Quiz 6 | Available 2:20 – 11:00 PM

3-20-2020

Basis of V is $S = \{v_1, \dots, v_n\} \subseteq V$

s.t. S is ① **Indep** ② $\langle S \rangle = V$

$L_A : \mathbb{R}^n \to \mathbb{R}^m$ by $L_A(X) = AX$

$\ker(L_A) = \{X \in \mathbb{R}^n \mid AX = 0^m_1\}$ Solve

Interp

$[A \mid 0^m_1] \xrightarrow{r.r.} [C \mid 0^m_1]$ RREF

$x_1 =$
$x_2 =$
$\vdots$
$x_n =$

n−r free var's

Basis of $\ker = \{K_1, \dots, K_{n-r}\}$

$\ker(L_A) = \{f_1 K_1 + f_2 K_2 + \dots + f_{n-r} K_{n-r} \mid f_i \in \mathbb{R}\}$

$= \langle K_1, \dots, K_{n-r} \rangle$

specific

For each $K_i \in \ker(L_A)$ have $AK_i = 0^m$,

$$= \sum_{j=1}^{n} x_j \, \mathrm{Col}_j(A) = 0^m, \quad K_i = \begin{bmatrix} x_1 \\ \vdots \\ x_n \end{bmatrix}$$

is a dep. rel on $\{\mathrm{Col}_1(A), \ldots, \mathrm{Col}_n(A)\}$

$L : V \longrightarrow W$ is one-to-one (inj) iff

$\dim(\mathrm{Ker}(L)) = 0$

If $L = L_A$ $\quad \mathrm{Ker}(L_A) = \mathrm{Nul}(A) = \{X \in \mathbb{R}^n \mid AX = 0\}$

$m \times n$

$$[A \mid 0^m_1] \xrightarrow{\text{r.r.}} [C \mid 0^m_1]$$

RREF

Interp
$$\left.\begin{array}{l} x_1 = \\ x_2 = \\ \vdots \\ x_n = \end{array}\right\} \text{ n-r free variables}$$

$\mathrm{Ker}(L_A) = \{f_1 K_1 + \cdots + f_{n-r} K_{n-r} \in \mathbb{R}^n \mid f_i \in \mathbb{R}\}$

has Basis $\{K_1, \ldots, K_{n-r}\}$

$L: V \to W$ is onto (surj) when

$\mathrm{Range}(L) = W$.

$\dim(V) = \dim(\mathrm{Ker}(L)) + \dim(\mathrm{Range}(L))$

$A \in \mathbb{R}^{13}_{4}$ $\quad L_A = T : \mathbb{R}^4 \to \mathbb{R}^{13}$

1to1 $\iff \dim \mathrm{Ker}(L_A) = 0$

$\dim(V) = \#$ vectors in any basis of V

$4 = 0 + 4$

Ex: $V = \mathbb{R}^2_2$

$$S = \Big\{ \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}, \begin{bmatrix} 2 & 2 \\ 1 & 1 \end{bmatrix}, \overbrace{\begin{bmatrix} 1 & 1 \\ 2 & 2 \end{bmatrix}, \begin{bmatrix} 1 & 1 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 \\ 1 & 1 \end{bmatrix}}^{\text{redundant}} \Big\} \subseteq \mathbb{R}^2_2$$

$$\quad v_1 \qquad\quad v_2 \qquad\quad v_3 \qquad\quad v_4 \qquad\quad v_5$$

has 5 vectors in $\mathbb{R}^2_2$ where $\dim(\mathbb{R}^2_2) = 4$

S must be dep. (too big for $\mathbb{R}^2_2$)

Dep. rel's? Redundant vectors?

Solve $\displaystyle\sum_{j=1}^{5} x_j v_j = \theta = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$

$$-1\ -1\ -2\ 0\ -1$$

$$\left[\begin{array}{ccccc|c} 1 & 2 & 1 & 1 & 0 & 0 \\ 1 & 2 & 1 & 1 & 0 & 0 \\ 1 & 1 & 2 & 0 & 1 & 0 \\ 1 & 1 & 2 & 0 & 1 & 0 \end{array}\right] \to \left[\begin{array}{ccccc|c} 1 & 2 & 1 & 1 & 0 & 0 \\ 1 & 1 & 2 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{array}\right]$$

$$\left[\begin{array}{ccccc|c} 1 & 1 & 2 & 0 & 1 & 0 \\ 0 & 1 & -1 & 1 & -1 & 0 \\ & & & & & \end{array}\right] \to \left[\begin{array}{ccccc|c} 1 & 0 & 3 & -1 & 2 & 0 \\ 0 & 1 & -1 & 1 & -1 & 0 \end{array}\right]$$

$0 \; -1 \; 1 \; -1 \; 1$

$$\begin{aligned} x_1 &= -3r + s - 2t \\ x_2 &= r - s + t \\ x_3 &= r \\ x_4 &= s \\ x_5 &= t \end{aligned} \Bigg\} \in \mathbb{R} \text{ free}$$

$$\boxed{\begin{aligned} r=1, s=0, t=0 &: -3v_1 + 1v_2 + 1v_3 = 0_2^2 \\ r=0, s=1, t=0 &: 1v_1 - 1v_2 + 1v_4 = 0_2^2 \\ r=0, s=0, t=1 &: -2v_1 + v_2 + v_5 = 0_2^2 \end{aligned}}$$

$$\begin{aligned} v_3 &= 3v_1 - v_2 \\ v_4 &= -v_1 + v_2 \\ v_5 &= 2v_1 - v_2 \end{aligned}$$

$\langle S \rangle = \langle v_1, v_2 \rangle \quad \dim(\langle S \rangle) = 2$

Coordinates: Let $S = \{v_1, \dots, v_n\}$ be any basis of V. So $\forall v \in V$, $v = \sum_{j=1}^{n} a_j v_j$ where $a_1, \dots a_n$ are uniquely determined by v.

Def: The **coordinate vector** of v w.r.t. S ~~are~~ is

$$[v]_S = \begin{bmatrix} a_1 \\ a_2 \\ \vdots \\ a_n \end{bmatrix} \in \mathbb{R}^n \text{ iff } v = \sum_{j=1}^{n} a_j v_j$$

$\boxed{Rep(v)_S}$

Ex: If $S = \{e_1, \dots, e_n\} \subseteq \mathbb{R}^n$ is std basis

then $v = \begin{bmatrix} a_1 \\ \vdots \\ a_n \end{bmatrix} = \sum_{j=1}^{n} a_j e_j$ so $[v]_S = \begin{bmatrix} a_1 \\ \vdots \\ a_n \end{bmatrix}$

Ex: $V = \mathbb{R}^2$, $T = \left\{ \underset{v_1}{\begin{bmatrix} 1 \\ 1 \end{bmatrix}}, \underset{v_2}{\begin{bmatrix} 1 \\ 2 \end{bmatrix}} \right\}$

If $v = \begin{bmatrix} a \\ b \end{bmatrix} \in \mathbb{R}^2$ find $[v]_T = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 2a-b \\ b-a \end{bmatrix}$: solve

$x_1 v_1 + x_2 v_2 = v$: solve $\left[\begin{array}{cc|c} 1 & 1 & a \\ 1 & 2 & b \end{array}\right] \to \left[\begin{array}{cc|c} 1 & 1 & a \\ 0 & 1 & b-a \end{array}\right] \xrightarrow{0\ -1\quad a-b} \left[\begin{array}{cc|c} 1 & 0 & 2a-b \\ 0 & 1 & b-a \end{array}\right]$

$\left[\begin{array}{c|c} T & v \end{array}\right]$

$x_1 = 2a-b$
$x_2 = b-a$

Check: $(2a-b)\begin{bmatrix} 1 \\ 1 \end{bmatrix} + (b-a)\begin{bmatrix} 1 \\ 2 \end{bmatrix} = \begin{bmatrix} a \\ b \end{bmatrix}$ ✓

<u>Def</u>: If S and T are both bases of V with $\dim(V)=n$

Then we call ${}_T P_S \in \mathbb{R}^n_n$ the transition matrix from S to T when

$${}_T P_S [v]_S = [v]_T \qquad \forall v \in V$$

Also, ${}_S P_T [v]_T = [v]_S$

<u>Th</u> For any bases S and T of V ($\dim V = n$) we can find a unique transition matrix ${}_T P_S$ by row reducing $[T|S] \xrightarrow[\text{to}]{\text{r.r}} [I_n | {}_T P_S]$

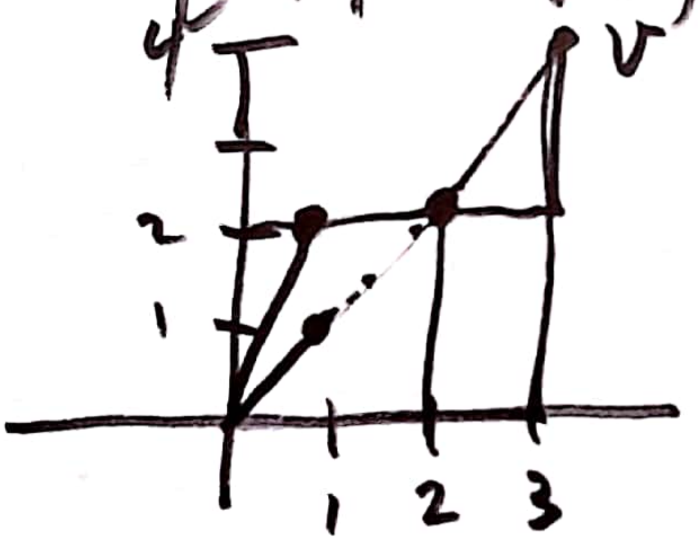
as columns

and ${}_S P_T = ({}_T P_S)^{-1}$

If $v = \begin{bmatrix} 3 \\ 4 \end{bmatrix} \in \mathbb{R}^2$ then $[v]_S = \begin{bmatrix} 3 \\ 4 \end{bmatrix}$

$$[v]_T = \begin{bmatrix} 6-4 \\ 4-3 \end{bmatrix} = \begin{bmatrix} 2 \\ 1 \end{bmatrix}$$

because $\begin{bmatrix} 3 \\ 4 \end{bmatrix} = 2\begin{bmatrix} 1 \\ 1 \end{bmatrix} + 1\begin{bmatrix} 1 \\ 2 \end{bmatrix}$



If $[v]_T = \begin{bmatrix}-1\\2\end{bmatrix}$ find v $\left([v]_S\right)$

It means $v = -1\underset{v_1}{\begin{bmatrix}1\\1\end{bmatrix}} + 2\underset{v_2}{\begin{bmatrix}1\\2\end{bmatrix}} = \begin{bmatrix}1\\3\end{bmatrix} = [v]_S$

${}_TP_S = \begin{bmatrix}2&-1\\-1&1\end{bmatrix}$ To find ${}_SP_T$ F.r $\left[\underset{S}{\begin{matrix}1&0\\0&1\end{matrix}}\;\middle|\;\underset{T}{\begin{matrix}1&1\\1&2\end{matrix}}\right]$ so

RREF

${}_SP_T = \begin{bmatrix}1&1\\1&2\end{bmatrix}$

${}_SP_T[v]_T = [v]_S$ so $\underset{{}_SP_T}{\begin{bmatrix}1&1\\1&2\end{bmatrix}}\underset{[v]_T}{\begin{bmatrix}-1\\2\end{bmatrix}} = \underset{[v]_S}{\begin{bmatrix}1\\3\end{bmatrix}}$

Relationship between $[v]_T$ and $[v]_S$:

$$[v]_T = \begin{bmatrix} 2a-b \\ b-a \end{bmatrix} = \underbrace{\begin{bmatrix} 2 & -1 \\ -1 & 1 \end{bmatrix}}_{{}_T P_S} \underbrace{\begin{bmatrix} a \\ b \end{bmatrix}}_{[v]_S}$$

Transition Matrix from

S to T

$$\underbrace{[v]_T}_{\in \mathbb{R}^2} = \underbrace{{}_T P_S}_{2\times 2} \underbrace{[v]_S}_{\in \mathbb{R}^2}$$

$${}_{S}P_{T}[v]_{T} = [v]_{S}$$

$$\begin{bmatrix} 1 & 1 \\ 1 & 2 \end{bmatrix} \begin{bmatrix} 2a-b \\ b-a \end{bmatrix} = \begin{bmatrix} a \\ b \end{bmatrix}$$

$$\begin{bmatrix} 1 & 1 \\ 1 & 2 \end{bmatrix} \begin{bmatrix} 2 & -1 \\ -1 & 1 \end{bmatrix} \begin{bmatrix} a \\ b \end{bmatrix}$$

$$= \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} a \\ b \end{bmatrix} = \begin{bmatrix} a \\ b \end{bmatrix}$$

inverses of each other.

<u>Exercise</u>: In $\mathbb{R}^3$ Let $S = \{e_1, e_2, e_3\}$ be the std basis and let $T = \left\{ \underbrace{\begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}}_{v_1}, \underbrace{\begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}}_{v_2}, \underbrace{\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}}_{v_3} \right\}$

For $v = \underbrace{\begin{bmatrix} a \\ b \\ c \end{bmatrix}}_{= [v]_S} \in \mathbb{R}^3$ find $[v]_T$

$$\left[\underbrace{\begin{matrix} 1 & 1 & 1 \\ 1 & 1 & 0 \\ 1 & 0 & 0 \end{matrix}}_{T} \middle| \underbrace{\begin{matrix} a \\ b \\ c \end{matrix}}_{v} \right] \xrightarrow{r.r.} \left[I_3 \middle| \underbrace{\begin{matrix} c \\ b-c \\ a-b \end{matrix}}_{[v]_T} \right]$$

Also find

${}_T P_S$: $[T \mid \underbrace{S}_{I_3}] \xrightarrow{r.r.} [I_3 \mid {}_T P_S]$

${}_S P_T$: $[S \mid T] \xrightarrow[=]{} [I_3 \mid {}_S P_T]$ so ${}_S P_T = \begin{bmatrix} 1 & 1 & 1 \\ 1 & 1 & 0 \\ 1 & 0 & 0 \end{bmatrix}$

Ex: $V = \mathbb{R}^2_2$ $\quad S = \left\{ \underset{v_1}{\begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}}, \underset{v_2}{\begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}}, \underset{v_3}{\begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix}}, \underset{v_4}{\begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}} \right\}$

is std basis. $\quad T = \left\{ \underset{w_1}{\begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}}, \underset{w_2}{\begin{bmatrix} 1 & 1 \\ 1 & 0 \end{bmatrix}}, \underset{w_3}{\begin{bmatrix} 1 & 1 \\ 0 & 0 \end{bmatrix}}, \underset{w_4}{\begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}} \right\}$

If $v = \begin{bmatrix} a & b \\ c & d \end{bmatrix} \in \mathbb{R}^2_2$

then $[v]_S \in \mathbb{R}^4$ when $v = \sum_{j=1}^{4} x_j \cdot v_j$ Solve

$= \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix}$

$$\begin{array}{c} I_4 \\ \left[\begin{array}{cccc|c} 1 & 0 & 0 & 0 & a \\ 0 & 1 & 0 & 0 & b \\ 0 & 0 & 1 & 0 & c \\ 0 & 0 & 0 & 1 & d \end{array}\right] \\ v_1 \; v_2 \; v_3 \; v_4 \quad v \end{array} \quad \begin{array}{l} x_1 = a \\ x_2 = b \\ x_3 = c \\ x_4 = d \end{array}$$

S as columns

$= \begin{bmatrix} a \\ b \\ c \\ d \end{bmatrix}$

Find $\cancel{\begin{bmatrix}a&b\\c&d\end{bmatrix}}$ $[v]_T$ for $v=\begin{bmatrix}a&b\\c&d\end{bmatrix}$

Solve $v=\sum_{j=1}^{4} x_j w_j$ that is.

$$\underbrace{\left[\begin{array}{cccc|c}1&1&1&1&a\\1&1&1&0&b\\1&1&0&0&c\\1&0&0&0&d\end{array}\right]}_{T \text{ as col's} \qquad v} \xrightarrow{\text{r.r.}} \underbrace{\left[\begin{array}{cccc|c}1&0&0&0&d\\0&1&0&0&c-d\\0&0&1&0&b-c\\0&0&0&1&a-b\end{array}\right]}_{I_4 \qquad [v]_T} \begin{array}{c}x_1\\x_2\\x_3\\x_4\end{array}$$

Check

$$\underbrace{d\begin{bmatrix}1&1\\1&1\end{bmatrix}}_{w_1}+\underbrace{(c-d)\begin{bmatrix}1&1\\1&0\end{bmatrix}}_{w_2}+\underbrace{(b-c)\begin{bmatrix}1&1\\0&0\end{bmatrix}}_{w_3}+\underbrace{(a-b)\begin{bmatrix}1&0\\0&0\end{bmatrix}}_{w_4}=\underbrace{\begin{bmatrix}a&b\\c&d\end{bmatrix}}_{v} \checkmark$$

$$[v]_T = \begin{bmatrix} d \\ c-d \\ b-c \\ a-b \end{bmatrix} = \underbrace{\begin{bmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & -1 \\ 0 & 1 & -1 & 0 \\ 1 & -1 & 0 & 0 \end{bmatrix}}_{{}_T P_S} \underbrace{\begin{bmatrix} a \\ b \\ c \\ d \end{bmatrix}}_{[v]_S}$$

$${}_S P_T = \begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 0 \\ 1 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{bmatrix}$$

$$\left[I_4 \mid T \right]$$