Monday: Apr. 13 : Math 304-6 Feingold / I

Example: Try to diagonalize $A = \begin{bmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{bmatrix} \in \mathbb{R}^3_3$.

Step ①: Find all possible e-values $\lambda \in \mathbb{R}$ s.t. $AX = \lambda X$ for $0^3_1 \neq X \in \mathbb{R}^3$. $X = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$

Suppose $AX = \begin{bmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \lambda \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$ then it means

$$\begin{bmatrix} x_1 + x_2 + x_3 \\ x_1 + x_2 + x_3 \\ x_1 + x_2 + x_3 \end{bmatrix} = \begin{bmatrix} \lambda x_1 \\ \lambda x_2 \\ \lambda x_3 \end{bmatrix} = \begin{bmatrix} \lambda & 0 & 0 \\ 0 & \lambda & 0 \\ 0 & 0 & \lambda \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \lambda I_3 X$$

same as $(*)\ (A - \lambda I_3) X = 0^3_1$ so looking for $\lambda \in \mathbb{R}$ s.t. $Nul(A - \lambda I_3)$ is <u>non-trivial</u>

$(*)$ has non-zero solutions.

2] $$[A-\lambda I_3 \,|\, 0^3_1] = \left[\begin{array}{ccc|c} 1-\lambda & 1 & 1 & 0 \\ 1 & 1-\lambda & 1 & 0 \\ 1 & 1 & 1-\lambda & 0 \end{array}\right]$$

$$-1 \quad -1 \quad \lambda-1$$

try row op's
$-R_3+R_1 \to R_1$
$-R_3+R_2 \to R_2$

get

$$\left[\begin{array}{ccc|c} -\lambda & 0 & \lambda & 0 \\ 0 & -\lambda & \lambda & 0 \\ 1 & 1 & 1-\lambda & 0 \end{array}\right] \to \left[\begin{array}{ccc|c} 1 & 1 & 1-\lambda & 0 \\ \lambda & 0 & -\lambda & 0 \\ 0 & \lambda & -\lambda & 0 \end{array}\right]$$

Two cases:
$\lambda = 0$ or
$\lambda \neq 0$

Case 1 $\lambda = 0$:

$$\boxed{\lambda_1 = 0}$$

$$\left[\begin{array}{ccc|c} 1 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right] \quad \begin{array}{l} x_1 = -r-s \\ x_2 = r \\ x_3 = s \end{array} \Big\} \text{free}$$

$$Nul(A-\lambda I_3) = Nul(A) = A_0 = \left\{ \begin{bmatrix} -r-s \\ r \\ s \end{bmatrix} \in \mathbb{R}^3 \,\middle|\, r,s \in \mathbb{R} \right\}$$

has basis $T_1 = \left\{ \underbrace{\begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix}}_{w_{11}}, \underbrace{\begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix}}_{w_{12}} \right\}$ $\quad g_{\lambda_1} = \dim(A_{\lambda_1}) = 2$

Case ②: $\lambda \neq 0$: $\left[\begin{array}{ccc|c} 1 & 1 & (1-\lambda) & 0 \\ \lambda & 0 & -\lambda & 0 \\ 0 & \lambda & -\lambda & 0 \end{array}\right] \to \left[\begin{array}{ccc|c} 1 & 1 & (1-\lambda) & 0 \\ 1 & 0 & -1 & 0 \\ 0 & 1 & -1 & 0 \end{array}\right] \to$ ⌊3

$\begin{array}{ccc} -1 & 0 & 1 \\ 0 & -1 & 1 \end{array}$

$\left[\begin{array}{ccc|c} 1 & 0 & -1 & 0 \\ 0 & 1 & -1 & 0 \\ 0 & 0 & 3-\lambda & 0 \end{array}\right]$ has free variables iff $\lambda = 3$. So only other e-value is $\boxed{\lambda_2 = 3}$ To get e-space $A_3 = Nul(A - 3I_3)$

solve $\left[\begin{array}{ccc|c} 1 & 0 & -1 & 0 \\ 0 & 1 & -1 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right]$ $x_1 = r$, $x_2 = r$, $x_3 = r$ free $\quad A_3 = \left\{ \begin{bmatrix} r \\ r \\ r \end{bmatrix} \in \mathbb{R}^3 \,\middle|\, r \in \mathbb{R} \right\}$

has basis $T_2 = \left\{ \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} = w_{21} \right\}$, $g_{\lambda_2} = \dim(A_3) = 1$

Is $T = T_1 \cup T_2 = \left\{ \begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} \right\}$ an e-basis of $\mathbb{R}^3$ for A? If so, find $P = {}_S P_T$, $D = P^{-1} A P$.

Check: $\underset{A}{\begin{bmatrix}1&1&1\\1&1&1\\1&1&1\end{bmatrix}}\underset{w_{11}}{\begin{bmatrix}-1\\1\\0\end{bmatrix}}=\begin{bmatrix}0\\0\\0\end{bmatrix}=0\begin{bmatrix}-1\\1\\0\end{bmatrix}$ $=0\,w_{11}$

$\underset{A}{\begin{bmatrix}1&1&1\\1&1&1\\1&1&1\end{bmatrix}}\underset{w_{12}}{\begin{bmatrix}-1\\0\\1\end{bmatrix}}=\begin{bmatrix}0\\0\\0\end{bmatrix}=0\begin{bmatrix}-1\\0\\1\end{bmatrix}$ $=0\,w_{12}$ and $\underset{A}{\begin{bmatrix}1&1&1\\1&1&1\\1&1&1\end{bmatrix}}\underset{w_{21}}{\begin{bmatrix}1\\1\\1\end{bmatrix}}=\begin{bmatrix}3\\3\\3\end{bmatrix}=3\begin{bmatrix}1\\1\\1\end{bmatrix}$ $=3\,w_{21}$

$\left[\begin{array}{ccc|ccc}1&0&0&-1&-1&1\\0&1&0&1&0&1\\0&0&1&0&1&1\end{array}\right]$ is in RREF so ${}_S P_T=\begin{bmatrix}-1&-1&1\\1&0&1\\0&1&1\end{bmatrix}=P$

$S \quad T$ (as columns)

$\left[\begin{array}{ccc|ccc}-1&-1&1&1&0&0\\1&0&1&0&1&0\\0&1&1&0&0&1\end{array}\right]\xrightarrow{+}\left[\begin{array}{ccc|ccc}1&0&1&0&1&0\\0&1&1&0&0&1\\0&0&3&1&1&1\end{array}\right]\rightarrow\left[\begin{array}{ccc|ccc}1&0&0&-\frac{1}{3}&\frac{2}{3}&-\frac{1}{3}\\0&1&0&-\frac{1}{3}&-\frac{1}{3}&\frac{2}{3}\\0&0&1&\frac{1}{3}&\frac{1}{3}&\frac{1}{3}\end{array}\right]$

$0 \quad 0 \quad -1 \quad -\frac{1}{3} \quad -\frac{1}{3} \quad -\frac{1}{3}$

$T \quad S$ (as columns)

${}_T P_S = \frac{1}{3}\begin{bmatrix}-1&2&-1\\-1&-1&2\\1&1&1\end{bmatrix}=P^{-1}$

$$P^{-1}AP = \frac{1}{3}\begin{bmatrix} -1 & 2 & -1 \\ -1 & -1 & 2 \\ 1 & 1 & 1 \end{bmatrix}\begin{bmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{bmatrix}\begin{bmatrix} -1 & -1 & 1 \\ 1 & 0 & 1 \\ 0 & 1 & 1 \end{bmatrix}$$

$$= \frac{1}{3}\begin{bmatrix} -1 & 2 & -1 \\ -1 & -1 & 2 \\ 1 & 1 & 1 \end{bmatrix}\begin{bmatrix} 0 & 0 & 3 \\ 0 & 0 & 3 \\ 0 & 0 & 3 \end{bmatrix} = \frac{1}{3}\begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 9 \end{bmatrix} = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 3 \end{bmatrix}$$

$$= D = \begin{bmatrix} \lambda_1 & 0 & 0 \\ 0 & \lambda_1 & 0 \\ 0 & 0 & \lambda_2 \end{bmatrix}$$ is diagonal, $\boxed{\begin{matrix}\lambda_1 = 0 \\ \lambda_2 = 3\end{matrix}}$

Note: $\lambda_1 = 0$ was repeated corresponding to $g_{\lambda_1} = 2$.

<u>Exercise</u>: Show that $A = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}$ can <u>not</u> be diagonalized using only real numbers.

Th: Let $L: V \to V$ have distinct e-values $\lambda_1, \dots, \lambda_r \in \mathbb{R}$ with corresponding e-vectors $w_1, \dots, w_r \in V$, so $L(w_i) = \lambda_i w_i$ for $1 \le i \le r$. Then $S = \{w_1, \dots, w_r\} \subseteq V$ is independent.

Proof. Suppose $\theta = \sum_{i=1}^{m} c_i w_i$ is a shortest dep. relation on S, so all $c_i \neq 0$, $1 \le i \le m$. We may have relabeled vectors for convenience.

Apply L to get $\theta = \sum_{i=1}^{m} c_i L(w_i) = \sum_{i=1}^{m} c_i \lambda_i w_i$

Could also just multiply the dep rel. by λ_1, get $\theta = \sum_{i=1}^{m} c_i \lambda_1 w_i$. Subtract 2^{nd} eq. from 1^{st}:

Get $\theta = \sum_{i=1}^{m} c_i(\lambda_i - \lambda_1) w_i = \sum_{i=2}^{m} c_i(\lambda_i - \lambda_1) w_i$

Note: $m \geq 2$ since $\theta = c_1 w_1$ can't happen
($c_1 \neq 0$, $w_1 \neq \theta$).

The $i=1$ term is $c_1(\lambda_1 - \lambda_1) w_1 = c_1(0) w_1 = \theta$.
For $2 \leq i \leq m$, $\lambda_i - \lambda_1 \neq 0$ (distinct e-values),
so $\theta = \sum_{i=2}^{m} c_i(\lambda_i - \lambda_1) w_i$ is a <u>shorter</u> dep. rel on S, contradicting "shortest" chosen. □

<u>Th</u>: Let $L: V \to V$ have distinct e-values $\lambda_1, \dots, \lambda_r \in \mathbb{R}$ and for $1 \leq i \leq r$ let $T_i = \{w_{i1}, \dots, w_{ig_i}\}$ be a basis of e-space L_{λ_i}.
Then $T = T_1 \cup T_2 \cup \dots \cup T_r$ is <u>independent</u>.

Note: $T = \{w_{11}, \dots, w_{1g_1}, w_{21}, \dots, w_{2g_2}, \dots, w_{r1}, \dots, w_{rg_r}\}$ is a list of $g_1 + g_2 + \dots + g_r$ e-vectors in V.

Proof. Since $w_{ij} \in T_i \subseteq L_{\lambda_i}$ for $1 \le i \le r$
and $1 \le j \le g_i$ we have $L(w_{ij}) = \lambda_i w_{ij}$.
Suppose T were dependent, so there is a "shortest" dep. rel. on T. If necessary, by relabeling sets T_i and vectors in T_i, we could write that dep. rel. as

$$\theta = \sum_{i=1}^{m} \sum_{j=1}^{h_i} c_{ij} w_{ij} \quad \text{where each } c_{ij} \neq 0.$$

$$= \sum_{i=1}^{m} w_i \quad \text{where } w_i = \sum_{j=1}^{h_i} c_{ij} w_{ij} \in L_{\lambda_i}$$

and each $w_i \neq \theta$ since T_i is indep and $c_{ij} \neq 0$.
But that contradicts the last Theorem, so T indep. □

This means T is a basis of V iff $g_1+g_2+\cdots+g_r = n = dim(V)$, so L is diag-able iff we get a basis of e-vectors for V, enough from each e-space L_{λ_i} to make an e-basis for all of $V=\langle T\rangle$.

Focus now on finding all distinct e-values $\lambda_1, \dots, \lambda_r$ for $L: V \to V$ or for $A \in \mathbb{R}^n_n$. Find all $\lambda \in \mathbb{R}$ s.t. $(L-\lambda I_V)(v)=\theta$ has solutions $v \neq \theta$, that is, s.t. $Ker(L-\lambda I_V) \neq \{\theta_V\}$. Let S be any basis of V, and $A = {}_S[L]_S$ so $A-\lambda I_n = {}_S[L-\lambda I_V]_S$.

$\mathrm{Nul}(A-\lambda I_n) \neq \{0_1^n\}$ iff $\mathrm{rank}(A-\lambda I_n) < n$

iff $A-\lambda I_n$ is <u>not invertible</u>.

Use determinants to study this, and for other uses.

<u>Def.</u> For $1 \leq n \in \mathbb{Z}$ (integer) let $S = \{1, 2, \dots, n\}$ and let

$S_n = \mathrm{Perm}(S) = \{f : S \to S \mid f \text{ is bijective}\}$

Notation: Write $f = \begin{pmatrix} 1 & 2 & \cdots & i & \cdots & n \\ f(1) & f(2) & \cdots & f(i) & \cdots & f(n) \end{pmatrix}$

like a table of values. <u>Examples:</u>

$S_2 = \left\{ \begin{pmatrix} 1 & 2 \\ 1 & 2 \end{pmatrix}, \begin{pmatrix} 1 & 2 \\ 2 & 1 \end{pmatrix} \right\}$ has only two elements.

$S_3 = \left\{ \begin{pmatrix} 1 & 2 & 3 \\ 1 & 2 & 3 \end{pmatrix}, \begin{pmatrix} 1 & 2 & 3 \\ 1 & 3 & 2 \end{pmatrix}, \begin{pmatrix} 1 & 2 & 3 \\ 2 & 1 & 3 \end{pmatrix}, \begin{pmatrix} 1 & 2 & 3 \\ 2 & 3 & 1 \end{pmatrix}, \begin{pmatrix} 1 & 2 & 3 \\ 3 & 1 & 2 \end{pmatrix}, \begin{pmatrix} 1 & 2 & 3 \\ 3 & 2 & 1 \end{pmatrix} \right\}$ has six elements.

For $f = \begin{pmatrix} 1 & 2 & \cdots & n \\ f(1) & f(2) & \cdots & f(n) \end{pmatrix}$ there are

# of choices: $n\,(n-1)\cdots(2)(1) = n!$ ("n factorial")

So S_n contains $n!$ distinct elements, each one a bijection from S to S.

Composition of any two elements of S_n is another one, so have a <u>binary operation</u> on S_n, $\circ$, composition.

<u>Example</u>: For $f = \begin{pmatrix} 1 & 2 & 3 \\ 2 & 3 & 1 \end{pmatrix}$ and $g = \begin{pmatrix} 1 & 2 & 3 \\ 2 & 1 & 3 \end{pmatrix}$

find $f \circ g$ and $g \circ f$.

$$g = \begin{pmatrix} 1 & 2 & 3 \\ 2 & 1 & 3 \end{pmatrix}$$
$$f = \begin{pmatrix} 1 & 2 & 3 \\ 2 & 3 & 1 \end{pmatrix}$$
$$f \circ g = \begin{pmatrix} 1 & 2 & 3 \\ 3 & 2 & 1 \end{pmatrix}$$

$$f = \begin{pmatrix} 1 & 2 & 3 \\ 2 & 3 & 1 \end{pmatrix}$$
$$g = \begin{pmatrix} 1 & 2 & 3 \\ 2 & 1 & 3 \end{pmatrix}$$
$$g \circ f = \begin{pmatrix} 1 & 2 & 3 \\ 1 & 3 & 2 \end{pmatrix}$$

Note: $f \circ g \neq g \circ f$ can happen.

<u>Note</u>: $I = I_S = \begin{pmatrix} 1 & 2 & \cdots & i & \cdots & n \\ 1 & 2 & \cdots & i & \cdots & n \end{pmatrix}$ the identity map on S is bijective so $I \in S_n$ and $\forall f \in S_n$, $f \circ I = f = I \circ f$, so have an identity element for $\circ$ in S_n.

<u>Also</u>: $\forall f \in S_n$, $f^{-1} \in S_n$ since bijections are invertible and their inverses are bijective.

Ex: $f = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 \\ 2 & 5 & 4 & 3 & 1 \end{pmatrix}$ has $f^{-1} = \begin{pmatrix} 2 & 5 & 4 & 3 & 1 \\ 1 & 2 & 3 & 4 & 5 \end{pmatrix}$

$\left(\begin{array}{l}\text{just permute columns to} \\ \text{get top row in order}\end{array}\right) = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 \\ 5 & 1 & 4 & 3 & 2 \end{pmatrix}$

<u>Th</u>: $(S_n, \circ, I)$ is a group under composition with id. elt. I.

Def: For $f=\begin{pmatrix}1 & \cdots & i & \cdots & j & \cdots & n\\ f(1) & \cdots & f(i) & \cdots & f(j) & \cdots & f(n)\end{pmatrix}\in S_n$

Say f has an **inversion** for the pair (i,j), $1\le i<j\le n$, when $f(i)>f(j)$.

EX: Inversions of $f=\begin{pmatrix}1 & 2 & 3\\ 2 & 1 & 3\end{pmatrix}\in S_3$ (only one) are marked below:

$f=\begin{pmatrix}1 & 2 & 3\\ 3 & 2 & 1\end{pmatrix}$ has 3 inversions

Def: Let $\mathrm{Inv}(f)$ = Total number of inversions in $f\in S_n$.

Def: For $f\in S_n$ let $\mathrm{sgn}(f)=(-1)^{\mathrm{Inv}(f)}\in\{1,-1\}$ (say f is even or odd)

<u>Def</u>: For $A=[a_{ij}]\in \mathbb{R}^n_n$ define

$$|A| = \det(A) = \sum_{\sigma\in S_n} \operatorname{sgn}(\sigma)\, a_{1\sigma(1)} a_{2\sigma(2)} \cdots a_{n\sigma(n)}$$

a sum ($n!$ terms) one term for each $\sigma \in S_n$
each term a product of n entries from A, one from each row, column number depends on σ.

<u>Ex</u>: $n=2$: $A = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix}$ $\quad S_2 = \left\{ \begin{pmatrix} 1 & 2 \\ 1 & 2 \end{pmatrix}, \begin{pmatrix} 1 & 2 \\ 2 & 1 \end{pmatrix} \right\}$

sgn(σ): $\quad 1 \quad -1$

$$\det(A) = (+1)a_{11}a_{22} + (-1)a_{12}a_{21}$$

$$= a_{11}a_{22} - a_{12}a_{21}$$

is usual 2×2 $\det(A)$ formula.

"Crosshatching"

EX: $n=3$: $A=[a_{ij}] \in \mathbb{R}^3_3$

$$S_3 = \left\{ \begin{pmatrix}1&2&3\\1&2&3\end{pmatrix}, \begin{pmatrix}1&2&3\\1&3&2\end{pmatrix}, \begin{pmatrix}1&2&3\\2&1&3\end{pmatrix}, \begin{pmatrix}1&2&3\\2&3&1\end{pmatrix}, \begin{pmatrix}1&2&3\\3&1&2\end{pmatrix}, \begin{pmatrix}1&2&3\\3&2&1\end{pmatrix} \right\}$$

sgn(σ): $\quad 1 \quad -1 \quad -1 \quad +1 \quad +1 \quad -1$

$$\det(A) = 1\,a_{11}a_{22}a_{33} + 1\,a_{12}a_{23}a_{31} + 1\,a_{13}a_{21}a_{32}$$
$$-1\,a_{11}a_{23}a_{32} - 1\,a_{12}a_{21}a_{33} - 1\,a_{13}a_{22}a_{31}$$

Crosshatching Method:

$$\begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix} \begin{matrix} a_{11} & a_{12} \\ a_{21} & a_{22} \\ a_{31} & a_{32} \end{matrix}$$

Warning: Crosshatching ONLY works for $n=2,3$

Products of three "+1" terms $\diagdown\diagdown\diagdown$

Products of three "-1" terms $///$

Ex: $\det\begin{bmatrix} 1 & -1 & 2 \\ 3 & 1 & -1 \\ 0 & 4 & 5 \end{bmatrix}\begin{matrix} 1 & -1 \\ 3 & 1 \\ 0 & 4 \end{matrix}$

$= (1)(1)(5) + (-1)(-1)(0) + (2)(3)(4)$
$\quad - (2)(1)(0) - (1)(-1)(4) - (-1)(3)(5)$

$= 5 + 24 + 4 + 15 = 48$

$A = \begin{bmatrix} 1 & -1 & 2 \\ 3 & 1 & -1 \\ 0 & 4 & 5 \end{bmatrix} \xrightarrow[\text{adder}]{} \begin{bmatrix} 1 & -1 & 2 \\ 0 & 4 & -7 \\ 0 & 4 & 5 \end{bmatrix} \xrightarrow{\text{adder}} \begin{bmatrix} 1 & -1 & 2 \\ 0 & 4 & -7 \\ 0 & 0 & 12 \end{bmatrix}$

$-3 \quad 3 \quad -6 \qquad\qquad 0 \quad -4 \quad 7 \qquad\qquad B$

$\det(B) = \begin{bmatrix} 1 & -1 & 2 \\ 0 & 4 & -7 \\ 0 & 0 & 12 \end{bmatrix}\begin{matrix} 1 & -1 \\ 0 & 4 \\ 0 & 0 \end{matrix} = (1)(4)(12) = 48$

How do row operations affect $\det(A)$?

Facts about $\det(A)$.

Th: Let $A = [a_{ij}] \in \mathbb{R}^n_n$. Then we have

(a) If A has a row of zeros then $\det(A) = 0$

(b) $\det(A^T) = \det(A)$

(c) If A has two identical rows, then $\det A = 0$

(d) If $\operatorname{rank}(A) < n$ then $\det(A) = 0$

(e) $\det(A) = 0$ implies $\operatorname{rank}(A) < n$

(f) $\det(A) = 0$ iff A is not invertible

Goal: Understand how elementary row operations affect $\det(A)$.