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Let V be a v.s. with basis $S = \{v_1, \ldots, v_n\}$ So $\forall v \in V \ \exists a_1, \ldots, a_n \in \mathbb{R}$ s.t. $v = \sum_{j=1}^{n} a_j v_j$ and $a_1, \ldots, a_n$ are uniquely det'd by v.

This gives a function

$$[\cdot]_S : V \longrightarrow \mathbb{R}^n \qquad n = \dim(V)$$

$$[v]_S = \begin{bmatrix} a_1 \\ a_2 \\ \vdots \\ a_n \end{bmatrix} \in \mathbb{R}^n$$

Th: $[\,]_S : V \longrightarrow \mathbb{R}^n$ is a bijective linear map.

<u>Proof</u>: Let $v = \sum_{j=1}^{n} a_j v_j$, $w = \sum_{j=1}^{n} b_j v_j \in V$

so $[v]_S = \begin{bmatrix} a_1 \\ \vdots \\ a_n \end{bmatrix}$ and $[w]_S = \begin{bmatrix} b_1 \\ \vdots \\ b_n \end{bmatrix}$ and

$$v + w = \sum_{j=1}^{n} (a_j + b_j) v_j \quad \text{so}$$

$$[v+w]_S = \begin{bmatrix} a_1 + b_1 \\ \vdots \\ a_n + b_n \end{bmatrix} = \begin{bmatrix} a_1 \\ \vdots \\ a_n \end{bmatrix} + \begin{bmatrix} b_1 \\ \vdots \\ b_n \end{bmatrix} = [v]_S + [w]_S$$

Also, $\forall c \in \mathbb{R}$, $c \cdot v = \sum_{j=1}^{n} (c a_j) v_j$ so

$$[cv]_S = \begin{bmatrix} c a_1 \\ \vdots \\ c a_n \end{bmatrix} = c \begin{bmatrix} a_1 \\ \vdots \\ a_n \end{bmatrix} = c [v]_S$$

To show $[\cdot]_S$ is injective, look at $\ker([\ ]_S) = \left\{ v \in V \,\middle|\, [v]_S = \begin{bmatrix} 0 \\ \vdots \\ 0 \end{bmatrix} = 0_1^n \right\}$ so $v \in \ker$ iff $v = \sum_{j=1}^{n} 0 v_j = \theta_V$ so $\ker = \{\theta_V\}$ is trivial, and $[\ ]_S$ is inj.

To show $[\ ]_S$ is surjective, $\forall \begin{bmatrix} a_1 \\ \vdots \\ a_n \end{bmatrix} \in \mathbb{R}^n$ find $v \in V$ s.t. $[v]_S = \begin{bmatrix} a_1 \\ \vdots \\ a_n \end{bmatrix}$. Just take $v = \sum_{j=1}^{n} a_j v_j$ then get it. $\blacksquare$

Example: $S = \left\{ \begin{bmatrix} 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \end{bmatrix} \right\} \subseteq \mathbb{R}^2$

$$\underset{e_1}{\begin{bmatrix} 1 \\ 0 \end{bmatrix}}, \quad \underset{e_2}{\begin{bmatrix} 0 \\ 1 \end{bmatrix}}$$

$\forall v = \begin{bmatrix} a \\ b \end{bmatrix} \in \mathbb{R}^2$

$= a \begin{bmatrix} 1 \\ 0 \end{bmatrix} + b \begin{bmatrix} 0 \\ 1 \end{bmatrix} = ae_1 + be_2$ so $[v]_S = \underbrace{\begin{bmatrix} a \\ b \end{bmatrix}}_{= v}$

$T = \left\{ \underset{v_1}{\begin{bmatrix} 1 \\ 2 \end{bmatrix}}, \underset{v_2}{\begin{bmatrix} 1 \\ 3 \end{bmatrix}} \right\}$, To write $v = x_1 v_1 + x_2 v_2$

solve $x_1 \begin{bmatrix} 1 \\ 2 \end{bmatrix} + x_2 \begin{bmatrix} 1 \\ 3 \end{bmatrix} = \begin{bmatrix} a \\ b \end{bmatrix}$ **Row reduce**

$$\left[\begin{array}{cc|c} 1 & 1 & a \\ 2 & 3 & b \end{array}\right] \to \left[\begin{array}{cc|c} 1 & 1 & a \\ 0 & 1 & b-2a \end{array}\right] \to \left[\begin{array}{cc|c} 1 & 0 & 3a-b \\ 0 & 1 & b-2a \end{array}\right] \quad \begin{array}{l} x_1 = 3a-b \\ x_2 = -2a+b \end{array}$$

$T \quad v$

$-2 \; -2 \; -2a$

$0 \; -1 \; 2a-b$

so $[v]_T = \begin{bmatrix} 3a-b \\ -2a+b \end{bmatrix}$

so $$[v]_T = \begin{bmatrix}3a-b\\-2a+b\end{bmatrix} = \underset{{}_T P_S}{\begin{bmatrix}3&-1\\-2&1\end{bmatrix}}\underset{[v]_S}{\begin{bmatrix}a\\b\end{bmatrix}}$$

Transition Matrix from S to T

$$\begin{tikzcd}
 & V=\mathbb{R}^2 \arrow[dl, "{[\ ]_S}"'] \arrow[dr, "{[\ ]_T}"] & \\
\mathbb{R}^2 \arrow[rr, "L_A", "A={}_T P_S"'] & & \mathbb{R}^2 \arrow[ll, bend left, "{}_S P_T"]
\end{tikzcd}$$

$${}_S P_T [v]_T = [v]_S$$

If $v = \begin{bmatrix}1\\2\end{bmatrix}$ then

$$[v]_T = \begin{bmatrix}3(1)-2\\-2(1)+2\end{bmatrix} = \begin{bmatrix}1\\0\end{bmatrix}$$

$$v = 1v_1 + 0v_2$$

If $w = \begin{bmatrix}1\\3\end{bmatrix}$ then

$$[w]_T = \begin{bmatrix}3(1)-3\\-2(1)+3\end{bmatrix} = \begin{bmatrix}0\\1\end{bmatrix}$$

$$w = 0v_1 + 1v_2$$

Generally, For $S=\{v_1,\dots,v_n\}$ basis of V

$$[v_j]_S = \begin{bmatrix}0\\ \vdots\\ 1\\ \vdots\\ 0\end{bmatrix} \leftarrow \text{row } j \quad \text{since } v_j = \underline{0}\,v_1+\dots+\underline{1}\,v_j+\dots+\underline{0}\,v_n$$

$= e_j$

j^{th} std basis vector.

If $L: V \longrightarrow W$, $\quad v=\sum_{j=1}^{n} a_j v_j$

$$L(v)=\sum_{j=1}^{n} a_j L(v_j)$$

so $\{L(v_1),\dots,L(v_n)\} = L(S)$ determines L.

$$\underbrace{{}_S P_T \begin{bmatrix} 3a-b \\ -2a+b \end{bmatrix}}_{[v]_T} = {}_S P_T\left({}_T P_S [v]_S\right) = \left(\underbrace{{}_S P_T\, {}_T P_S}_{= I_2}\right)[v]_S$$

$$= I_2 \quad [v]_S$$

$$ {}_S P_T = \left({}_T P_S\right)^{-1}$$

$$\begin{array}{c} \left[\begin{array}{cc|cc} 3 & -1 & 1 & 0 \\ -2 & 1 & 0 & 1 \end{array}\right] \\ \begin{array}{cc|cc} 2 & 0 & 2 & 2 \end{array} \\ {}_S P_T \quad I_2 \end{array} \longrightarrow \begin{array}{c} \left[\begin{array}{cc|cc} 1 & 0 & 1 & 1 \\ 0 & 1 & 2 & 3 \end{array}\right] \\ I_2 \quad T \end{array}$$

Th: Let $S = \{v_1, \dots, v_n\}$ and
$T = \{w_1, \dots, w_n\}$ be bases of V.

Then there are invertible $n \times n$ matrices ${}_T P_S$ and ${}_S P_T$ such that $\forall v \in V$

$$\boxed{{}_T P_S [v]_S = [v]_T} \quad \text{and} \quad {}_S P_T [v]_T = [v]_S$$

and ${}_T P_S = ({}_S P_T)^{-1}$ and to find them:

Interp

solve $[T \mid S] \xrightarrow{r.r.} [I_n \mid {}_T P_S]$ $\quad [v_j]_T = \mathrm{Col}_j({}_T P_S)$

"as columns"

and $[S \mid T] \xrightarrow{r.r.} [I_n \mid {}_S P_T]$ $\quad [w_j]_S = \mathrm{Col}_j({}_S P_T)$

Proof: ${}_T P_S [v_j]_S = [v_j]_T$ for $1 \le j \le n$

Need:

$${}_T P_S \, e_j = \mathrm{Col}_j({}_T P_S)$$

Solving

$$\sum_{i=1}^{n} x_{ij} w_i = v_j$$

Can do this. Then

$\forall v \in V$ $\quad v = \sum a_j v_j$

$${}_T P_S [v]_S = {}_T P_S \left[\sum a_j v_j\right]_S = {}_T P_S \left(\sum a_j [v_j]_S\right)$$

$$= \sum a_j \left({}_T P_S [v_j]_S\right) = \sum a_j [v_j]_T = \left[\sum a_j v_j\right]_T$$

$$= [v]_T$$

Ex: $V = \mathbb{R}^2_2$ $\quad S = \left\{ \underbrace{\begin{bmatrix}1&0\\0&0\end{bmatrix}}_{v_1}, \underbrace{\begin{bmatrix}0&1\\0&0\end{bmatrix}}_{v_2}, \underbrace{\begin{bmatrix}0&0\\1&0\end{bmatrix}}_{v_3}, \underbrace{\begin{bmatrix}0&0\\0&1\end{bmatrix}}_{v_4} \right\}$ std

$T = \left\{ \underbrace{\begin{bmatrix}1&1\\1&1\end{bmatrix}}_{w_1}, \underbrace{\begin{bmatrix}1&1\\1&0\end{bmatrix}}_{w_2}, \underbrace{\begin{bmatrix}1&1\\0&0\end{bmatrix}}_{w_3}, \underbrace{\begin{bmatrix}1&0\\0&0\end{bmatrix}}_{w_4} \right\}$ $\quad v = \begin{bmatrix}a&b\\c&d\end{bmatrix}$

$v = a v_1 + b v_2 + c v_3 + d v_4$ so $[v]_S = \begin{bmatrix}a\\b\\c\\d\end{bmatrix} \in \mathbb{R}^4$

Find $[v]_T = \begin{bmatrix}x_1\\x_2\\x_3\\x_4\end{bmatrix}$ s.t. $\sum_{j=1}^{4} x_j w_j = v$

$$= \begin{bmatrix}(x_1+x_2+x_3+x_4) & (x_1+x_2+x_3)\\ (x_1+x_2) & x_1\end{bmatrix}$$

Solve:

$$\underbrace{\left[\begin{array}{cccc|c}1&1&1&1&a\\1&1&1&0&b\\1&1&0&0&c\\1&0&0&0&d\end{array}\right]}_{T \quad\quad v \text{ as columns}} \xrightarrow{\text{r.r.}} \left[\begin{array}{cccc|c}1&0&0&0&d\\0&1&0&0&c-d\\0&0&1&0&b-c\\0&0&0&1&a-b\end{array}\right]$$

<u>Check</u>:

$d w_1 + (c-d) w_2 + (b-c) w_3 + (a-b) w_4 = v$

So $[v]_T = \begin{bmatrix} d \\ c-d \\ b-c \\ a-b \end{bmatrix} = \underbrace{\begin{bmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & -1 \\ 0 & 1 & -1 & 0 \\ 1 & -1 & 0 & 0 \end{bmatrix}}_{{}_TP_S} \underbrace{\begin{bmatrix} a \\ b \\ c \\ d \end{bmatrix}}_{[v]_S}$

Algor.

$$\left[\begin{array}{cccc|cccc} 1 & 1 & 1 & 1 & 1 & 0 & 0 & 0 \\ 1 & 1 & 1 & 0 & 0 & 1 & 0 & 0 \\ 1 & 1 & 0 & 0 & 0 & 0 & 1 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{array}\right] \xrightarrow{\text{r.r.}} \left[\begin{array}{cccc|cccc} 1 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 & 0 & 0 & 1 & -1 \\ 0 & 0 & 1 & 0 & 0 & 1 & -1 & 0 \\ 0 & 0 & 0 & 1 & 1 & -1 & 0 & 0 \end{array}\right]$$

$T \qquad S$

as col's

Check ${}_TP_S = ({}_SP_T)^{-1}$

$$\left[\begin{array}{c|cccc} & 1 & 1 & 1 & 1 \\ I_4 & 1 & 1 & 1 & 0 \\ & 1 & 1 & 0 & 0 \\ & 1 & 0 & 0 & 0 \end{array}\right]$$

$S \qquad T$

already RREF

$= {}_SP_T$.

$$\begin{tikzcd}
 & v \in V = \mathbb{R}^2_2 \arrow[dl, "{[v]_S}"'] \arrow[dr, "{[v]_T}"] & \\
\mathbb{R}^4 \arrow[rr, shift left, "{}_T P_S"] & & \mathbb{R}^4 \arrow[ll, shift left, "{}_S P_T"]
\end{tikzcd}$$

$${}_T P_S [v]_S = [v]_T$$

$$\underset{n\times n}{{}_S P_T}\ \underset{n\times 1}{[v]_T} = \underset{n\times 1}{[v]_S}$$

In general:

$$\begin{tikzcd}
 & V \arrow[dl, "{[\ ]_S}"'] \arrow[dr, "{[\ ]_T}"] & \\
\mathbb{R}^n \arrow[rr, shift left, "{}_T P_S"] & & \mathbb{R}^n \arrow[ll, shift left, "{}_S P_T"]
\end{tikzcd}$$

$${}_T P_S = \left({}_S P_T\right)^{-1}$$