# Matrix Realizations of Hyperbolic Triangle Groups

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For her senior year in high school, the first author signed up to take the newly created, Science Research Class. This class was designed for students with a passion for a specific subject area, and with the desire to discover more about it. Elizabeth's passion is mathematics. The class instructors, with the help of Prof. Kenneth McLeod, State University of New York at Binghamton, paired each student with a SUNY-B professor. The two authors of this article were paired in that way and decided to work on a specific mathematical project. For this project, we studied some non-Euclidean (hyperbolic) geometry and group theory (the algebra of symmetry). M.C. Escher's ([1]) intriguing pictures of hyperbolic geometry ([3]) inside of a circle motivated us to study these subjects, which go far beyond the usual Euclidean geometry and algebra learned in high school. Looking past the artistic decorations of Escher's *Circle Limit IV*, for example, one sees a tessellation of the hyperbolic plane into congruent triangles. The interior angles, $\frac{\pi}{r}$, $\frac{\pi}{s}$, $\frac{\pi}{t}$, of each triangle add up to less than $\pi$ radians.

Through reflections in the sides of such triangles as mentioned above, several examples of groups can be generated. We studied such hyperbolic triangle groups, denoted by $T(r, s, t)$, through the utilization of $2 \times 2$ matrices and the corresponding linear fractional maps of the complex numbers. Our goal was to discover the three generators of the hyperbolic triangle groups

as explicit $2 \times 2$ matrices using basic geometry and some calculus. In this article we show how this was accomplished for the specific examples $T(3,4,3)$, $T(3,4,4)$, and $T(3,4,5)$.

## Definition of a Group

A *group* $G$ is a set closed under an operation, denoted by $\bullet$, that is, $\bullet : G \times G \to G$ and $G \times G = \{(g_1, g_2) \mid g_1, g_2 \in G\}$, such that the following properties are satisfied: the operation is associative, "There exists an identity element" $\exists e \in G$ such that $e \bullet g = g = g \bullet e$, "for all" $\forall g \in G$, $\exists g^{-1} \in G$ such that $g \bullet g^{-1} = e = g^{-1} \bullet g$. Some examples of sets are: the natural numbers $\mathbb{N} = \{0, 1, 2, 3, \cdots\}$, the integers $\mathbb{Z} = \{\cdots, -2, -1, 0, 1, 2, \cdots\}$, the even integers $2\mathbb{Z} = \{2n \mid n \in \mathbb{Z}\} = \{m \in \mathbb{Z} \mid m \ is \ even\}$, the rational numbers $\mathbb{Q} = \{\frac{m}{n} \mid m, n \in \mathbb{Z}, n \neq 0\}$, and the real numbers $\mathbb{R} = \{m.d_1 d_2 d_3 \cdots \mid m \in \mathbb{Z}, 0 \leq d_i \leq 9, 1 \leq i \in \mathbb{N}\}$. Under the operation of addition, $+$, all of these examples, except $\mathbb{N}$ are groups. Each group has its own notation which includes its operation and identity element, and in general the notation is $(G, \bullet, e)$. Some specific examples include the integers under addition $(\mathbb{Z}, +, 0)$ and the nonzero rational numbers under multiplication $(\mathbb{Q} - \{0\}, \cdot, 1)$.

## Group Acting on a Set

A group may act on a set as follows. Let $(G, \cdot, e)$ be a group and $S$ be a set. $G$ will act on $S$ when we have a function $\bullet : G \times S \to S$. The domain is the set of ordered pairs, the Cartesian product $G \times S = \{(g, s) \mid g \in G, s \in S\}$ and we write the function as $\bullet(g, s) = g \bullet s$. For $G$ to act on $S$, the following conditions must be satisfied: 1. $\forall g_1, g_2 \in G$, $\forall s \in S$, we have $(g_1 \bullet (g_2 \bullet s)) = (g_1 \cdot g_2) \bullet s$, and 2. $\forall s \in S$, $e \bullet s = s$. For example, $G = \mathbb{Z} \times \mathbb{Z}$ is a group where the operation addition is given by $(m_1, n_1) + (m_2, n_2) = (m_1 + m_2, n_1 + n_2)$ so the identity element is $(0, 0)$. Also, we have the additive inverse of $(m, n)$ is $(-m, -n) \in \mathbb{Z} \times \mathbb{Z}$ since $(m, n) + (-m, -n) = (0, 0)$. This operation is

associative

$$\begin{aligned}((m_1,n_1)+(m_2,n_2))+(m_3,n_3) &= ((m_1+m_2)+m_3,(n_1+n_2)+n_3)\\ =(m_1+(m_2+m_3),n_1+(n_2+n_3)) &= (m_1,n_1)+((m_2,n_2)+(m_3,n_3))\end{aligned}$$

because of the associative property of addition in $\mathbb{Z}$. $G=\mathbb{Z}\times\mathbb{Z}$ acts on $S=\mathbb{R}\times\mathbb{R}$ by:

$$(m,n)\bullet(x,y)=(m+x,n+y).$$

This is an action because it satisfies

$$\begin{aligned}(m_1,n_1)\bullet((m_2,n_2)\bullet(x,y)) &= (m_1,n_1)\bullet(m_2+x,n_2+y)\\ =(m_1+(m_2+x),n_1+(n_2+y)) &= ((m_1+m_2)+x),(n_1+n_2)+y))\\ =(m_1+m_2,n_1+n_2)\bullet(x,y) &= ((m_1,n_1)+(m_2,n_2))\bullet(x,y)\end{aligned}$$

and $(0,0)\bullet(x,y)=(0+x,0+y)=(x,y)$. Another example of a group acting on a set is the group of symmetries of a Euclidean equilateral triangle where the group operation is composition of functions. This group consists of flips (mirror reflections) and rotations of 120 degrees, acting on the triangle. Doing a sequence of flips and/or rotations in different orders can give different results.

## Equivalence Relation on a Set, and Equivalence Classes

We now discuss what it means to have a relation on a set. Let $S$ be a set, then a *relation* on $S$ is a subset $R\subseteq S\times S$ so we say $s_1,s_2\in S$ are $R$ related if $(s_1,s_2)\in R$. A notation for this relation would be $s_1\sim s_2$ when $(s_1,s_2)\in R$. For example, "Equality", denoted $=$, is the relation $E=\{(s,s)\mid s\in S\}$. Three possible properties of a relation are as follows: $\forall s_1,s_2,s_3\in S$, 1. Reflexive: $s_1\sim s_1$, 2. Symmetric: if $s_1\sim s_2$ then $s_2\sim s_1$, 3. Transitive: if $s_1\sim s_2$ and $s_2\sim s_3$ then $s_1\sim s_3$ . A relation is an *equivalence relation* when it is reflexive, symmetric, and transitive. Given a set with an equivalence relation $\sim$, define the *equivalence class* of an element $s\in S$ to be $[s]=\{x\in S\mid x\sim s\}$, and note that all elements of an equivalence

class are $\sim$ related to each other. Then any two equivalence classes are either identical or disjoint, and $S$ is the disjoint union of its distinct equivalence classes.

## Orbits of a Group Acting on a Set as Equivalence Classes

Suppose group $G$ acts on set $S$. There is an equivalence relation $\sim$ defined on $S$ by saying $s_1 \sim s_2$ when $g \bullet s_1 = s_2$ for some $g \in G$. Under that relation an equivalence class is also called an *orbit*, and the orbit of $s$ is denoted by $O_s = \{g \bullet s \in S \mid g \in G\}$. Any two elements in the same orbit are related to each other. An example of an orbit, where $G = \mathbb{Z} \times \mathbb{Z}$ acting on $S = \mathbb{R} \times \mathbb{R}$, is

$$\begin{aligned} O_{(0,0)} &= \{(m,n) \bullet (0,0) \mid m,n \in \mathbb{Z}\} = \{(m,n) \mid (m,n) \in \mathbb{Z} \times \mathbb{Z}\} \\ &= \mathbb{Z} \times \mathbb{Z} = O_{(a,b)}, \quad \forall a,b \in \mathbb{Z}, \end{aligned}$$

and another orbit is

$$O_{(1/2,3/4)} = \{(\frac{1}{2} + m, \frac{3}{4} + n) \mid m,n \in \mathbb{Z}\} = O_{(a+1/2,b+3/4)}, \quad \forall a,b \in \mathbb{Z}.$$

Denote by $S/\sim$ the set of equivalence classes, so $S/\sim = \{O_s \mid s \in S\}$ is the set of orbits. A *fundamental domain* is a subset of $S$ consisting of exactly one element from each orbit. For $G = \mathbb{Z} \times \mathbb{Z}$ acting on $S = \mathbb{R} \times \mathbb{R}$, a fundamental domain is $\{(x,y) \mid 0 \leq x < 1, 0 \leq y < 1\}$, so that $S/\sim$ is a torus $T^2 = S^1 \times S^1$.

## Complex Numbers

A group that is significant to the project is the *complex numbers*, $\mathbb{C} = \{z = a + b\mathbf{i} \mid a,b \in \mathbb{R}\}$. The *real part* of $z$ is $Re(z) = a$ and the *imaginary part* of $z$ is $Im(z) = b$. The complex numbers have operations $+$ and $\cdot$ defined by :

$$\begin{aligned} (a + b\mathbf{i}) + (c + d\mathbf{i}) &= (a+c) + (b+d)\mathbf{i} \\ (a + b\mathbf{i}) \cdot (c + d\mathbf{i}) &= (ac - bd) + (ad + bc)\mathbf{i}. \end{aligned}$$

These equations follow from the usual algebraic laws obeyed by these operations, so they are commutative, associative, and distributive, and $\mathbf{i}^2 = -1$. The group of real numbers is a subset of the group of complex numbers ($\mathbb{R} \subset \mathbb{C}$). A *complex conjugation* can be defined as the function $- : \mathbb{C} \to \mathbb{C}$ where its value on $z$ is $\bar{z} = a - bi$. So $z \cdot \bar{z} = (a + b\mathbf{i}) \cdot (a - b\mathbf{i}) = a^2 + b^2 \geq 0$ and $z \cdot \bar{z} = 0$ iff $z = 0$. When $z \neq 0$ then $z \cdot \left(\frac{\bar{z}}{a^2+b^2}\right) = \frac{z \cdot \bar{z}}{a^2+b^2} = 1$, so $z^{-1} = \frac{\bar{z}}{a^2+b^2} \in \mathbb{C}$.

## 2x2 Matrices, Determinants, and Matrix Multiplication

Certain groups of $2 \times 2$ *matrices* will be used in this project. Let

$$\mathbb{R}_{2\times 2} = \left\{ A = \begin{bmatrix} a & b \\ c & d \end{bmatrix} \;\middle|\; a, b, c, d \in \mathbb{R} \right\}$$

be the set of all $2 \times 2$ real matrices. For any such $A$, the *determinant* of $A$ is defined to be $\det A = ad - bc$. The *special linear group* is

$$SL(2, \mathbb{R}) = \{A \in \mathbb{R}_{2\times 2} \mid \det\ A = 1\}.$$

The *general linear group* is

$$GL(2, \mathbb{R}) = \{A \in \mathbb{R}_{2\times 2} \mid \det\ A \neq 0\}.$$

In between these two are the groups

$$SL^{\pm 1}(2, \mathbb{R}) = \{A \in \mathbb{R}_{2\times 2} \mid \det\ A = \pm 1\}$$

and

$$GL^{>0}(2, \mathbb{R}) = \{A \in \mathbb{R}_{2\times 2} \mid \det\ A > 0\}.$$

Let $B = \begin{bmatrix} \alpha & \beta \\ \gamma & \delta \end{bmatrix}$ and define the *matrix product* of $A$ and $B$ to be

$$AB = \begin{bmatrix} \mathrm{a} & \mathrm{b} \\ \mathrm{c} & \mathrm{d} \end{bmatrix} \begin{bmatrix} \alpha & \beta \\ \gamma & \delta \end{bmatrix} = \begin{bmatrix} (a\alpha + b\gamma) & (a\beta + b\delta) \\ (c\alpha + d\gamma) & (c\beta + d\delta) \end{bmatrix} \in \mathbb{R}_{2\times 2}.$$

Also define $rA = \begin{bmatrix} ra & rb \\ rc & rd \end{bmatrix} \in \mathbb{R}_{2\times 2}$ for any $r \in \mathbb{R}$.

To see why all of these sets are groups, first we check associativity of matrix multiplication, which can be proven directly from the definition. For any, $A, B, C \in \mathbb{R}_{2\times 2}$ we have

$$\begin{aligned}
(AB)C &= \left( \begin{bmatrix} a & b \\ c & d \end{bmatrix} \begin{bmatrix} \alpha & \beta \\ \gamma & \delta \end{bmatrix} \right) \begin{bmatrix} w & x \\ y & z \end{bmatrix} \\
&= \begin{bmatrix} (a\alpha + b\gamma) & (a\beta + b\delta) \\ (c\alpha + d\gamma) & (c\beta + d\delta) \end{bmatrix} \begin{bmatrix} w & x \\ y & z \end{bmatrix} \\
&= \begin{bmatrix} (a\alpha + b\gamma)w + (a\beta + b\delta)y & (a\alpha + b\gamma)x + (a\beta + b\delta)z \\ (c\alpha + d\gamma)w + (c\beta + d\delta)y & (c\alpha + d\gamma)x + (c\beta + d\delta)z \end{bmatrix} \\
&= \begin{bmatrix} (a\alpha w + b\gamma w + a\beta y + b\delta y) & (a\alpha x + b\gamma x + a\beta z + b\delta z) \\ (c\alpha w + d\gamma w + c\beta y + d\delta y) & (c\alpha x + d\gamma x + c\beta z + d\delta z) \end{bmatrix}
\end{aligned}$$

and we have

$$\begin{aligned}
A(BC) &= \begin{bmatrix} a & b \\ c & d \end{bmatrix} \left( \begin{bmatrix} \alpha & \beta \\ \gamma & \delta \end{bmatrix} \begin{bmatrix} w & x \\ y & z \end{bmatrix} \right) \\
&= \begin{bmatrix} a & b \\ c & d \end{bmatrix} \begin{bmatrix} (\alpha w + \beta y) & (\alpha x + \beta z) \\ (\gamma w + \delta y) & (\gamma x + \delta z) \end{bmatrix} \\
&= \begin{bmatrix} a(\alpha w + \beta y) + b(\gamma w + \delta y) & a(\alpha x + \beta z) + b(\gamma x + \delta z) \\ c(\alpha w + \beta y) + d(\gamma w + \delta y) & c(\alpha x + \beta z) + d(\gamma x + \delta z) \end{bmatrix} \\
&= \begin{bmatrix} (a\alpha w + a\beta y + b\gamma w + b\delta y) & (a\alpha x + a\beta z + b\gamma x + b\delta z) \\ (c\alpha w + c\beta y + d\gamma w + d\delta y) & (c\alpha x + c\beta z + d\gamma x + d\delta z) \end{bmatrix} \\
&= (AB)C.
\end{aligned}$$

To show closure of these sets we will use the following formula true for any

$A, B \in \mathbb{R}_{2\times 2}$. We have

$$\begin{aligned}
\det(AB) &= (a\alpha + b\gamma)(c\beta + d\delta) - (a\beta + b\delta)(c\alpha + d\gamma) \\
&= a\alpha c\beta + a\alpha d\delta + b\gamma c\beta + b\gamma d\delta - a\beta c\alpha - a\beta d\gamma - b\delta c\alpha - b\delta d\gamma \\
&= a\alpha d\delta + b\gamma c\beta - a\beta d\gamma - b\delta c\alpha \\
&= (ad - bc)(\alpha\delta - \beta\gamma) = (\det\ A)(\det\ B).
\end{aligned}$$

$SL(2,\mathbb{R})$ is closed under matrix multiplication because if $\det\ A = 1 = \det\ B$ then $\det(AB) = (1)(1) = 1$. $GL(2,\mathbb{R})$ is closed because if $\det\ A \neq 0$, $\det\ B \neq 0$ then $\det(AB) = (\det\ A)(\det\ B) \neq 0$. $SL^{\pm 1}(2,\mathbb{R})$ is closed because if $\det\ A = \pm 1$ and $\det\ B = \pm 1$ then $\det(AB) = (\det\ A)(\det\ B) = (\pm 1)(\pm 1) = \pm 1$. $GL^{>0}(2,\mathbb{R})$ is closed because if $\det\ A > 0$ and $\det\ B > 0$ then $\det(AB) = (\det\ A)(\det\ B) > 0$. The identity element for matrix multiplication, $I = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$, is in all four of these sets. To see that these four sets are groups we only need to check their closure under inverses. If $A$ has an inverse, $A^{-1}$, such that $AA^{-1} = I = A^{-1}A$, then $(\det\ A)(\det\ A^{-1}) = \det(AA^{-1}) = \det(I) = 1$ so $\det\ A \neq 0$ and $\det(A^{-1}) = (\det\ A)^{-1}$. Conversely, if $\det\ A \neq 0$ there is a simple formula for the inverse of $A$, namely,

$$A^{-1} = \frac{1}{\det\ A} \begin{bmatrix} d & -b \\ -c & a \end{bmatrix}.$$

If $A \in G = GL(2,\mathbb{R})$ then $\det\ A \neq 0$ so $\det(A^{-1}) = (\det\ A)^{-1} \neq 0$ so $A^{-1} \in GL(2,\mathbb{R})$. If $A \in G = SL(2,\mathbb{R})$ then $\det\ A = 1$ so $\det(A^{-1}) = (\det\ A)^{-1} = 1$ so $A^{-1} \in SL(2,\mathbb{R})$. $A \in G = SL^{\pm 1}(2,\mathbb{R})$ then $\det\ A = \pm 1$ so $\det(A^{-1}) = (\det\ A)^{-1} = \pm 1$ so $A^{-1} \in SL^{\pm 1}(2,\mathbb{R})$. If $A \in G = GL^{>0}(2,\mathbb{R})$ then $\det\ A > 0$ so $\det(A^{-1}) = (\det\ A)^{-1} > 0$ so $A^{-1} \in GL^{>0}(2,\mathbb{R})$. In each case $A^{-1} \in G$, so each set is a group.

## Linear Fractional Maps on the Upper Half-Plane

For the project, we will use special functions $\mathbb{C} \to \mathbb{C}$ called *linear fractional maps*. If $A \in GL(2, \mathbb{R})$ so $ad - bc \neq 0$, define the function

$$f_A(z) = \frac{az+b}{cz+d} \text{ if } ad - bc > 0, \text{but } f_A(z) = \frac{a\bar{z}+b}{c\bar{z}+d} \text{ if } ad - bc < 0.$$

We have

$$Im\left(\frac{az+b}{cz+d}\right) = \frac{(ad-bc)Im(z)}{(cz+d)(c\bar{z}+d)} \text{ and } Im\left(\frac{a\bar{z}+b}{c\bar{z}+d}\right) = -\frac{(ad-bc)Im(z)}{(cz+d)(c\bar{z}+d)},$$

so if $Im(z) > 0$ then $Im(f_A(z)) > 0$. Therefore, we may consider these as functions on the *upper half plane*, $f_A : \mathcal{H} \to \mathcal{H}$ where $\mathcal{H} = \{z \in \mathbb{C} \mid Im(z) > 0\}$. Let $A, B \in GL^{>0}(2, \mathbb{R})$ with notation as before, so $f_A(z) = \frac{az+b}{cz+d}$ and $f_B(z) = \frac{\alpha z+\beta}{\gamma z+\delta}$. Then we can see that $f_A \circ f_B = f_{AB}$ by comparing these two functions;

$$\begin{aligned}(f_A \circ f_B)(z) &= f_A(f_B(z)) = f_A\left(\frac{az+b}{cz+d}\right) = \frac{\alpha\frac{az+b}{cz+d} + \beta}{\gamma\frac{az+b}{cz+d} + \delta} \\ &= \frac{a(\alpha z+\beta) + b(\gamma z+\delta)}{c(\alpha z+\beta) + (\gamma z+\delta)} = \frac{(a\alpha + b\gamma)z + (a\beta + b\delta)}{(c\alpha + d\gamma)z + (c\beta + d\delta)} = f_{(AB)}(z).\end{aligned}$$

If det $A < 0$ or det $B < 0$ we still get $f_A \circ f_B = f_{AB}$.

Note that for $S = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}$ and $T = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}$ in $SL(2, \mathbb{R})$, we have $f_S(z) = \frac{-1}{z}$ and $f_T(z) = z + 1$, so that $f_S$ exchanges the interior and exterior of the unit semicircle, $|z| = 1$, in $\mathcal{H}$, and $f_T$ is horizontal translation by 1. In fact, $S$ and $T$ are in the group

$$SL(2, \mathbb{Z}) = \left\{ A = \begin{bmatrix} a & b \\ c & d \end{bmatrix} \;\middle|\; a, b, c, d \in \mathbb{Z}, ad - bc = 1 \right\}.$$

Furthermore, we see that $f_A(z) = \frac{az+b}{cz+d} = \frac{-az-b}{-cz-d} = f_{-A}(z)$ so $A$ and $-A$ determine the same map. This can be reflected in the group $GL(2, \mathbb{R})$ by introducing an equivalence relation $A \sim B$ when $B = \pm A$, and the resulting set of equivalence classes is the group $PGL(2, \mathbb{R})$.

## Euclidean and Hyperbolic Triangle Groups

We will now give an example of a Euclidean triangle group acting on $\mathbb{R}^2$. For this example let

$$R_x \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} x \\ -y \end{bmatrix}, \quad R_y \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} -x \\ y \end{bmatrix}, \quad \text{and} \quad R_z \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 1-y \\ 1-x \end{bmatrix},$$

and note that these are reflections with respect to the lines $y = 0$, $x = 0$, and $x + y = 1$, respectively. The set of all possible finite sequences of compositions of these maps is a group, called the group *generated* by these maps, and because the generators are reflections in the three sides of a Euclidean triangle, it is naturally called a *Euclidean triangle group*. There is a standard notation for these triangle groups which gives the relations that completely describe the group. Doing a reflection twice gives the identity map, but the product of two distinct reflections is a rotation by some angle determined by the angle between the two sides. For example, the $x$-axis and the $y$-axis are perpendicular, at angle $\pi/2$ radians, so the product of the two corresponding reflections, $R_x R_y$, is rotation by twice that angle, $\pi$ radians. Therefore, $(R_x R_y)^2$ is rotation by $2\pi$ radians, which is the identity map, providing a relation in the triangle group. The other relations can be found similarly, and we find that this particular triangle group has the description

$$T(2,4,4) = \langle R_x, R_y, R_z | R_x^2 = R_y^2 = R_z^2 = (R_x R_y)^2 = (R_x R_z)^4 = (R_y R_z)^4 = I \rangle.$$

This is an example of a group, $T(2,4,4)$, acting on a set, $\mathbb{R}^2$. The elements of the group reflect the shaded triangle shown in Figure 1 (a fundamental domain). The entire set $\mathbb{R}^2$ is then covered by infinitely many copies of that triangle, providing a *tesselation* of the Euclidean plane by those triangles. Figure 2 illustrates only some of the triangles, giving a partial tesselation of $\mathbb{R}^2$ by this group.

Figure 1: $T(2,4,4)$ Fundamental Domain

Figure 2: $T(2,4,4)$ Partial Tesselation of $\mathbb{R}^2$

For this project we did analogous calculations with linear fractional maps for certain hyperbolic triangle groups ([3]) acting on the *hyperbolic plane*, a model of hyperbolic geometry explained below. We were inspired by several beautiful pictures, two of which are shown in Figures 3 and 4. Most of the background math needed to understand what we did has been explained in the earlier sections. We still need to discuss the two most common models for hyperbolic geometry, and then we can find specific matrices in the group $SL^{\pm 1}(2, \mathbb{R})$ which generate particular hyperbolic triangle groups which give tesselations of one of those models.

Figure 3: Tesselation of disk by $T(3, 4, 5)$

Figure 4: Tesselation of disk by $T(2, 3, \infty)$

There are two commonly used models for hyperbolic geometry, one being the upper half plane $\mathcal{H}$, and the other is the interior of the unit circle in $\mathbb{C}$, $\mathcal{D} = \{z \in \mathbb{C} \mid |z| < 1\}$. A hyperbolic straight line in $\mathcal{H}$ is either a vertical ray, $\{x + \mathbf{i}y \in \mathcal{H} \mid y > 0\}$ for fixed $x \in \mathbb{R}$, or a semicircle $C(a, b) = \{x + \mathbf{i}y \in \mathcal{H} \mid (x-c)^2 + y^2 = r^2, y > 0\}$ with center $c = \frac{a+b}{2}$, radius $r = \frac{b-a}{2}$, touching the real axis orthogonally at the points $a$ and $b$ in $\mathbb{R}$ where $a < b$. A hyperbolic straight line in $\mathcal{D}$ is either a diameter, $\{r(\cos\theta + \mathbf{i}\sin\theta) \in \mathcal{D} | -1 < r < 1\}$, or an arc of a circle meeting the unit circle orthogonally at two points $a + b\mathbf{i}$ and $c + d\mathbf{i}$. Using the tangent lines at these points one finds the center of such a circle to be $\frac{1}{ad-bc}((d-b) + \mathbf{i}(a-c))$ and its radius to be $\left|\frac{1-ac-bd}{ad-bc}\right|$. The proof of these results is as follows.

The center of the circle we want is the point where the two tangent lines meet. Several steps were taken to figure out this point. First, using implicit differentiation we found the slope of the tangent line at $x + \mathbf{i}y$ to be $\frac{-x}{y}$. This allows us to write the equations of the tangent lines in point-slope form. The equation of the tangent line at the point $a + b\mathbf{i}$ is $y - b = \frac{-a}{b}(x - a)$ and the equation of the tangent line at point $c + d\mathbf{i}$ is $y - d = \frac{-c}{d}(x - c)$. To find the

center of the circle, we solve for $x$ as shown in the following:

$$\begin{aligned}
[b - \frac{a}{b}(x-a)](bd) &= [d - \frac{c}{d}(x-c)](bd) \\
b^2d - ad(x-a) &= bd^2 - bc(x-c) \\
b^2d - adx + a^2d &= bd^2 - bcx + bc^2 \\
b^2d - bd^2 + a^2d - bc^2 &= adx - bcx \\
(a^2+b^2)d - b(c^2+d^2) &= (ad-bc)x \\
d - b &= (ad-bc)x \\
\frac{d-b}{ad-bc} &= x
\end{aligned}$$

since $a^2 + b^2 = 1 = c^2 + d^2$, and from this we found $y = \frac{a-c}{ad-bc}$.

All the axioms of Euclidean geometry, except for the *parallel postulate*, are satisfied in these two models. But given a line and a point not on it, there are infinitely many lines through that point parallel to that given line. These two models are equivalent because they are connected by an invertible linear fractional map $f : \mathcal{H} \to \mathcal{D}$ where $f(z) = \mathbf{i}\left(\frac{z-\mathbf{i}}{z+\mathbf{i}}\right) = w$ for any $z \in \mathcal{H}$ can be associated with the $2\times 2$ complex matrix $M = \frac{1}{\sqrt{2}}\begin{bmatrix} \mathbf{i} & 1 \\ 1 & \mathbf{i} \end{bmatrix}$ with determinant $-1$, and $f^{-1} : \mathcal{D} \to \mathcal{H}$ where $f^{-1}(w) = -\mathbf{i}\left(\frac{w+\mathbf{i}}{w-\mathbf{i}}\right) = z$ for any $w \in \mathcal{D}$ can be associated with the $2 \times 2$ complex matrix $M^{-1} = \frac{1}{\sqrt{2}}\begin{bmatrix} -\mathbf{i} & 1 \\ 1 & -\mathbf{i} \end{bmatrix}$ with determinant $-1$. The function $f$ also takes the boundary of $\mathcal{H}$, $z \in \mathbb{R}$, to the boundary of $\mathcal{D}$, $|w| = 1$, with the understanding that as $z \to \pm\infty$, $f(z) \to \mathbf{i}$. In Figures 5 and 6 we see the tesselations of $\mathcal{H}$ that correspond to those of $\mathcal{D}$ shown in Figures 3 and 4. Although we have been greatly inspired by the pictures in $\mathcal{D}$, we will do all the rest of our work in $\mathcal{H}$. The matrices needed to express the results as groups acting on $\mathcal{D}$ can be easily found by the method just described, but they would involve complex numbers.

Figure 5: Tesselation of $\mathcal{H}$ by $T(3, 4, 5)$

Figure 6: Tesselation of $\mathcal{H}$ by $T(2, 3, \infty)$

## Generators for Specific Hyperbolic Triangle Groups

With this background we can now describe the hyperbolic triangle groups studied in this project. In Figures 7 and 8, one in $\mathcal{H}$ and one in $\mathcal{D}$, we see three hyperbolic straight lines forming a hyperbolic triangle whose angles are $\frac{\pi}{2}$, $\frac{\pi}{3}$, $\frac{\pi}{\infty} = 0$. In this case the triangle includes one point on the boundary. The group generated by reflections in the three sides of this triangle is the hyperbolic triangle group

$$T(2,3,\infty) = \langle R_1, R_2, R_3 | R_1^2 = R_2^2 = R_3^2 = (R_1R_2)^2 = (R_1R_3)^3 = I \rangle.$$

Figure 7: $T(2,3,\infty)$ Fundamental Domain in $\mathcal{H}$

Figure 8: $T(2,3,\infty)$ Fundamental Domain in $\mathcal{D}$

Note that $R_2R_3$ is not of finite order, that is $(R_2R_3)^k \neq I$ for all $0 < k \in \mathbb{Z}$. There are three matrices

$$A_1 = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}, \quad A_2 = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}, \quad A_3 = \begin{bmatrix} -1 & 1 \\ 0 & 1 \end{bmatrix}$$

in $SL^{\pm 1}(2,\mathbb{R})$ corresponding to maps on $\mathcal{H}$

$$f_{A_1}(z) = \frac{1}{\bar{z}}, \quad f_{A_2}(z) = -\bar{z}, \quad f_{A_3}(z) = 1 - \bar{z},$$

that are generators of $T(2,3,\infty)$ on $\mathcal{H}$. For $A \in SL^{\pm 1}(2,\mathbb{R})$, the linear fractional map $g_B : \mathcal{D} \to \mathcal{D}$ corresponds to $f_A : \mathcal{H} \to \mathcal{H}$ when $g_B(w) = f(f_A(f^{-1}(w)))$. In the case when $A = A_1$ we get $g_{B_1}(w) = \bar{w} = \frac{\mathbf{i}\bar{w}+0}{0\bar{w}+\mathbf{i}}$ which is associated with the matrix $B_1 = \begin{bmatrix} \mathbf{i} & 0 \\ 0 & \mathbf{i} \end{bmatrix}$ having det $B_1 = -1$ , and this is clearly reflection with respect to $y = 0$ in $\mathcal{D}$. In the case when $A = A_2$ we get $g_{B_2}(w) = -\bar{w} = \frac{1\bar{w}+0}{0\bar{w}-1}$ which is associated with the matrix $B_2 = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}$ having det $B_2 = -1$, and this is reflection with respect to $x = 0$ in $\mathcal{D}$. In the last case when $A = A_3$ we get $g_{B_3}(w) = -\bar{w} = \frac{(\mathbf{i}+2)\bar{w}-1}{1\bar{w}+(\mathbf{i}-2)}$ which we

associate with the matrix $B_3 = \begin{bmatrix} \frac{\mathbf{i}+2}{2} & \frac{-1}{2} \\ \frac{1}{2} & \frac{\mathbf{i}-2}{2} \end{bmatrix}$ having det $B_3 = -1$, and this is reflection with respect to the hyperbolic line labeled $R_3$ in Figure 8. Note that $B_j$, $j = 1, 2, 3$, are in the group

$$SL^{\pm 1}(2, \mathbb{C}) = \left\{ B = \begin{bmatrix} \alpha & \beta \\ \gamma & \delta \end{bmatrix} \in \mathbb{C}_{2\times 2} \;\middle|\; \det\ B = \pm 1 \right\}.$$

Inversion with respect to a semicircle $C(a, b)$ in $\mathcal{H}$ is given in the following.

THEOREM 1. Let $a < b \in \mathbb{R}$, $r = \frac{b-a}{2}$, $c = \frac{a+b}{2}$, $C(a, b) = \{x + \mathbf{i}y \in \mathcal{H} | (x - c)^2 + y^2 = r^2, y > 0\}$ and $\rho(a, b) = \frac{1}{b-a} \begin{bmatrix} a+b & -2ab \\ 2 & -a-b \end{bmatrix} = \frac{1}{r} \begin{bmatrix} c & r^2 - c^2 \\ 1 & -c \end{bmatrix}$. Then inversion with respect to $C(a, b)$ in $\mathcal{H}$ is the linear fractional map $f_{\rho(a,b)}(z) = \frac{c\bar{z} + (r^2 - c^2)}{\bar{z} - c}$.

*Proof.* First consider the special case $a = -r$, $b = r$ giving a circle of radius r centered at 0. This is obtained by stretching

$$\begin{bmatrix} r & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} r^{-1} & 0 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 0 & r \\ r^{-1} & 0 \end{bmatrix} = \rho(-r, r),$$

from $A_1$ where $a = -1$, $b = 1$. The general case follows by using translation $T_c = \begin{bmatrix} 1 & c \\ 0 & 1 \end{bmatrix}$. We have

$$\begin{aligned} \begin{bmatrix} 1 & c \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 0 & r \\ r^{-1} & 0 \end{bmatrix} \begin{bmatrix} 1 & -c \\ 0 & 1 \end{bmatrix} &= \begin{bmatrix} cr^{-1} & r \\ r^{-1} & 0 \end{bmatrix} \begin{bmatrix} 1 & -c \\ 0 & 1 \end{bmatrix} \\ &= \begin{bmatrix} cr^{-1} & -c^2 r^{-1} + r \\ r^{-1} & -cr^{-1} \end{bmatrix} = \frac{1}{r} \begin{bmatrix} c & r^2 - c^2 \\ 1 & -c \end{bmatrix} = \rho(a, b). \quad \square \end{aligned}$$

The goal of this project is to find matrices that generate the hyperbolic triangle groups $T(3, 4, t)$ for $t = 3, 4, 5$, in $\mathcal{H}$. We keep $A_2$ corresponding to line $R_2$ in Figure 7 as one side. Next we find a hyperbolic line $C_1 = C(a, b)$ passing through the point $\mathbf{i}$ whose angle with $R_2 = C_2$ is $\frac{\pi}{4}$ at that point.

The equation of such a circle is $(x-c)^2+y^2=r^2$, and the slope of a tangent line to it at $x+\mathbf{i}y$ is $\frac{dy}{dx}=\frac{c-x}{y}$. We want that slope to be $-1$ when $x=0$ and $y=1$, so $c=-1$. Then using $(0+1)^2+1^2=r^2$ we get $r=\sqrt{2}$ so $a=-1-\sqrt{2}$ and $b=-1+\sqrt{2}$. This gives the matrix $\rho(a,b)=\rho(-1-\sqrt{2},-1+\sqrt{2})=\frac{1}{\sqrt{2}}\begin{bmatrix}-1 & 1\\ 1 & 1\end{bmatrix}$ corresponding to the map $f_{\rho(a,b)}(z)=\frac{-\bar{z}+1}{\bar{z}+1}$ which is inversion with respect to $C_1=C(a,b)$. We now find a third hyperbolic line $C_3=C(a,b)$ for some new $a$ and $b$ which intersects $R_2=C_2$ at some point between 0 and $\mathbf{i}$ at the angle $\frac{\pi}{3}$, and intersects $C_1$ at the angle $\frac{\pi}{t}$. We will say the center at $C_3=C(a,b)$ is $c=\frac{a+b}{2}$ with radius $r=\frac{b-a}{2}$, so we want the slope of the tangent line at $x=0$ to be $\frac{dy}{dx}=\frac{c}{y}=\frac{1}{\sqrt{3}}$ so $y^2=3c^2=r^2-c^2$ so $r=2c$. Using this information we can now find point $b$ so $\frac{b-a}{2}=2\frac{a+b}{2}$ so $b-a=2a+2b$ so $b=-3a$, giving $c=-a$ and $r=-2a$.

We now find the condition on $a$ so that $C_1$ and $C_3$ meet at angle $\frac{\pi}{t}$ for $t=3,4$, or 5. The equation of $C_1$ is $(x+1)^2+y^2=2$ and the equation of $C_3$ is $(x+a)^2+y^2=4a^2$. Call $x_0+\mathbf{i}y_0$ the point in $\mathcal{H}$ where $C_1$ and $C_3$ intersect, so $x_0^2+2x_0+1+y_0^2=2$ and $x_0^2+2ax_0+a^2+y_0^2=4a^2$. Taking the difference of these equations gives $2x_0+1-2ax_0-a^2=2-4a^2$ and solving for $x_0$ gives $x_0=\frac{1-3a^2}{2(1-a)}$. Tangent vectors to the circles $C_1$ and $C_3$ at $x_0$ are given by

$$v_1=\begin{bmatrix} y_0 \\ -1-x_0\end{bmatrix} \quad \text{and } v_2=\begin{bmatrix} y_0 \\ -a-x_0\end{bmatrix}.$$

These vectors have lengths $||v_1||=\sqrt{y_0^2+(-1-x_0)^2}=\sqrt{2}$ and $||v_2||=\sqrt{y_0^2+(-a-x_0)^2}=r=-2a>0$. If $\theta$ is the angle between the tangent vector $v_1$ and $v_2$ then

$$\begin{aligned}\cos(\theta) &= \frac{v_1\cdot v_2}{(||v_1||)(||v_2||)}=\frac{y_0^2+(-1-x_0)(-a-x_0)}{\sqrt{2}(-2a)}\\ &= \frac{y_0^2+x_0^2+(a+1)x_0+a}{-2a\sqrt{2}}=\frac{1-2x_0+(a+1)x_0+a}{-2a\sqrt{2}}\\ &= \frac{(a-1)x_0+a+1}{-2a\sqrt{2}}.\end{aligned}$$

Here we have used the "dot product" of vectors in $\mathbb{R}^2$ ([2], p. 4,6),

$$\begin{bmatrix} x_1 \\ y_1 \end{bmatrix} \cdot \begin{bmatrix} x_2 \\ y_2 \end{bmatrix} = x_1x_2 + y_1y_2.$$

First consider the case when $t = 3$ so $\theta = \frac{\pi}{3}$. Then $\cos(\pi/3) = \frac{1}{2} = \frac{(a-1)x_0+a+1}{-2a\sqrt{2}}$ means $-a\sqrt{2} = (a-1)x_0 + a + 1$ if and only if $x_0 = \frac{a(1+\sqrt{2})+1}{1-a}$. Since we already have a formula for $x_0$ in terms of $a$, we get

$$\frac{a(1+\sqrt{2})+1}{1-a} = \frac{1-3a^2}{2(1-a)}$$

which means $2a(1+\sqrt{2}) + 2 = 1 - 3a^2$, that is, $a$ satisfies the quadratic equation $3a^2 + 2(1+\sqrt{2})a + 1 = 0$. Therefore, we have

$$\begin{aligned} a &= \frac{-2(1+\sqrt{2}) \pm \sqrt{4(1+\sqrt{2})^2 - 12}}{6} = \frac{-2(1+\sqrt{2}) \pm \sqrt{8\sqrt{2}}}{6} \\ &= \frac{-1-\sqrt{2} \pm \sqrt[4]{8}}{3}. \end{aligned}$$

Although the quadratic equation has two solutions only one of them gives a semicircle that crosses $C_2$ between 0 and $\mathbf{i}$. That is the solution $a = \frac{-1-\sqrt{2}+\sqrt[4]{8}}{3} \approx -0.244$ and the resulting $C_3$ is shown in Figure 9.

Figure 9: $T(3,4,3)$ Fundamental Domain in $\mathcal{H}$

Next we will consider the case $t = 4$ so $\theta = \frac{\pi}{4}$. Then $\cos(\pi/4) = \frac{1}{\sqrt{2}} = \frac{(a-1)x_0+a+1}{-2a\sqrt{2}}$ means $-2a = (a-1)x_0 + (a+1)$ if and only if $x_0 = \frac{3a+1}{1-a}$. We get

$$\frac{3a+1}{1-a} = \frac{1-3a^2}{2(1-a)}$$

which means $6a + 2 = 1 - 3a^2$, that is, $a$ satisfies the quadratic equation $3a^2 + 6a + 1 = 0$. Therefore, we have

$$a = \frac{-6 \pm \sqrt{36-12}}{6} = \frac{-6 \pm 2\sqrt{6}}{6} = \frac{-3 \pm \sqrt{6}}{3} = -1 \pm \sqrt{\frac{2}{3}}.$$

Again, although the quadratic equation has two solutions only one of them gives a semicircle that crosses $C_2$ between 0 and $\mathbf{i}$. That is the solution $a = -1 + \sqrt{\frac{2}{3}} \approx -0.184$ and the resulting $C_3$ is shown in Figure 10.

Figure 10: $T(3,4,4)$ Fundamental Domain in $\mathcal{H}$

The last case we will consider is $t = 5$ so $\theta = \frac{\pi}{5}$. Recall that the "golden ratio" $\tau = \frac{1+\sqrt{5}}{2}$ satisfies $\tau^2 = \tau + 1$. We have $\cos(\pi/5) = \frac{1+\sqrt{5}}{4} = \frac{\tau}{2} = \frac{(a-1)x_0+a+1}{-2a\sqrt{2}}$ means $a\sqrt{2}\tau = (a-1)x_0 - (a+1)$ if and only if $x_0 = \frac{a(\sqrt{2}\tau+1)+1}{1-a}$. We then get

$$\frac{a(\sqrt{2}\tau+1)+1}{1-a} = \frac{1-3a^2}{2(1-a)}$$

which means $2a(\sqrt{2}\tau+1)+2 = 1-3a^2$ so $a$ satisfies the quadratic equation $3a^2 + 2(\sqrt{2}\tau+1)a + 1 = 0$. Therefore, we have

$$\begin{aligned} a &= \frac{-2(\sqrt{2}\tau+1) \pm \sqrt{4(\sqrt{2}\tau+1)^2 - 12}}{6} \\ &= \frac{-(\sqrt{2}\tau+1) \pm \sqrt{(\sqrt{2}\tau+1)^2 - 3}}{3} \\ &= \frac{-(\sqrt{2}\tau+1) \pm \sqrt{2\tau+2+2\sqrt{2}\tau-2}}{3} \\ &= \frac{-(\sqrt{2}\tau+1) \pm \sqrt{2\tau+2\sqrt{2}\tau}}{3} \\ &= \frac{-(\sqrt{2}\tau+1) \pm \sqrt{(\sqrt{2}+1)(\sqrt{5}+1)}}{3}. \end{aligned}$$

The value of $a$ which gives a semicircle that crosses $C_2$ between 0 and $\mathbf{i}$ is $a = \frac{-(\sqrt{2}\tau+1)+\sqrt{(\sqrt{2}+1)(\sqrt{5}+1)}}{3} \approx -0.164$ and the resulting $C_3$ is shown in Figure 11.

Figure 11: $T(3,4,5)$ Fundamental Domain in $\mathcal{H}$

We finish with a summary of our results.

THEOREM 2. The hyperbolic triangle groups $T(3,4,t)$ for $t = 3,4,5$

acting on the upper half-plane $\mathcal{H}$ by linear fractional maps are each generated by three specific maps $f_{M_1}$, $f_{M_2}$ and $f_{M_3}$ associated with the three matrices

$$M_1 = \frac{1}{\sqrt{2}} \begin{bmatrix} -1 & 1 \\ 1 & 1 \end{bmatrix}, \quad M_2 = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}, \quad M_3 = \frac{1}{-2a} \begin{bmatrix} -a & 3a^2 \\ 1 & a \end{bmatrix}$$

where $a = \frac{-1-\sqrt{2}+\sqrt[4]{8}}{3}$ if $t = 3$, $a = -1 + \sqrt{\frac{2}{3}}$ if $t = 4$, and $a = \frac{-(\sqrt{2}\tau+1)+\sqrt{(\sqrt{2}+1)(\sqrt{5}+1)}}{3}$ if $t = 5$.

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# References


[1] M. C. Escher and J. L. Locher, *The World of M. C. Escher*, New Concise NAL Edition, Harry N. Abrams, Inc, Publishers, New York 1971, Distributed by New American Library.

[2] S. Lipschutz, M. Lipson, *Schaums Outlines, Linear Algebra, Third Ed.*, McGraw Hill, 2001.

[3] W. Magnus, *Noneuclidean Tesselations and their Groups*, Pure, Appl. Math, Vol. **61**. New York, London, Academic Press, 1974.

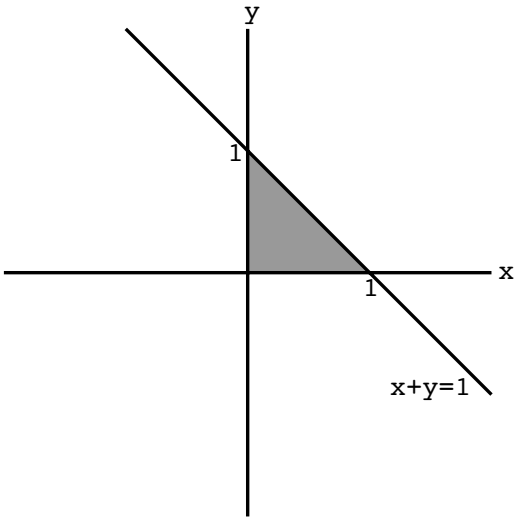


Figure 1: $T(2,4,4)$ Fundamental Domain

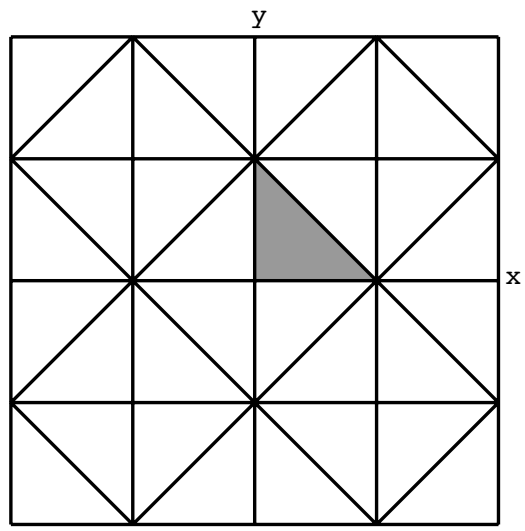


Figure 2: $T(2, 4, 4)$ Partial Tesselation of $\mathbb{R}^2$

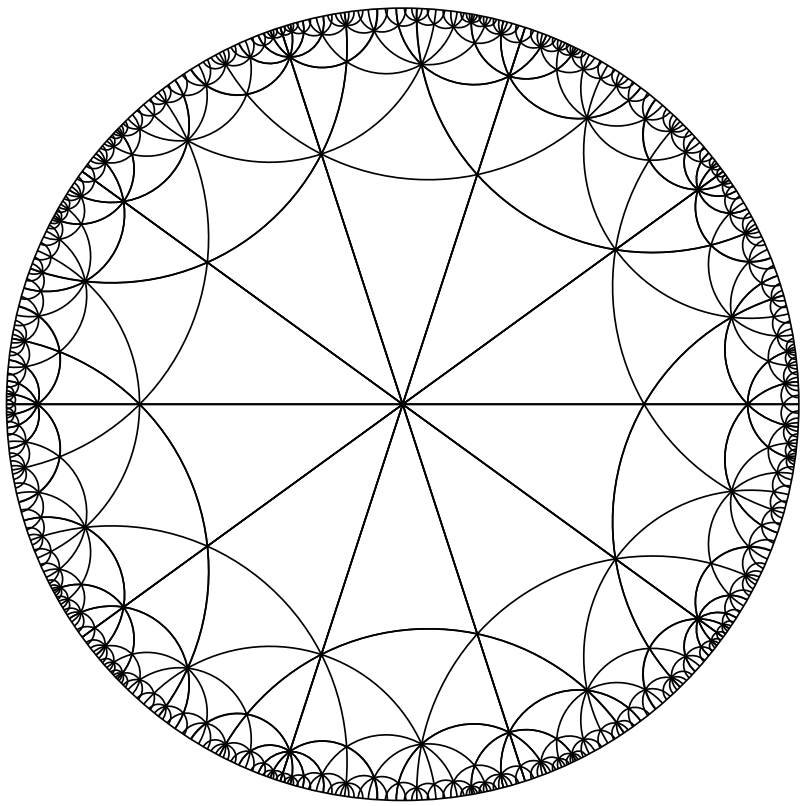

Figure 3: Tesselation of disk by $T(3, 4, 5)$

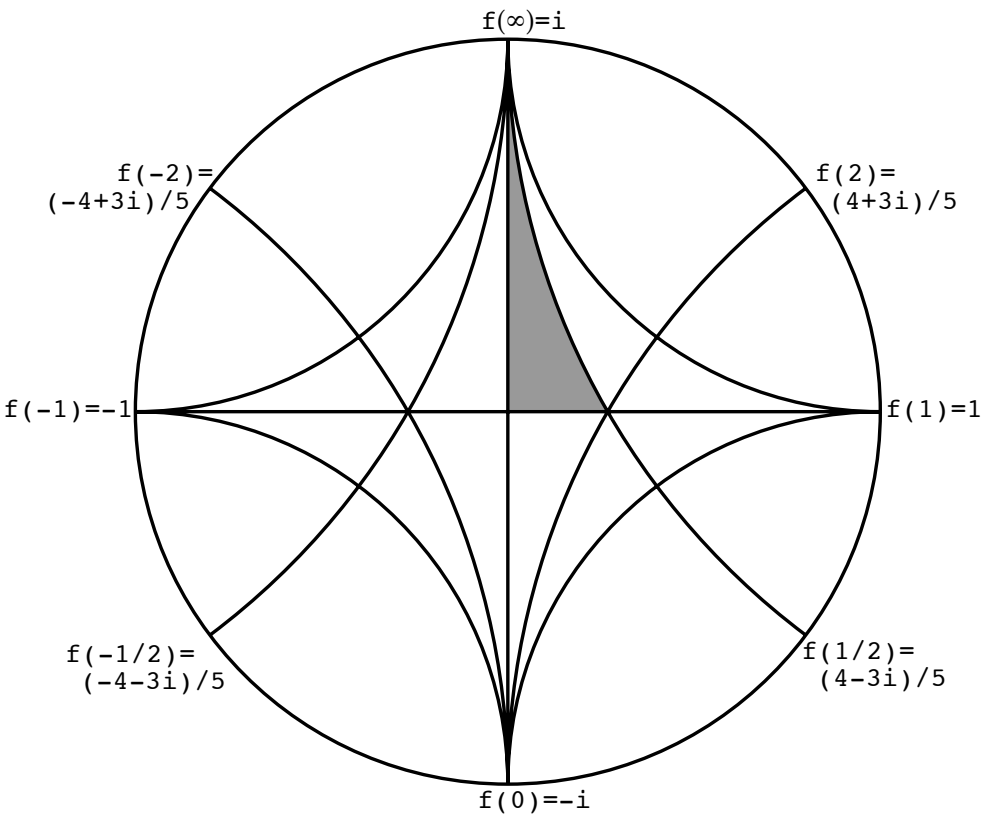


Figure 4: Tesselation of disk by $T(2, 3, \infty)$

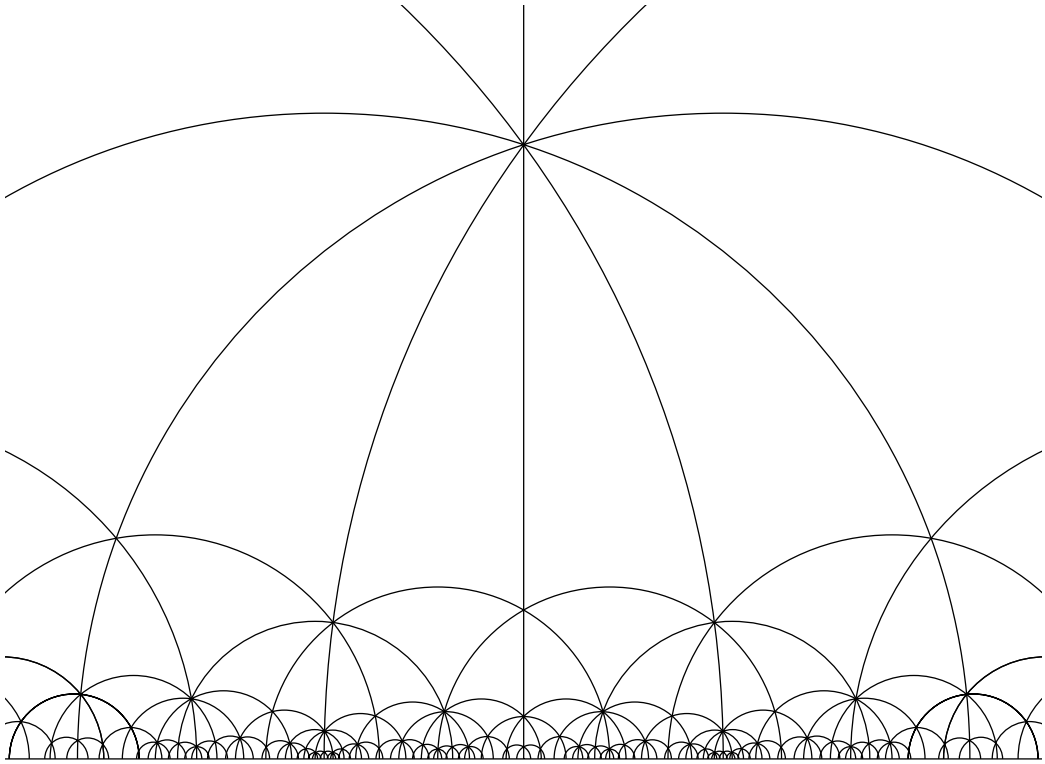

Figure 5: Tesselation of $\mathcal{H}$ by $T(3, 4, 5)$

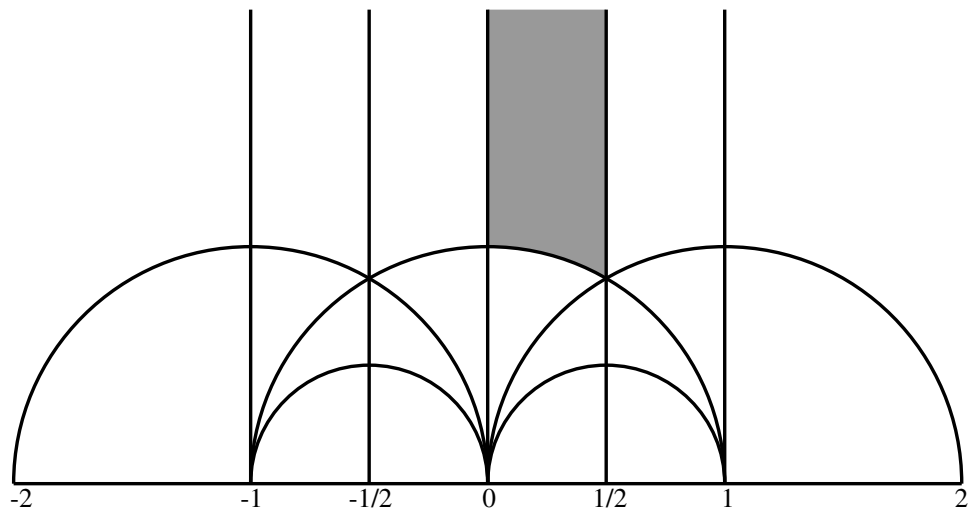


Figure 6: Tesselation of $\mathcal{H}$ by $T(2, 3, \infty)$

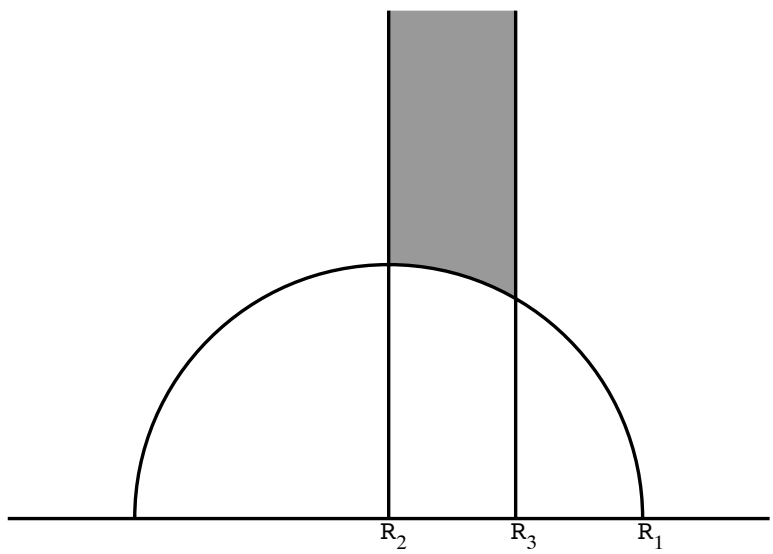


Figure 7: $T(2, 3, \infty)$ Fundamental Domain in $\mathcal{H}$

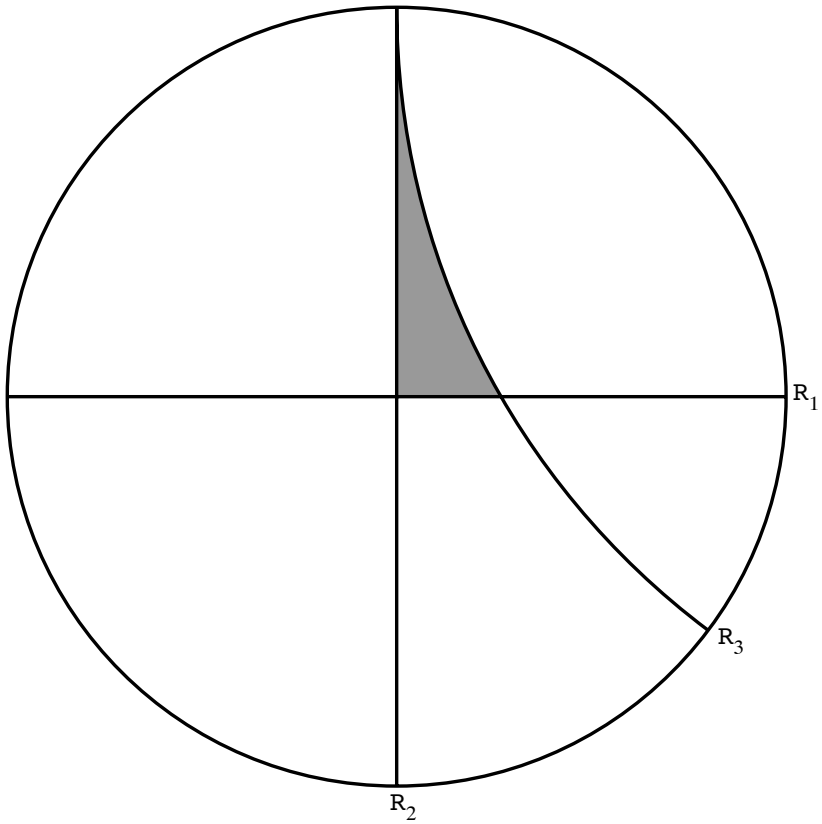


Figure 8: $T(2, 3, \infty)$ Fundamental Domain in $\mathcal{D}$

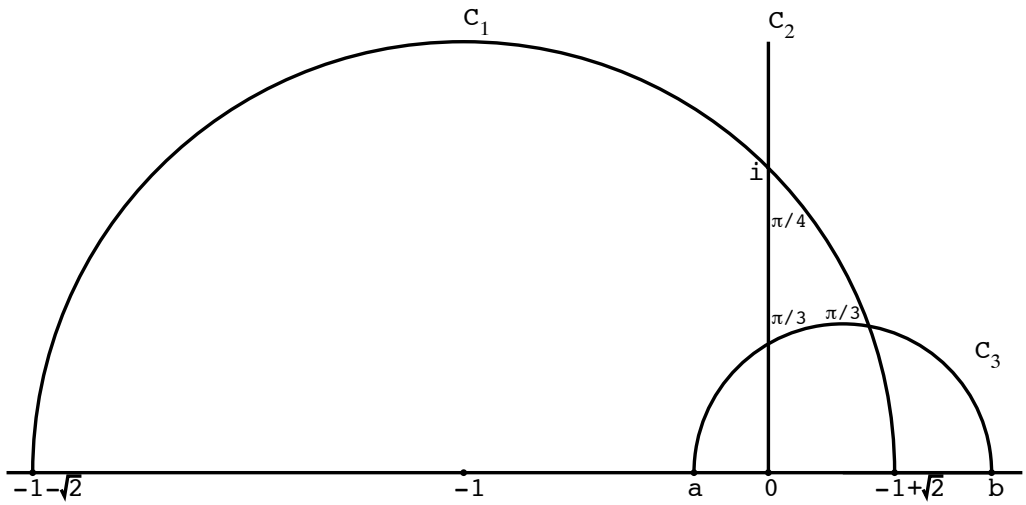


Figure 9: $T(3,4,3)$ Fundamental Domain in $\mathcal{H}$

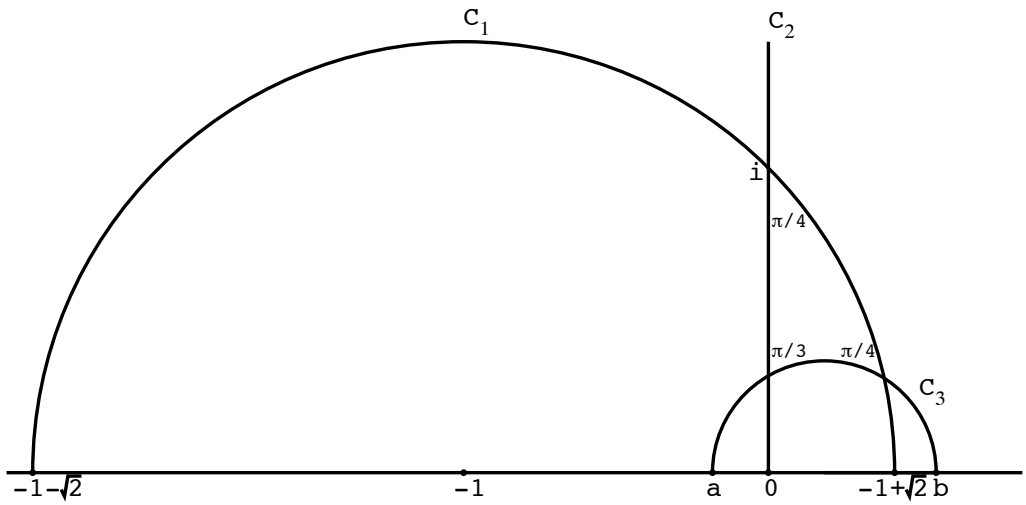


Figure 10: $T(3,4,4)$ Fundamental Domain in $\mathcal{H}$

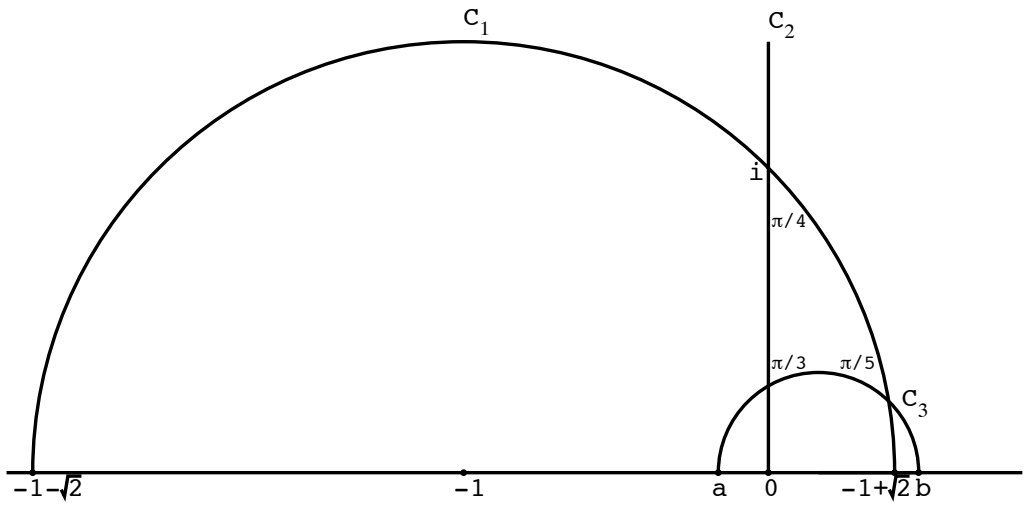


Figure 11: $T(3,4,5)$ Fundamental Domain in $\mathcal{H}$