Th: Let $S = \{v_1, \dots, v_n\}$ and $S' = \{v_1', \dots, v_n'\}$ be bases of V, and $T = \{w_1, \dots, w_m\}$ and $T' = \{w_1', \dots, w_m'\}$ be bases of W, and let ${}_S P_{S'} \in \mathbb{R}^n_n$ be the invertible transition matrix from S' to S so ${}_S P_{S'} [v]_{S'} = [v]_S$, $\forall v \in V$, and let ${}_{T'} Q_T \in \mathbb{R}^m_m$ be the invertible transition matrix from T to T' so ${}_T Q_{T'} [w]_{T'} = [w]_T$, $\forall w \in W$.

Let ${}_T[L]_S \in \mathbb{R}^m_n$ represent $L : V \to W$ from S to T

and ${}_{T'}[L]_{S'} \in \mathbb{R}^m_n$ " ...... " S' to T'

So ${}_T[L]_S [v]_S = \boxed{[L(v)]}_T$ and ${}_{T'}[L]_{S'} [v]_{S'} = [L(v)]_{T'}$

then ${}_{T'}[L]_{S'} = {}_{T'}Q_T \left({}_T[L]_S\right) {}_S P_{S'}$

Diagram summarizing last theorem:

$$
\begin{tikzcd}
 & V \arrow[r, "L"] \arrow[d, "{[\ ]_S}"] \arrow[dl, "{[\ ]_{S'}}"'] & W \arrow[d, "{[\ ]_T}"'] \arrow[dr, "{[\ ]_{T'}}"] & \\
 & \mathbb{R}^n \arrow[r, "{{}_T[L]_S}", "m\times n"'] & \mathbb{R}^m \arrow[dr, "{{}_{T'}Q_T}", "m\times m"'] & \\
\mathbb{R}^n \arrow[ur, "{{}_S P_{S'}}", "n\times n"'] \arrow[rrr, "{{}_{T'}[L]_{S'}}", "m\times n"'] & & & \mathbb{R}^m
\end{tikzcd}
$$

Given equations: ① ${}_S P_{S'}[v]_{S'} = [v]_S$

② ${}_{T'}Q_T\,[w]_T = [w]_{T'}$ ③ ${}_T[L]_S\,[v]_S = [L(v)]_T$

④ ${}_{T'}[L]_{S'}\,[v]_{S'} = [L(v)]_{T'}$

Prove Goal: ${}_{T'}[L]_{S'} = {}_{T'}Q_T\left({}_T[L]_S\right){}_S P_{S'}$

Proof: ${}_{T'}[L]_{S'}[v]_{S'} \overset{④}{=} [L(v)]_{T'} \overset{(2)}{=} {}_{T'}Q_T[L(v)]_T$ 3

$\overset{③}{=} {}_{T'}Q_{T\,T}[L]_S[v]_S \overset{①}{=} {}_{T'}Q_{T\,T}[L]_{S\,S}P_{S'}[v]_{S'}$ is

true $\forall v \in V$, so use $v = v_j' \in S'$ so that

$[v_j']_{S'} = e_j$ and we get

$Col_j({}_{T'}[L]_{S'}) = {}_{T'}[L]_{S'}e_j = {}_{T'}[L]_{S'}[v_j']_{S'} =$

$({}_{T'}Q_{T\,T}[L]_{S\,S}P_{S'})[v_j']_{S'} = Col_j({}_{T'}Q_{T\,T}[L]_{S\,S}P_{S'})$

for all $1 \leq j \leq n$, so

$${}_{T'}[L]_{S'} = {}_{T'}Q_{T\,T}[L]_{S\,S}P_{S'}. \quad \square$$

Example 3 continued from 3-30-2020:

$L : \mathbb{R}^2_2 \to \mathbb{R}^2$ is $L\begin{bmatrix} a & b \\ c & d \end{bmatrix} = \begin{bmatrix} a+b \\ c+d \end{bmatrix}$

S = std basis of $\mathbb{R}^2_2$ $\quad T$ = std basis of $\mathbb{R}^2$

$S' = \left\{ \underset{v_1'}{\begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}}, \underset{v_2'}{\begin{bmatrix} 1 & 1 \\ 1 & 0 \end{bmatrix}}, \underset{v_3'}{\begin{bmatrix} 1 & 1 \\ 0 & 0 \end{bmatrix}}, \underset{v_4'}{\begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}} \right\}$ and $T' = \left\{ \underset{w_1'}{\begin{bmatrix} 1 \\ 1 \end{bmatrix}}, \underset{w_2'}{\begin{bmatrix} 1 \\ -1 \end{bmatrix}} \right\}$.

Find ${}_{T'}[L]_{S'}$ directly (without using trans. mat's)

$L(S')$: $L\begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix} = \begin{bmatrix} 2 \\ 2 \end{bmatrix}$

$L\begin{bmatrix} 1 & 1 \\ 1 & 0 \end{bmatrix} = \begin{bmatrix} 2 \\ 1 \end{bmatrix}$

$L\begin{bmatrix} 1 & 1 \\ 0 & 0 \end{bmatrix} = \begin{bmatrix} 2 \\ 0 \end{bmatrix}$

$L\begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$

$$\underset{T' \qquad\quad L(S')}{\left[\begin{array}{cc|cccc} 1 & 1 & 2 & 2 & 2 & 1 \\ 1 & -1 & 2 & 1 & 0 & 0 \end{array}\right]} \xrightarrow{\text{r.r.}}$$

$$\left[\begin{array}{cc|cccc} 1 & 1 & 2 & 2 & 2 & 1 \\ 0 & -2 & 0 & -1 & -2 & -1 \end{array}\right] \to \left[\begin{array}{cc|cccc} 1 & 0 & 2 & \frac{3}{2} & 1 & \frac{1}{2} \\ 0 & 1 & 0 & \frac{1}{2} & 1 & \frac{1}{2} \end{array}\right]$$

$$0 \quad -1 \quad 0 \quad -\tfrac{1}{2} \quad -1 \quad -\tfrac{1}{2}$$

$${}_{T'}[L]_{S'} = \begin{bmatrix} 2 & \frac{3}{2} & 1 & \frac{1}{2} \\ 0 & \frac{1}{2} & 1 & \frac{1}{2} \end{bmatrix}$$

Does ${}_{T'}[L]_{S'}\,[v]_{S'} = [L(v)]_{T'}$?

Find $[v]_{S'}$: Solve $\sum_{j=1}^{4} x_j v_j' = v$

$$x_1\begin{bmatrix}1&1\\1&1\end{bmatrix}+x_2\begin{bmatrix}1&1\\1&0\end{bmatrix}+x_3\begin{bmatrix}1&1\\0&0\end{bmatrix}+x_4\begin{bmatrix}1&0\\0&0\end{bmatrix}=\begin{bmatrix}a&b\\c&d\end{bmatrix}$$

$$\begin{array}{c}\left[\begin{array}{cccc|c}1&1&1&1&a\\1&1&1&0&b\\1&1&0&0&c\\1&0&0&0&d\end{array}\right]\\ \begin{array}{cc}\qquad T' & \qquad\quad v\end{array}\end{array}\xrightarrow{\text{r.r.}}\left[\begin{array}{cccc|c}1&0&0&0&d\\0&1&0&0&c-d\\0&0&1&0&b-c\\0&0&0&1&a-b\end{array}\right] \text{ so } [v]_{S'}=\begin{bmatrix}d\\c-d\\b-c\\a-b\end{bmatrix}$$

$${}_{T'}[L]_{S'}\,[v]_{S'} = \begin{bmatrix}2&3/2&1&1/2\\0&1/2&1&1/2\end{bmatrix}\begin{bmatrix}d\\c-d\\b-c\\a-b\end{bmatrix} =$$

$$\begin{bmatrix}2d+\frac{3}{2}(c-d)+1(b-c)+\frac{1}{2}(a-b)\\0\cdot d+\frac{1}{2}(c-d)+1(b-c)+\frac{1}{2}(a-b)\end{bmatrix}=\begin{bmatrix}\frac{1}{2}a+\frac{1}{2}b+\frac{1}{2}c+\frac{1}{2}d\\\frac{1}{2}a+\frac{1}{2}b-\frac{1}{2}c-\frac{1}{2}d\end{bmatrix}$$

$$= [L(v)]_{T'} \text{ since } \frac{1}{2}(a+b+c+d)\begin{bmatrix}1\\1\end{bmatrix}+\frac{1}{2}(a+b-c-d)\begin{bmatrix}1\\-1\end{bmatrix}=\begin{bmatrix}a+b\\c+d\end{bmatrix}$$

Find ${}_S P_{S'}$: $\left[\begin{array}{cccc|cccc} 1 & 0 & 0 & 0 & 1 & 1 & 1 & 1 \\ 0 & 1 & 0 & 0 & 1 & 1 & 1 & 0 \\ 0 & 0 & 1 & 0 & 1 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 1 & 0 & 0 & 0 \end{array}\right]$ is in RREF

$\qquad\qquad\qquad\qquad S \qquad\qquad\qquad S'$

$$= \begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 0 \\ 1 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{bmatrix}.$$

Find ${}_{T'}Q_T$: $\left[\begin{array}{cc|cc} 1 & 1 & 1 & 0 \\ 1 & -1 & 0 & 1 \end{array}\right]$

$\qquad\qquad\qquad\qquad T' \qquad\quad T$

$$\xrightarrow{\text{r.r.}} \left[\begin{array}{cc|cc} 1 & 0 & \frac{1}{2} & \frac{1}{2} \\ 0 & 1 & \frac{1}{2} & -\frac{1}{2} \end{array}\right] \quad \text{so } {}_{T'}Q_T = \frac{1}{2}\begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix}$$

(see page 11 of notes from 3-30)

Check: ${}_{T'}Q_T \; {}_T[L]_S \; {}_S P_{S'} =$

$$\frac{1}{2}\begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix}\begin{bmatrix} 1 & 1 & 0 & 0 \\ 0 & 0 & 1 & 1 \end{bmatrix}\begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 0 \\ 1 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{bmatrix} = \frac{1}{2}\begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix}\begin{bmatrix} 2 & 2 & 2 & 1 \\ 2 & 1 & 0 & 0 \end{bmatrix}$$

$$= \frac{1}{2}\begin{bmatrix} 4 & 3 & 2 & 1 \\ 0 & 1 & 2 & 1 \end{bmatrix} = \begin{bmatrix} 2 & \frac{3}{2} & 1 & \frac{1}{2} \\ 0 & \frac{1}{2} & 1 & \frac{1}{2} \end{bmatrix}$$

$\therefore = {}_{T'}[L]_{S'}$ checks!

Topics for Quiz 7 on Friday, 4-3;

Given $L: V \to W$, bases S, S' in V

" T, T' in W

find ${}_T[L]_S$, ${}_{T'}[L]_{S'}$ directly:

$[T \mid L(S)] \xrightarrow{r.r.} [I_m \mid {}_T[L]_S]$ and

$[T' \mid L(S')] \xrightarrow{r.r.} [I_m \mid {}_{T'}[L]_{S'}]$ <u>and</u> using

transition matrices:

$${}_{T'}[L]_{S'} = {}_{T'}Q_T \left({}_T[L]_S\right) {}_S P_{S'}$$

Implications of today's results:

Write $A = {}_T[L]_S$ and $B = {}_{T'}[L]_{S'}$

and $P = {}_S P_{S'}$ and $Q = {}_{T'}Q_T$ so

$$\underset{m\times n}{B} = \underset{(m\times m)}{Q}\,\underset{(m\times n)}{A}\,\underset{(n\times n)}{P} \quad \text{for invertible matrices } P, Q$$

Know $Q = E_g E_{g-1} \cdots E_1$ and $P = F_1 F_2 \cdots F_h$

for elem. matrices E_i, F_j of size $m\times m$ and $n\times n$

$$B = \underbrace{\left(E_g\left(\cdots\left(E_1 A\right)\right)\right)}_{\text{row operations done to } A}\underbrace{F_1 F_2 \cdots F_h}_{\text{do column operations}}$$

says B is obtained from A by row and/or column op's.

Suppose QA is RREF, then what is the "nicest" $(QA)P$ you could get by doing so column op's to QA?

Ex:

$$\begin{bmatrix} 1 & 2 & 0 & 3 & 5 \\ 0 & 0 & 1 & -1 & 2 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix} \xrightarrow[\text{op.s}]{\text{col.}} \begin{bmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix} \rightarrow \begin{bmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix} = \left[\begin{array}{c|c} I_r & 0 \\ \hline 0 & 0 \end{array}\right]$$

RREF   switch

where $r = \text{rank}(A)$

So **best** $B = QAP$ is "Block Iden Form"

BIF

Can find S', T' s.t.

$B = Q A P$ is in BIF, but not very important for us.

<u>Def</u>: For $A, B \in \mathbb{R}^m_n$, say $A \underset{\text{row/col}}{\sim} B$ when B can be obtained from A by a finite sequence of row and/or column operations.

This is true when $\exists P \in \mathbb{R}^n_n$, $\exists Q \in \mathbb{R}^m_m$ both invertible, s.t. $B = QAP$, and we say A and B are "equivalent" or "row/col eq."

<u>Th</u>. This relation on $\mathbb{R}^m_n$ is an <u>equivalence relation</u> (reflexive, Symm, Transitive)

Important special cases:

Suppose $L : V \longrightarrow V$ then have

$$\begin{tikzcd} V \arrow[r, "L"] \arrow[d, "{[\ ]_S}"] & V \arrow[d, "{[\ ]_{S'}}"] \\ \mathbb{R}^n \arrow[r, "{{}_{S'}[L]_S}"] & \mathbb{R}^n \end{tikzcd}$$

Suppose $L = I_V$

$L(v) = v$
$\forall v \in V$

$$\begin{tikzcd} & V \arrow[dl, "{[\ ]_S}"'] \arrow[dr, "{[\ ]_{S'}}"] & \\ \mathbb{R}^n \arrow[rr, "{{}_{S'}[I_V]_S}", "{= {}_{S'}P_S}"'] & & \mathbb{R}^n \end{tikzcd}$$

Then $\quad {}_{S'}[L]_S \, [v]_S = [L(v)]_{S'}$

$$\boxed{{}_{S'}[I_V]_S} \, [v]_S = [v]_{S'}$$

$= {}_{S'}P_S$ is the transition matrix from S to S'.

Linear Operator on V

S and T are bases of V

$${}_T P_S = \left({}_S P_T\right)^{-1}$$

$$\begin{tikzcd}
 & V \arrow[r, "L"] \arrow[d, "{[\ ]_S}"] \arrow[ddl, "{[\ ]_T}"'] & V \arrow[d, "{[\ ]_S}"'] \arrow[ddr, "{[\ ]_T}"] & \\
 & \mathbb{R}^n \arrow[r, "{{}_S[L]_S}", "{n\times n}"'] & \mathbb{R}^n \arrow[dr, "{{}_T P_S}", "{n\times n}"'] & \\
\mathbb{R}^n \arrow[ur, "{{}_S P_T}", "{n\times n}"'] \arrow[rrr, "{{}_T[L]_T}", "{n\times n}"'] & & & \mathbb{R}^n
\end{tikzcd}$$

$${}_T[L]_T = {}_T P_S \; {}_S[L]_S \; {}_S P_T$$

$$B = P^{-1} \; A \; P$$

say B is <u>similar</u> to A, $B \underset{\text{sim}}{\sim} A$

Can we find "nice" B? diagonal?