

Fri. 3-27-2020

Let  $V = \text{Poly}_2 = \{a_0 + a_1 t + a_2 t^2 \mid a_i \in \mathbb{R}\}$   $\perp$

$S = \{1, t, t^2\}$   $T = \{t^2, t^2+t, t^2+t+1\}$

std basis of  $V$ .  $p = at^2 + bt + c \cdot 1$ .

$$[p]_S = \begin{bmatrix} c \\ b \\ a \end{bmatrix} \quad [p]_T = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} \quad \text{s.t.} \quad =$$

$$p = x_1(t^2) + x_2(t^2+t) + x_3(t^2+t+1)$$
$$= (x_1 + x_2 + x_3)t^2 + (x_2 + x_3)t + x_3 \cdot 1$$

solve  $\begin{bmatrix} 1 & 1 & 1 & | & a \\ 0 & 1 & 1 & | & b \\ 0 & 0 & 1 & | & c \end{bmatrix} \xrightarrow{\text{r.r.}} \begin{bmatrix} 1 & 0 & 0 & | & a-b \\ 0 & 1 & 0 & | & b-c \\ 0 & 0 & 1 & | & c \end{bmatrix}$

$x_1 = a - b$   
 $x_2 = b - c$   
 $x_3 = c$

$$[P]_T = \begin{bmatrix} a-b \\ b-c \\ c \end{bmatrix} = \begin{bmatrix} 0 & -1 & 1 \\ -1 & 1 & 0 \\ 1 & 0 & 0 \end{bmatrix} \begin{bmatrix} c \\ b \\ a \end{bmatrix}$$

L2

$T P_S \swarrow [P]_S$

$$\begin{array}{l} t^2 \\ t \\ 1 \end{array} \begin{bmatrix} 1 & 1 & 1 & | & 0 & 0 & 1 \\ 0 & 1 & 1 & | & 0 & 1 & 0 \\ 0 & 0 & 1 & | & 1 & 0 & 0 \end{bmatrix} \xrightarrow{\text{r.r.}} \begin{bmatrix} 1 & 0 & 0 & | & 0 & -1 & 1 \\ 0 & 1 & 0 & | & -1 & 1 & 0 \\ 0 & 0 & 1 & | & 1 & 0 & 0 \end{bmatrix}$$

$T$ 
 $S$ 
 $I_3$ 
 $T P_S$

"as columns"

Isomorphism: (property of  $L$ ) [3]

$L: V \rightarrow W$  is an isom. when  $L$  is  
a bij. lin. map.

Isomorphic: (relation between vector  
spaces)

Say  $V \cong W$  ( $V$  is isom. to  $W$ ) when

$\exists L: V \rightarrow W$  which is an isomorphism

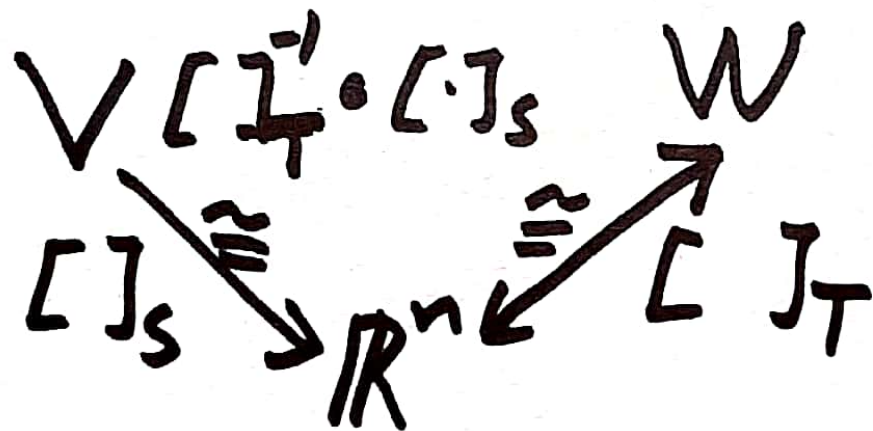
Th: If  $\dim(V) = n$  Then  $V \cong \mathbb{R}^n$

Pf. [  $\cdot$  ]  $I_S: V \rightarrow \mathbb{R}^n$  is an isom. for  
any basis  $S$  of  $V$ .

Q If  $\dim(V) = n = \dim(W)$  then  $\square$

$$V \cong W.$$

Pf: Let  $S$  be a basis of  $V$  and  
" " " " " " " "  $W$  so



Claim:

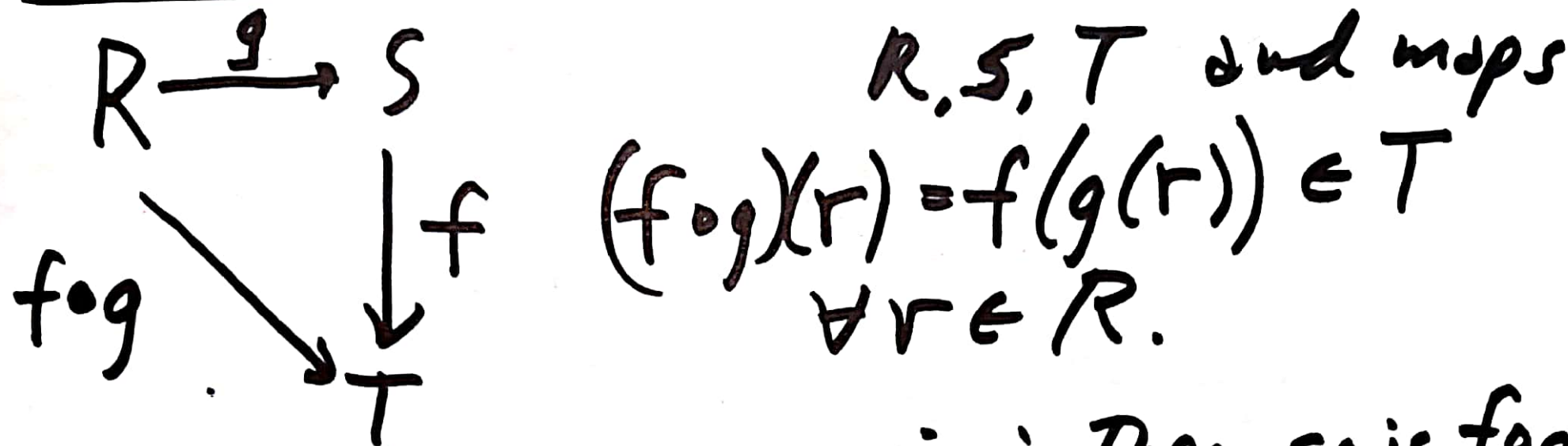
Composition of  $[ ]_T^{-1}$  and  $[ ]_S$

is a big.

So  $[ ]_T^{-1} \circ [ ]_S : V \xrightarrow{\cong} W$  is an isom.



Theory of functions: For sets  $\mathbb{L}$



- Th (a) If  $f$  and  $g$  are inj then so is  $f \circ g$
- (b) If  $f$  and  $g$  are surj " " " "
- (c) ----- " bij " " " "
- (d) If  $f$  is bij then so is  $f^{-1}$ .

P.F. (a) Suppose  $f$  and  $g$  are inj. so  $\lfloor$

$$f(a_1) = f(a_2) \Rightarrow a_1 = a_2 \quad \text{for } a_1, a_2 \in S$$

$$g(r_1) = g(r_2) \Rightarrow r_1 = r_2 \quad \text{for } r_1, r_2 \in R$$

Suppose  $(f \circ g)(r_1) = (f \circ g)(r_2)$  so

$$\underbrace{f(g(r_1))}_{a_1} = \underbrace{f(g(r_2))}_{a_2}$$

so by inj. of  $f$  get

$$g(r_1) = g(r_2) \quad \text{so by inj. of } g \text{ get } r_1 = r_2$$

so  $f \circ g$  is inj.  $\checkmark$

(b) Suppose  $f$  and  $g$  are surj.  $\square$

①  $\forall t \in T, \exists a \in S$  s.t.  $f(a) = t$  and

②  $\forall a \in S, \exists r \in R$  s.t.  $g(r) = a$ .

want:  $\forall t \in T$ , find  $r \in R$  s.t.  $(f \circ g)(r) = t$

By ①  $\forall t \in T, \exists a \in S$  s.t.  $f(a) = t$

By ②  $\forall a \in S, \exists r \in R$  s.t.  $g(r) = a$  so

so  $(f \circ g)(r) = f(g(r)) = f(a) = t$  ✓

(c) Follows from (a) and (b).

(d) If  $f$  is bij so is  $f^{-1}$  [8]

$f$  is inj and surj so  $f^{-1}: T \rightarrow S$   
is defined by  $f^{-1}(t) = a$  if  $t = f(a)$

$a \in S \xrightarrow{f} t \in T$  If  $f^{-1}(t_1) = f^{-1}(t_2)$   
 $\xleftarrow{f^{-1}}$  then  $f(f^{-1}(t_1)) = f(f^{-1}(t_2))$

shows  $f^{-1}$  is inj  $t_1$   $t_2$

To show  $f^{-1}$  is surj,  
 $\forall a \in S$  show  $\exists t \in T$  s.t.  $f^{-1}(t) = a$   
Can use  $t = f(a)$  get  $f^{-1}(f(a)) = a$

□



Back to Lin Alg:  
If  $L: V \rightarrow W$  is an isom (bij, lin) L9  
Then  $L^{-1}: W \rightarrow V$  is also isom (bij, lin.)

Pf: Know  $L^{-1}$  is bij from general  
function theory. Check  $L^{-1}$  is lin.

$$\forall w_1, w_2 \in W \quad L^{-1}(w_i) = v_i \quad i=1, 2$$

means  $w_i = L(v_i)$  so

$$w_1 + w_2 = L(v_1) + L(v_2) = L(v_1 + v_2) \quad \text{so}$$

$$L^{-1}(w_1 + w_2) = v_1 + v_2 = L^{-1}(w_1) + L^{-1}(w_2)$$

$$\forall c \in \mathbb{R}, \quad L^{-1}(cw_1) = cL^{-1}(w_1)$$

want

Say  $L^{-1}(w_1) = v_1$  so  $w_1 = L(v_1)$  10

so  $cw_1 = cL(v_1) = L(cv_1)$  so

$L^{-1}(cw_1) = cv_1 = cL^{-1}(w_1)$   $\square$  ✓

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Th: If  $\dim(V) = n \neq m = \dim(W)$

then  $V \not\cong W$ .

Pf: If  $L: V \rightarrow W$  were an isom.  
(bij, inj and surg) then  $\ker(L) = \{0_V\}$   
 $\dim \ker(L) = 0$  so  $\dim(V) = \dim(\text{Range}(L))$   
 $= \dim(W)$  contradicts  $n \neq m$ .  $\square$

Fact:  $\cong$  is reflexive, Symm and Transitive, so it is an equivalence rel.

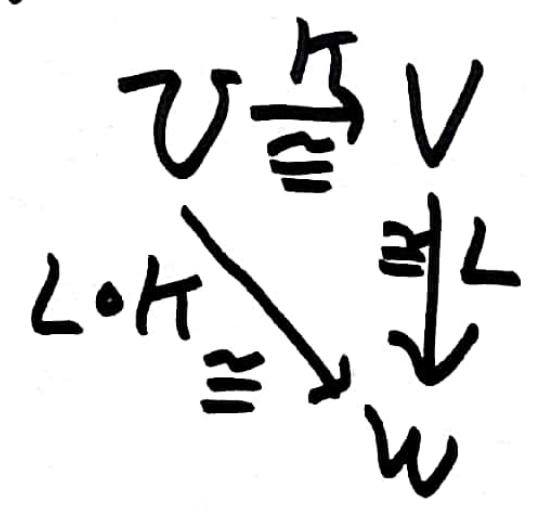
①  $V \cong V$  since  $I_V: V \rightarrow V$  is an isom.

② If  $V \cong W$ ,  $\exists L: V \rightarrow W$  isom so  
 $\exists L^{-1}: W \rightarrow V$  isom so  $W \cong V$ .

③ If  $U \cong V$  and  $V \cong W$  then  
 $K: U \rightarrow V$  and  $L: V \rightarrow W$  isom's

so  $L \circ K: U \rightarrow W$  is an isom

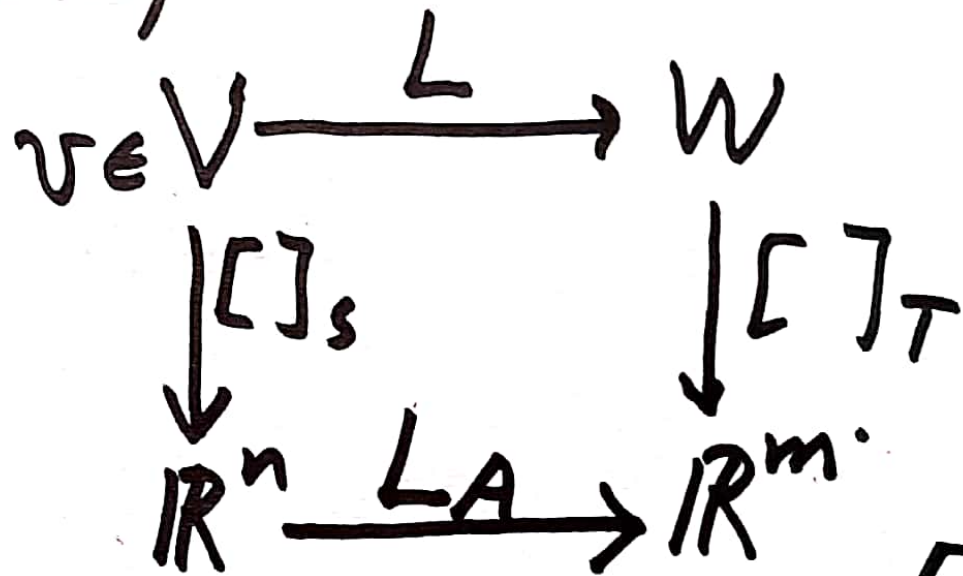
so  $U \cong W$ .





Suppose  $L: V \rightarrow W$  and  $\underline{12}$   
 $S = \{v_1, \dots, v_n\}$  is a basis of  $V$  and  
 $T = \{w_1, \dots, w_m\}$  is a basis of  $W$  so have

$\dim(V) = n$   
 $\dim(W) = m$



Want to  
find

$A \in \mathbb{R}^{m \times n}$  s.t.

$$\underset{m \times n}{A} \underset{n \times 1}{[v]}_S = \underset{m \times 1}{[L(v)]}_T$$

Th: Can Always find such an  $A$ , call it  
matrix representing  $L$  from  $S$  to  $T$ ,  $A = [L]_{S,T}$