Fri. 3-27-2020

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Let $V = Poly_2 = \{a_0 + a_1 t + a_2 t^2 \mid a_i \in \mathbb{R}\}$

$S = \{1, t, t^2\}$ std basis of V. $T = \{t^2, t^2+t, t^2+t+1\}$

$p = at^2 + bt + c\,1.$

$$[p]_S = \begin{bmatrix} c \\ b \\ a \end{bmatrix} \qquad [p]_T = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} \text{ s.t.}$$

$=$

$$p = x_1(t^2) + x_2(t^2+t) + x_3(t^2+t+1)$$
$$= (x_1+x_2+x_3)t^2 + (x_2+x_3)t + x_3 \cdot 1$$

solve $\left[\begin{array}{ccc|c} 1 & 1 & 1 & a \\ 0 & 1 & 1 & b \\ 0 & 0 & 1 & c \end{array}\right]$ $\xrightarrow{r.r}$ $\left[\begin{array}{ccc|c} 1 & 0 & 0 & a-b \\ 0 & 1 & 0 & b-c \\ 0 & 0 & 1 & c \end{array}\right]$ $\begin{array}{l} x_1 = a-b \\ x_2 = b-c \\ x_3 = c \end{array}$

$\quad T \qquad\quad p$

$$[p]_T = \begin{bmatrix} a-b \\ b-c \\ c \end{bmatrix} = \underbrace{\begin{bmatrix} 0 & -1 & 1 \\ -1 & 1 & 0 \\ 1 & 0 & 0 \end{bmatrix}}_{{}_T P_S} \underbrace{\begin{bmatrix} c \\ b \\ a \end{bmatrix}}_{[p]_S}$$

$$\begin{array}{c} t^2 \\ t \\ 1 \end{array} \underbrace{\left[\begin{array}{ccc|ccc} 1 & 1 & 1 & 0 & 0 & 1 \\ 0 & 1 & 1 & 0 & 1 & 0 \\ 0 & 0 & 1 & 1 & 0 & 0 \end{array}\right]}_{T \qquad\qquad S} \xrightarrow{\text{r.r.}} \underbrace{\left[\begin{array}{ccc|ccc} 1 & 0 & 0 & 0 & -1 & 1 \\ 0 & 1 & 0 & -1 & 1 & 0 \\ 0 & 0 & 1 & 1 & 0 & 0 \end{array}\right]}_{I_3 \qquad\qquad {}_T P_S}$$

"as columns"

Isomorphism: (property of L)
$L: V \to W$ is an isom. when L is a bij. lin. map.

Isomorphic: (relation between vector spaces)

Say $V \cong W$ (V is isom. to W) when $\exists L: V \to W$ which is an isomorphism

Th: If $\dim(V) = n$ Then $V \cong \mathbb{R}^n$

Pf. $[\cdot]_S : V \to \mathbb{R}^n$ is an isom. for any basis S of V.

Th If $\dim(V) = n = \dim(W)$ then $V \cong W$.

<u>Pf</u>: Let S be a basis of V and
" T " " " " W so

$$\begin{tikzcd} V \arrow[rr, "{[\ ]_T^{-1} \circ [\cdot]_S}"] \arrow[dr, "{[\ ]_S}"', "\cong"] & & W \arrow[dl, leftrightarrow, "{[\ ]_T}", "\cong"'] \\ & \mathbb{R}^n & \end{tikzcd}$$

<u>Claim:</u> Composition of $[\ ]_T^{-1}$ and $[\ ]_S$ is a <u>bij</u>.

So $[\ ]_T^{-1} \circ [\ ]_S : V \xrightarrow{\cong} W$ is an isom.

Theory of functions:

$$\begin{tikzcd} R \arrow[r, "g"] \arrow[dr, "f\circ g"'] & S \arrow[d, "f"] \\ & T \end{tikzcd}$$

For sets R, S, T and maps

$$(f\circ g)(r) = f(g(r)) \in T$$

$\forall r \in R$.

Th (a) If f and g are inj then so is $f\circ g$

(b) If f and g are surj " " " "

(c) ------------ bij " " " "

(d) If f is bij then so is f^{-1}.

Pf. (a) Suppose f and g are inj. so 6

$f(s_1) = f(s_2) \Rightarrow s_1 = s_2$ for $s_1, s_2 \in S$

$g(r_1) = g(r_2) \Rightarrow r_1 = r_2$ for $r_1, r_2 \in R$

Suppose $(f \circ g)(r_1) = (f \circ g)(r_2)$ so

$$f(\underbrace{g(r_1)}_{s_1}) = f(\underbrace{g(r_2)}_{s_2})$$ so by inj of f get

$g(r_1) = g(r_2)$ so by inj of g get $r_1 = r_2$

so $f \circ g$ is inj.

(b) Suppose f and g are surj. 7

① $\forall t \in T, \exists s \in S$ s.t. $f(s)=t$ and

② $\forall s \in S, \exists r \in R$ s.t. $g(r)=s$.

<u>Want:</u> $\forall t \in T$, find $r \in R$ s.t. $(f \circ g)(r)=t$

By ① $\forall t \in T, \exists s \in S$ s.t. $f(s)=t$

By ② $\forall s \in S, \exists r \in R$ s.t. $g(r)=s$ so

So $(f \circ g)(r) = f(g(r)) = f(s) = t$ ✓

(c) Follows from (a) and (b).

(d) If f is bij so is f^{-1}

f is inj and surj so $f^{-1}: T \to S$ is defined by $f^{-1}(t) = s$ iff $t = f(s)$

$$s \in S \underset{f^{-1}}{\overset{f}{\rightleftarrows}} t \in T$$

If $f^{-1}(t_1) = f^{-1}(t_2)$ then $\underbrace{f(f^{-1}(t_1))}_{t_1} = \underbrace{f^{-1}(f(t_2))}_{t_2}$

shows f^{-1} is inj

To show f^{-1} is surj, $\forall s \in S$ show $\exists t \in T$ s.t. $f^{-1}(t) = s$

Can use $t = f(s)$ get $f^{-1}(f(s)) = s$ ✓

$\square$

Back to Lin Alg:

If $L: V \to W$ is an isom (bij, lin)

Then $L^{-1}: W \to V$ is also isom (bij, lin.)

Pf: Know L^{-1} is bij from general function theory. Check L^{-1} is lin.

$\forall w_1, w_2 \in W$ $\quad L^{-1}(w_i) = v_i \quad i = 1, 2$

means $\quad w_i = L(v_i)$ so

$w_1 + w_2 = L(v_1) + L(v_2) = L(v_1 + v_2)$ so

$L^{-1}(w_1 + w_2) = v_1 + v_2 = L^{-1}(w_1) + L^{-1}(w_2)$

$\forall c \in \mathbb{R}$, $\underbrace{L^{-1}(cw_1) = cL^{-1}(w_1)}_{\text{want}}$

Say $L^{-1}(w_1) = v_1$ so $w_1 = L(v_1)$

so $cw_1 = cL(v_1) = L(cv_1)$ so

$L^{-1}(cw_1) = cv_1 = cL^{-1}(w_1)$ $\square$ ✓

Th: If $\dim(V) = n \neq m = \dim(W)$

then $V \not\cong W$.

Pf: If $L: V \to W$ were an isom.

(bij, inj and surj) then $\mathrm{Ker}(L) = \{\theta_V\}$

$\dim \mathrm{Ker}(L) = 0$ so $\dim(V) = \dim(\mathrm{Range}(L))$

$= \dim(W)$ contradicts $n \neq m$. $\square$

Fact: $\cong$ is reflexive, symm and transitive, so it is an equivalence rel.

① $V \cong V$ since $I_V : V \to V$ is an isom.

② If $V \cong W$, $\exists L : V \to W$ isom so $\exists L^{-1} : W \to V$ isom so $W \cong V$.

③ If $U \cong V$ and $V \cong W$ then $K : U \to V$ and $L : V \to W$ isom's

so $L \circ K : U \to W$ is an isom

so $U \cong W$.

$$\begin{tikzcd} U \arrow[r, "K", "\cong"'] \arrow[dr, "L \circ K"', "\cong"] & V \arrow[d, "\exists L"] \\ & W \end{tikzcd}$$

Suppose $L : V \longrightarrow W$ and $S = \{v_1, \ldots, v_n\}$ is a basis of V and $T = \{w_1, \ldots, w_m\}$ is a basis of W so have

$$\boxed{\begin{array}{l} \dim(V) = n \\ \dim(W) = m \end{array}}$$

$$\begin{tikzcd} v \in V \arrow[r, "L"] \arrow[d, "{[\ ]_S}"] & W \arrow[d, "{[\ ]_T}"] \\ \mathbb{R}^n \arrow[r, "L_A"] & \mathbb{R}^m \end{tikzcd}$$

<u>Want to</u> <u>find</u> $A \in \mathbb{R}^m_n$ s.t.

$$\underset{m\times n}{A}\ \underset{n\times 1}{[v]_S} = \underset{m\times 1}{[L(v)]_T}$$

<u>Th</u>: Can Always find such an A, call it matrix representing L from S to T, $A = {}_T[L]_S$