2})^k\,v_0, \lambda_k = \lambda_0 - k(\varepsilon_l + \frac{1}{2}\bar{d})$
	$k \leqslant -1, a_1(-\frac{1}{2})^{-k}\,v_0, \lambda_k = \lambda_0 - k(\varepsilon_1 - \frac{1}{2}\bar{d})$
	These are with respect to simple root vectors:
	$\alpha_0 \leftrightarrow \sum_{n \in \mathbb{Z}} :a_l(-n)\,a_1^*(1+n):$
	$\alpha_i \leftrightarrow \sum_{n \in \mathbb{Z}} :a_i(-n)\,a_{i+1}^*(n):, 1 \leqslant i \leqslant l-1$

By $\operatorname{ch}(V_k)$, we mean the $\mathbf{g}$-character of V_k, that is,

$$\operatorname{ch}(V_k) = \sum_{\mu} \dim(V_{k,\mu})\, e^{\mu}, \tag{6.10}$$

where μ runs through the weight lattice of $\mathbf{g}$ and $V_{k,\mu}$ is the μ-weight space of V_k. Using the formula

$$\operatorname{ch}(V_1 \otimes V_2) = \operatorname{ch}(V_1)\operatorname{ch}(V_2), \tag{6.11}$$

we obtain the characters

$$\operatorname{ch}(V_{\pm}(\mathbb{Z}^l)) = q^{\pm l/8} e^{r\omega_l} \prod_{i=1}^{l} \prod_{n=1}^{\infty} (1 \pm q^n e^{\varepsilon_i})^{\pm 1} (1 \pm q^{n-1} e^{-\varepsilon_i})^{\pm 1}, \tag{6.12}$$

where $r = +1$ in the fermionic case and $-\frac{1}{2}$ in the bosonic case, and

$$\operatorname{ch}(V_{\pm}((\mathbb{Z}+\tfrac{1}{2})^l)) = \prod_{i=1}^{l} \prod_{n=1}^{\infty} (1 \pm q^{n-1/2} e^{\varepsilon_i})^{\pm 1} (1 \pm q^{n-1/2} e^{-\varepsilon_i})^{\pm 1}. \tag{6.13}$$

The characters of the extensions $V_{\pm}^S(Z^l)$ are determined by the fact that

$$\operatorname{ch}(V(S)) = \prod_{0<n \in S} (1 + q^n) \tag{6.14}$$

in the nontwisted cases, and

$$\operatorname{ch}(V(S)) = \prod_{0<n \in S} (1 \pm q^n)^{\pm 1} \tag{6.15}$$

in the twisted cases, where $\pm$ is $+$ for $n \in \mathbb{Z}$, and $-$ for $n \in \mathbb{Z} + \frac{1}{2}$.

Some of the fermionic representations also admit another construction known as the vertex representation [3]. Comparison of the characters for these representations implies different variations of the Jacobi triple product identity (see, e.g., [2]).

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