If such a matrix A can be found s.t. $\underset{m\times n}{A}\underset{n\times 1}{[v]_S} = \underset{m\times 1}{[L(v)]_T}$ $\forall v \in V$ then must be true for $v_j \in S$, $1 \le j \le n$. So

$$A[v_j]_S = [L(v_j)]_T$$

$= Ae_j = \mathrm{Col}_j(A)$ So to get $\mathrm{Col}_j(A)$ find $[L(v_j)]_T$ by solving $\sum_{i=1}^{m} x_{ij} w_i = L(v_j)$

by $\left[\underset{\text{as columns}}{T} \middle| \underbrace{L(v_1)L(v_2)\cdots L(v_n)}_{L(S)}\right] \xrightarrow{r.r.} \left[I_m \middle| A\right]$

We can always do the row reduction

$[T \mid L(S)] \xrightarrow{r.r.} [I_m \mid {}_T[L]_S]$ giving the

matrix $A = {}_T[L]_S$ s.t. ${}_T[L]_S [v_j]_S = [L(v_j)]_T$

If this holds for $v_j \in S$ then

$\forall v \in V, \exists a_j \in \mathbb{R}$ s.t. $v = \sum_{j=1}^{n} a_j \cdot v_j$ so

$[v]_S = \begin{bmatrix} a_1 \\ \vdots \\ a_n \end{bmatrix}$ and ${}_T[L]_S [v]_S = {}_T[L]_S \begin{bmatrix} a_1 \\ \vdots \\ a_n \end{bmatrix}$

$= {}_T[L]_S \left[\sum_{j=1}^{n} a_j \cdot v_j\right]_S = {}_T[L]_S \sum_{j=1}^{n} a_j [v_j]_S =$

$$\sum_{j=1}^{n} a_j \underset{m\times n}{{}_T[L]_S} \underset{n\times 1}{[v_j]_S} = \sum_{j=1}^{n} a_j \underset{m\times 1}{[L(v_j)]_T}.$$

$$= \left[\sum_{j=1}^{n} a_j L(v_j)\right]_T = \left[L\left(\sum_{j=1}^{n} a_j v_j\right)\right]_T = [L(v)]_T$$

So $\forall v \in V$, $\underset{m\times n}{{}_T[L]_S} \underset{n\times 1}{[v]_S} = \underset{m\times 1}{[L(v)]_T}$. $\square$

Usually write diagram:

$$\begin{tikzcd} V \arrow[r, "L"] \arrow[d, "{[\ ]_S}"] & W \arrow[d, "{[\ ]_T}"] \\ \mathbb{R}^n \arrow[r, "{{}_T[L]_S}", "m \times n"'] & \mathbb{R}^m \end{tikzcd}$$

when $\boxed{{}_T[L]_S [v]_S = [L(v)]_T}$

Example 1: Let $L: P_2 \longrightarrow P_1$ be

$L(p(t)) = p'(t)$ (derivative).

Let $S = \{1, t, t^2\}$ be basis of P_2 and

$T = \{1, t\}$ " " " P_1

$$\begin{array}{ccc} P_2 & \xrightarrow{L} & P_1 \\ {[\;]_S} \downarrow & & \downarrow [\;]_T \\ \mathbb{R}^3 & \xrightarrow[2\times 3]{{}_T[L]_S} & \mathbb{R}^2 \end{array}$$

To get ${}_T[L]_S = A$ solve

$$[T \mid L(S)] \xrightarrow{r.r.} [I_2 \mid \underbrace{{}_T[L]_S}_{2\times 3}]$$

"as columns"

First calculate $L(S)$:

$L(S)$: $L(1) = 0$
$L(t) = 1$
$L(t^2) = 2t$

$$\begin{array}{c} 1 \\ t \end{array}\left[\begin{array}{cc|ccc} 1 & 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 & 2 \end{array}\right]$$

already RREF so

T $L(S)$ as columns

${}_T[L]_S = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 2 \end{bmatrix}$

Check that for $v = a1 + bt + ct^2$

$$\begin{bmatrix}0&1&0\\0&0&2\end{bmatrix}\begin{bmatrix}a\\b\\c\end{bmatrix} = \begin{bmatrix}b\\2c\end{bmatrix}$$ means $L(v) = b1 + 2ct$

$${}_T[L]_S \, [v]_S = [L(v)]_T$$ $= v'(t)$ <u>checks</u>

<u>Example 2:</u> Let $L : \mathbb{R}^2 \longrightarrow \mathbb{R}^2$ s.t. $L\begin{bmatrix}x\\y\end{bmatrix} = \begin{bmatrix}y\\x\end{bmatrix}$

Let $S = \left\{ \underbrace{\begin{bmatrix}1\\0\end{bmatrix}}_{v_1}, \underbrace{\begin{bmatrix}0\\1\end{bmatrix}}_{v_2} \right\} = T$

$$\begin{tikzcd}
\mathbb{R}^2 \arrow[r, "L"] \arrow[d, "{[\ ]_S}"] & \mathbb{R}^2 \arrow[d, "{[\ ]_S}"] \\
\mathbb{R}^2 \arrow[r, "{}_S[L]_S"', "2\times 2"] & \mathbb{R}^2
\end{tikzcd}$$

Let $v = \begin{bmatrix}a\\b\end{bmatrix} \in \mathbb{R}^2$ so

$[v]_S = \begin{bmatrix}a\\b\end{bmatrix}$ want

$${}_S[L]_S \underbrace{\begin{bmatrix}a\\b\end{bmatrix}}_{[v]_S} = [L(v)]_S = L(v)$$

Algorithm: $[S \mid L(S)] \xrightarrow{r.r.} [I_2 \mid {}_S[L]_S]$ L7

$\begin{bmatrix} 1 & 0 & 0 & 1 \\ 0 & 1 & 1 & 0 \end{bmatrix}$ already in RREF so ${}_S[L]_S = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$.

$S \quad L(S)$

Check $\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} a \\ b \end{bmatrix} = \begin{bmatrix} b \\ a \end{bmatrix} = L(v)$

${}_S[L]_S \, [v]_S = [L(v)]_S =$

Short way: $L \begin{bmatrix} x \\ y \end{bmatrix} = \begin{pmatrix} y \\ x \end{pmatrix} = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \begin{pmatrix} x \\ y \end{pmatrix}$

$L = L_A \qquad {}_S[L]_S = A$

What if we use $T = \left\{ \underset{v_1'}{\begin{bmatrix}1\\1\end{bmatrix}}, \underset{v_2'}{\begin{bmatrix}1\\-1\end{bmatrix}} \right\}$?

Find ${}_T[L]_T$:

$$[T \mid L(T)] = \left[\begin{array}{cc|cc}1&1&1&-1\\1&-1&1&1\end{array}\right] \xrightarrow{r.r.} \left[\begin{array}{cc|cc}1&1&1&-1\\0&-2&0&2\end{array}\right] \to$$

$$\begin{array}{cc} T & L(T) \end{array}$$

$$\to \left[\begin{array}{cc|cc}1&1&1&-1\\0&-1&0&1\end{array}\right] \longrightarrow \left[\begin{array}{cc|cc}1&0&1&0\\0&1&0&-1\end{array}\right]$$ says

$$\begin{array}{cc} I_2 & {}_T[L]_T \end{array}$$

${}_T[L]_T\,[v]_T = [L(v)]_T$ comes down to

Check: ✓

$$\begin{bmatrix}1&0\\0&-1\end{bmatrix}\begin{bmatrix}\frac{1}{2}(a+b)\\ \frac{1}{2}(a-b)\end{bmatrix} = \begin{bmatrix}\frac{1}{2}(a+b)\\ \frac{1}{2}(-a+b)\end{bmatrix}$$

$$\frac{1}{2}(a+b)\underset{v_1'}{\begin{bmatrix}1\\1\end{bmatrix}} + \frac{1}{2}(-a+b)\underset{v_2'}{\begin{bmatrix}1\\-1\end{bmatrix}} = \begin{bmatrix}b\\a\end{bmatrix}$$

$v = \begin{bmatrix} a \\ b \end{bmatrix} \in \mathbb{R}^2$ find $[v]_T = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$ by

$$\begin{array}{c} -1 \quad -1 \quad\; -a \\ \left[\begin{array}{cc|c} 1 & 1 & a \\ 1 & -1 & b \end{array}\right] \\ T \qquad\quad v \end{array} \longrightarrow \left[\begin{array}{cc|c} 1 & 1 & a \\ 0 & -2 & b-a \end{array}\right] \longrightarrow \left[\begin{array}{cc|c} 1 & 1 & a \\ 0 & -1 & \frac{1}{2}b - \frac{1}{2}a \end{array}\right] +$$

$$\longrightarrow \left[\begin{array}{cc|c} 1 & 0 & \frac{1}{2}a + \frac{1}{2}b \\ 0 & 1 & \frac{1}{2}a - \frac{1}{2}b \end{array}\right] \quad \begin{array}{l} x_1 = \frac{1}{2}(a+b) \\ x_2 = \frac{1}{2}(a-b) \end{array}$$

$$[v]_T = \begin{bmatrix} \frac{1}{2}(a+b) \\ \frac{1}{2}(a-b) \end{bmatrix} \qquad \frac{1}{2}(a+b)\begin{bmatrix} 1 \\ 1 \end{bmatrix} + \frac{1}{2}(a-b)\begin{bmatrix} 1 \\ -1 \end{bmatrix} = \begin{bmatrix} a \\ b \end{bmatrix} \checkmark$$

Summary diagram:

$$\begin{tikzcd}
& \mathbb{R}^2 \arrow[r, "L"] \arrow[d, "{[\ ]_S}"] \arrow[ddl, bend right, "{[\ ]_T}"'] & \mathbb{R}^2 \arrow[d, "{[\ ]_S}"] \arrow[ddr, bend left, "{[\ ]_T}"] & \\
& \mathbb{R}^2 \arrow[r, "{{}_S[L]_S}", "{\begin{bmatrix}0&1\\1&0\end{bmatrix}}"'] & \mathbb{R}^2 \arrow[dr, "{{}_TP_S}"] & \\
\mathbb{R}^2 \arrow[ur, "{{}_SP_T}"] \arrow[rrr, "{{}_T[L]_T}", "{\begin{bmatrix}1&0\\0&-1\end{bmatrix}}"'] & & & \mathbb{R}^2
\end{tikzcd}$$

Claim:

$${}_T[L]_T = \left({}_TP_S\right)\,{}_S[L]_S\left({}_SP_T\right)$$

Check:

$${}_S P_T : \begin{array}{c} \left[\begin{array}{cc|cc} 1 & 0 & 1 & 1 \\ 0 & 1 & 1 & -1 \end{array}\right] \\ \quad S \qquad\quad T \end{array} \text{ so } {}_S P_T = \begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix}$$

$${}_T P_S : \begin{array}{c} -1 \;\; -1 \quad -1 \;\;\; 0 \\ \left[\begin{array}{cc|cc} 1 & 1 & 1 & 0 \\ 1 & -1 & 0 & 1 \end{array}\right] \\ \quad T \qquad\quad S \end{array} \rightarrow \left[\begin{array}{cc|cc} 1 & 1 & 1 & 0 \\ 0 & -2 & -1 & 1 \end{array}\right] \rightarrow \left[\begin{array}{cc|cc} 1 & 1 & 1 & 0 \\ 0 & -1 & -\frac{1}{2} & \frac{1}{2} \end{array}\right]$$

$$\rightarrow \left[\begin{array}{cc|cc} 1 & 0 & \frac{1}{2} & \frac{1}{2} \\ 0 & 1 & \frac{1}{2} & -\frac{1}{2} \end{array}\right] \text{ so } {}_T P_S = {}_S P_T^{-1} = \frac{1}{2}\begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix}$$

Check: $\frac{1}{2}\begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix}\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}\begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix} = \frac{1}{2}\begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix}\begin{bmatrix} 1 & -1 \\ 1 & 1 \end{bmatrix}$

$$= \frac{1}{2}\begin{bmatrix} 2 & 0 \\ 0 & -2 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} = {}_T[L]_T \quad \checkmark$$

Example 3: Let $L : \mathbb{R}_2^2 \to \mathbb{R}^2$ be $L\begin{bmatrix} a & b \\ c & d \end{bmatrix} = \begin{bmatrix} a+b \\ c+d \end{bmatrix}$

$$T = \left\{ \begin{bmatrix} 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \end{bmatrix} \right\}$$

$$T' = \left\{ \begin{bmatrix} 1 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ -1 \end{bmatrix} \right\}$$

$$\begin{tikzcd}
 & \mathbb{R}_2^2 \arrow[r, "L"] \arrow[d, "{[\ ]_S}"] \arrow[ddl, "{[\ ]_{S'}}"'] & \mathbb{R}^2 \arrow[d, "{[\ ]_T}"] \arrow[ddr, bend left] & \\
 & \mathbb{R}^4 \arrow[r, "{{}_T[L]_S}", "2\times 4"'] & \mathbb{R}^2 \arrow[dr, "{{}_{T'}Q_T}"] & \\
\mathbb{R}^4 \arrow[ur, "{{}_S P_{S'}}", "4\times 4"'] \arrow[rrr, "{{}_{T'}[L]_{S'}}", "2\times 4"'] & & & \mathbb{R}^2
\end{tikzcd}$$

$\underline{L(S):}$

$L\begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$

$L\begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$

$L\begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix} = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$

$L\begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$

$$\left[\begin{array}{cc|cccc} 1 & 0 & 1 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 & 1 & 1 \end{array}\right]$$

T $L(S)$ as columns

in RREF so

$${}_T[L]_S = \begin{bmatrix} 1 & 1 & 0 & 0 \\ 0 & 0 & 1 & 1 \end{bmatrix}$$

$$L(v) = \begin{bmatrix} a+b \\ c+d \end{bmatrix} = (a+b)\begin{bmatrix} 1 \\ 0 \end{bmatrix} + (c+d)\begin{bmatrix} 0 \\ 1 \end{bmatrix}$$

$$\begin{bmatrix} 1 & 1 & 0 & 0 \\ 0 & 0 & 1 & 1 \end{bmatrix}\begin{bmatrix} a \\ b \\ c \\ d \end{bmatrix} = \begin{bmatrix} a+b \\ c+d \end{bmatrix} \checkmark$$

$${}_T[L]_S \quad [v]_S = [L(v)]_T$$