Math 304-2 Linear Algebra Fall 2024 Exam 1 Feingold

Show all work for each problem unless instructed otherwise.

(1) (15 Points) Let $L_A : \mathbb{F}^5 \to \mathbb{F}^4$ be the linear function $L_A(X) = AX$ associated with the matrix $A = \begin{bmatrix} 1 & 1 & 1 & 1 & 1 \\ 1 & 2 & 3 & 4 & 5 \\ 3 & 4 & 6 & 7 & 8 \\ 5 & 7 & 10 & 12 & 14 \end{bmatrix}$.

(a) Find the subspace $\mathrm{Ker}(L_A) = \{X \in \mathbb{F}^5 \mid L_A(X) = 0\}$ as a **set of vectors in terms of free variables** and **find a basis** for it.

(b) Find the subspace $\mathrm{Range}(L_A) = \{B = L_A(X) \in \mathbb{F}^4 \mid X \in \mathbb{F}^5\}$ by giving **consistency conditions** on the entries of the vectors $B = [b_i]$ in it.

(c) Determine whether L_A is injective and whether L_A is surjective. Briefly explain why.

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(2) (15 Points) Let $T : \mathbb{F}^3 \to \mathbb{F}^2$ and $S : \mathbb{F}^2 \to \mathbb{F}^3$ be the functions

$$T \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} x_1 + 2x_2 - 3x_3 \\ x_1 - x_2 + 2x_3 \end{bmatrix} \quad \text{and} \quad S \begin{bmatrix} y_1 \\ y_2 \end{bmatrix} = \begin{bmatrix} y_1 - y_2 \\ y_1 + y_2 \\ 3y_1 + 2y_2 \end{bmatrix}.$$

(a) Find the matrices A and B such that $S(Y) = AY$ and $T(X) = BX$.

(b) Use **composition of functions, not matrix multiplication** to find the **formula** for the composition $(S \circ T) \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$.

(c) Use your answer to part (b) to find the matrix C such that $(S \circ T)(X) = CX$.

(d) What should be the relationship between the matrices A, B and C? Check that your matrices satisfy that relation.

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(3) (15 Points) Answer each question separately. **No justifications are needed**.

(a) For a **nonzero** matrix $A \in \mathbb{F}^4_7$, find all the possible values of $r = rank(A)$.

(b) For $A \in \mathbb{F}^m_n$ what condition on $rank(A) = r$ is equivalent to the **non-homogeneous** linear system $AX = B$ being **inconsistent** for some $B \in \mathbb{F}^m$?

(c) For $A \in \mathbb{F}^m_n$ what condition on $rank(A) = r$ is equivalent to the **homogeneous** linear system $AX = 0^m_1$ having **nontrivial** solutions?

(d) If $A \in \mathbb{F}^m_n$ and $AX = 0^m_1$ has only the trivial solution, what is the **most** you can say about the relation between m and n?

(e) What conditions on m, n and $rank(A) = r$ would mean that $A \in \mathbb{F}^m_n$ is invertible?

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(4) (15 Points) For $A \in \mathbb{F}_n^m$, let $L_A : \mathbb{F}^n \to \mathbb{F}^m$ be the linear function $L_A(X) = AX$. For $1 \leq j \leq n$ let $\mathbf{e}_j \in \mathbb{F}^n$ be the matrix with 1 in row j and 0 in all other rows.
No justifications are needed for these questions.

(a) What relation between m and n would guarantee that L_A is **not injective**?

(b) What relation between m and n would guarantee that L_A is **not surjective**?

(c) If $\text{rank}(A) = n$ what does that tell you about L_A?

(d) If $\text{rank}(A) = m$ what does that tell you about L_A?

(e) What is the relationship between A and $L_A(\mathbf{e}_j)$ for $1 \leq j \leq n$?

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(5) (15 Points) **No justifications are needed for these questions.**

(a) Let $L : \mathbb{F}^3 \to \mathbb{F}^2$ be linear with $L(\mathbf{e}_1) = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$, $L(\mathbf{e}_2) = \begin{bmatrix} 1 \\ 2 \end{bmatrix}$, $L(\mathbf{e}_3) = \begin{bmatrix} -2 \\ 3 \end{bmatrix}$. Find $A \in \mathbb{F}_3^2$ such that $L(X) = L_A(X) = AX$ for all $X \in \mathbb{F}^3$.

(b) For $A \in \mathbb{F}_n^3$ find matrix E such that $B = EA$ is the matrix obtained from A by doing to A the elementary row operation $5Row_2(A) + Row_1(A) \to Row_1(A)$.

(c) If $S = \{v_1, \cdots, v_k\}$ is an **independent** subset of $\mathbb{F}^m$, what is the relationship between k and m?

(d) For $A \in \mathbb{F}_n^n$ suppose $[A|I_n]$ row reduces to $[C|D]$ with C in RREF. When A is invertible what is C and what is D?

(e) If $S \subseteq T \subseteq V$ and T is **dependent** in vector space V, what can you say about S?

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(1) (15 Points) Let $L_A : \mathbb{F}^5 \to \mathbb{F}^4$ be the linear function $L_A(X) = AX$ associated with the matrix $A = \begin{bmatrix} 1 & 1 & 1 & 1 & 1 \\ 1 & 2 & 3 & 4 & 5 \\ 3 & 4 & 6 & 7 & 8 \\ 5 & 7 & 10 & 12 & 14 \end{bmatrix}$.

(a) Find the subspace $\mathrm{Ker}(L_A) = \{X \in \mathbb{F}^5 \mid L_A(X) = 0\}$ as a **set of vectors in terms of free variables** and **find a basis** for it.

Solution: (6 pts) To find $\mathrm{Ker}(L_A)$ we must solve a linear system by row reducing

$$\left[\begin{array}{ccccc|c} 1 & 1 & 1 & 1 & 1 & 0 \\ 1 & 2 & 3 & 4 & 5 & 0 \\ 3 & 4 & 6 & 7 & 8 & 0 \\ 5 & 7 & 10 & 12 & 14 & 0 \end{array}\right] \text{ to } \left[\begin{array}{ccccc|c} 1 & 0 & 0 & -1 & -2 & 0 \\ 0 & 1 & 0 & 1 & 2 & 0 \\ 0 & 0 & 1 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{array}\right] \text{ so } \begin{array}{c} x_1 = r + 2s \\ x_2 = -r - 2s \\ x_3 = -r - s \\ x_4 = r \in \mathbb{F} \\ x_5 = s \in \mathbb{F} \end{array}$$

$$\mathrm{Ker}(L_A) = \left\{ \begin{bmatrix} r + 2s \\ -r - 2s \\ -r - s \\ r \\ s \end{bmatrix} \in \mathbb{F}^5 \;\middle|\; r, s \in \mathbb{F} \right\} \text{ has basis } \left\{ \begin{bmatrix} 1 \\ -1 \\ -1 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 2 \\ -2 \\ -1 \\ 0 \\ 1 \end{bmatrix} \right\}.$$

(b) Find the subspace $\mathrm{Range}(L_A) = \{B = L_A(X) \in \mathbb{F}^4 \mid X \in \mathbb{F}^5\}$ by giving **consistency conditions** on the entries of the vectors $B = [b_i]$ in it.

Solution: (5 pts) $B = [b_i] \in \mathrm{Range}(L_A)$ iff the following system is consistent:

$$\left[\begin{array}{ccccc|c} 1 & 1 & 1 & 1 & 1 & b_1 \\ 1 & 2 & 3 & 4 & 5 & b_2 \\ 3 & 4 & 6 & 7 & 8 & b_3 \\ 5 & 7 & 10 & 12 & 14 & b_4 \end{array}\right] \to \left[\begin{array}{ccccc|c} 1 & 0 & 0 & -1 & -2 & -2b_2 + b_3 \\ 0 & 1 & 0 & 1 & 2 & 3b_1 + 3b_2 - 2b_3 \\ 0 & 0 & 1 & 1 & 1 & -2b_1 - b_2 + b_3 \\ 0 & 0 & 0 & 0 & 0 & -b_1 - b_2 - b_3 + b_4 \end{array}\right] \begin{array}{c} \text{is consistent iff} \\ 0 = -b_1 - b_2 - b_3 + b_4 \\ \text{iff} \\ b_1 + b_2 + b_3 = b_4 \end{array}$$

(c) Determine whether L_A is injective and whether L_A is surjective. Briefly explain why.

Solution: (4 pts) L_A is **not injective** since by (a) more than one vector in $\mathbb{F}^5$ is sent to the zero vector, and L_A is **not surjective** since by (b) not all vectors of $\mathbb{F}^4$ are in $\mathrm{Range}(L_A)$.

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(2) (15 Points) Let $T : \mathbb{F}^3 \to \mathbb{F}^2$ and $S : \mathbb{F}^2 \to \mathbb{F}^3$ be the functions

$$T \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} x_1 + 2x_2 - 3x_3 \\ x_1 - x_2 + 2x_3 \end{bmatrix} \quad \text{and} \quad S \begin{bmatrix} y_1 \\ y_2 \end{bmatrix} = \begin{bmatrix} y_1 - y_2 \\ y_1 + y_2 \\ 3y_1 + 2y_2 \end{bmatrix}.$$

(a) Find the matrices A and B such that $S(Y) = AY$ and $T(X) = BX$.

Solution: (4 pts) $S(Y) = \begin{bmatrix} y_1 - y_2 \\ y_1 + y_2 \\ 3y_1 + 2y_2 \end{bmatrix} = \begin{bmatrix} 1 & -1 \\ 1 & 1 \\ 3 & 2 \end{bmatrix} \begin{bmatrix} y_1 \\ y_2 \end{bmatrix} = AY$ and

$$T(X) = \begin{bmatrix} x_1 + 2x_2 - 3x_3 \\ x_1 - x_2 + 2x_3 \end{bmatrix} = \begin{bmatrix} 1 & 2 & -3 \\ 1 & -1 & 2 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = BX.$$

(b) Use composition of functions, not matrix multiplication, to find the formula for the composition $(S \circ T) \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$.

Solution: (5 pts) $(S \circ T) \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = S \left(T \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} \right) = S \begin{bmatrix} x_1 + 2x_2 - 3x_3 \\ x_1 - x_2 + 2x_3 \end{bmatrix}$

$$= \begin{bmatrix} (x_1 + 2x_2 - 3x_3) - (x_1 - x_2 + 2x_3) \\ (x_1 + 2x_2 - 3x_3) + (x_1 - x_2 + 2x_3) \\ 3(x_1 + 2x_2 - 3x_3) + 2(x_1 - x_2 + 2x_3) \end{bmatrix} = \begin{bmatrix} 3x_2 - 5x_3 \\ 2x_1 + x_2 - x_3 \\ 5x_1 + 4x_2 - 5x_3 \end{bmatrix}.$$

(c) Use your answer to part (b) to find the matrix C such that $(S \circ T)(X) = CX$.

Solution: (3 pts) $(S \circ T)(X) = \begin{bmatrix} 3x_2 - 5x_3 \\ 2x_1 + x_2 - x_3 \\ 5x_1 + 4x_2 - 5x_3 \end{bmatrix} = \begin{bmatrix} 0 & 3 & -5 \\ 2 & 1 & -1 \\ 5 & 4 & -5 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = CX.$

(d) What should be the relationship between the matrices A, B and C? Check that your matrices satisfy that relation.

Solution: (3 pts) The relationship should be that $AB = C$ (matrix multiplication). Check:

$$AB = \begin{bmatrix} 1 & -1 \\ 1 & 1 \\ 3 & 2 \end{bmatrix} \begin{bmatrix} 1 & 2 & -3 \\ 1 & -1 & 2 \end{bmatrix} = \begin{bmatrix} (1-1) & (2+1) & (-3-2) \\ (1+1) & (2-1) & (-3+2) \\ (3+2) & (6-2) & (-9+4) \end{bmatrix} = \begin{bmatrix} 0 & 3 & -5 \\ 2 & 1 & -1 \\ 5 & 4 & -5 \end{bmatrix} = C.$$

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(3) (15 Points, 3 pts each) Answer each question separately. No justifications are needed.

(a) For a **nonzero** matrix $A \in \mathbb{F}_7^4$, what are the possible values of $r = rank(A)$?

Solution: If $A \in \mathbb{F}_7^4$ is not the zero matrix, $rank(A)$ is the number of leading ones in its RREF, so $1 \leq rank(A) \leq 4 = Min(4, 7)$ since each leading one occupies a row, there is at least one, and no more than the number of rows.

(b) For $A \in \mathbb{F}_n^m$ what condition on $rank(A) = r$ is equivalent to the **non-homogeneous** linear system $AX = B$ being **inconsistent** for some $B \in \mathbb{F}^m$?

Solution: For $r = rank(A) < m$ we have $AX = B$ is inconsistent for some B because $[A|B]$ row reduces to $[C|D]$ with C in RREF, and C has at least one row of zeros, giving a consistency condition.

(c) For $A \in \mathbb{F}_n^m$ what condition on $rank(A) = r$ is equivalent to the **homogeneous** linear system $AX = 0_1^m$ having **nontrivial** solutions?

Solution: When $r = rank(A) < n$ the linear system $AX = 0$ has nontrivial solutions since there are $n - r > 0$ free variables corresponding to non-pivot columns in the RREF.

(d) If $A \in \mathbb{F}_n^m$ and $AX = 0_1^m$ has only the trivial solution, what is the **most** you can say about the relation between m and n?

Solution: If $n > m$ then $AX = 0$ would have at least one free variable, giving nontrivial solutions. By the contrapositive, $n > m$ is false so we must have $n \leq m$.

(e) What conditions on m, n and $rank(A)$ would mean that $A \in \mathbb{F}_n^m$ is invertible?

Solution: A is invertible when $m = n = rank(A)$.

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(4) (15 Points, 3 pts each) For $A \in \mathbb{F}_n^m$, let $L_A : \mathbb{F}^n \to \mathbb{F}^m$ be the linear function $L_A(X) = AX$. For $1 \leq j \leq n$ let $\mathbf{e}_j \in \mathbb{F}^n$ be the matrix with 1 in row j and 0 in all other rows. No justifications are needed for these questions.

(a) What relation between m and n would guarantee that L_A is **not injective**?

Solution: If $n > m$ then L_A is not injective since there would be free variables in the solution to $L_A(X) = 0_1^m$, giving a nontrivial kernel.

(b) What relation between m and n would guarantee that L_A is **not surjective**?

Solution: If $n < m$ then L_A is not surjective since more equations than variables guarantees a row of zeros in the RREF of A, giving a consistency condition for $AX = B$.

(c) If $\text{rank}(A) = n$ what does that tell you about L_A?

Solution: If $\text{rank}(A) = n$ then L_A is injective since n leading ones in the RREF means a leading one in each column and there are no free variables in $AX = 0_1^m$.

(d) If $\text{rank}(A) = m$ what does that tell you about L_A?

Solution: If $\text{rank}(A) = m$ then L_A is surjective since m leading ones in the RREF means no zero rows so $AX = B$ is always consistent.

(e) What is the relationship between A and $L_A(\mathbf{e}_j)$ for $1 \leq j \leq n$?

Solution: For $1 \leq j \leq n$, $L_A(\mathbf{e}_j) = Col_j(A)$ is the j^{th} column of A.

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(5) (15 Points, 3 pts each) No justifications are needed for these questions.

(a) Let $L : \mathbb{F}^3 \to \mathbb{F}^2$ be linear with $L(\mathbf{e}_1) = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$, $L(\mathbf{e}_2) = \begin{bmatrix} 1 \\ 2 \end{bmatrix}$, $L(\mathbf{e}_3) = \begin{bmatrix} -2 \\ 3 \end{bmatrix}$. Find $A \in \mathbb{F}^2_3$ such that $L(X) = L_A(X) = AX$ for all $X \in \mathbb{F}^3$.

Solution: The $A \in \mathbb{F}^2_3$ such that $L(X) = L_A(X) = AX$ for all $X \in \mathbb{F}^3$ would have to satisfy $L(\mathbf{e}_j) = A\mathbf{e}_j = Col_j(A)$ for $j = 1, 2, 3$, so $A = \begin{bmatrix} 1 & 1 & -2 \\ 1 & 2 & 3 \end{bmatrix}$.

(b) For $A \in \mathbb{F}^3_n$ find matrix E such that $B = EA$ is the matrix obtained from A by doing to A the elementary row operation $5Row_2(A) + Row_1(A) \to Row_1(A)$.

Solution: $E = \begin{bmatrix} 1 & 5 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$ obtained by doing the row operation to I_3.

(c) If $S = \{v_1, \cdots, v_k\}$ is an **independent** subset of $\mathbb{F}^m$, what is the relationship between k and m?

Solution: $k \leq m$ since m is the maximum size of an independent set in $\mathbb{F}^m$.

(d) For $A \in \mathbb{F}^n_n$ suppose $[A|I_n]$ row reduces to $[C|D]$ with C in RREF. When A is invertible what is C and what is D?

Solution: A is invertible when $C = I_n$, in which case $D = A^{-1}$.

(e) If $S \subseteq T \subseteq V$ and T is **dependent** in vector space V, what can you say about S?

Solution: S could be either independent or dependent.
