Exercise: $A = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}$ If $X \in \mathbb{R}^2$ were an e-vector for A with e-value $\lambda \in \mathbb{R}$, then $(A - \lambda I_2)X = 0^2_1$ would have non-zero solutions

But $\begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix} - \begin{bmatrix} \lambda & 0 \\ 0 & \lambda \end{bmatrix} = \begin{bmatrix} -\lambda & 1 \\ -1 & -\lambda \end{bmatrix}$ and

$$\left[\begin{array}{cc|c} -\lambda & 1 & 0 \\ -1 & -\lambda & 0 \end{array}\right] \longrightarrow \left[\begin{array}{cc|c} 1 & \lambda & 0 \\ 0 & \lambda^2+1 & 0 \end{array}\right] \longrightarrow \left[\begin{array}{cc|c} 1 & 0 & 0 \\ 0 & 1 & 0 \end{array}\right]$$

$+$

$\lambda \quad \lambda^2$

since $\lambda^2 + 1 \geq 1 > 0$

so $AX = \lambda X$ has only solution $X = 0^2_1$

Alternatively, $\det \begin{bmatrix} -\lambda & 1 \\ -1 & -\lambda \end{bmatrix} = \underline{(-\lambda)^2 - (-1)(1) = \lambda^2 + 1} \neq 0$

so $\begin{bmatrix} -\lambda & 1 \\ -1 & -\lambda \end{bmatrix}$ is invertible for all $\lambda \in \mathbb{R}$.

This A cannot be diagonalized "over $\mathbb{R}$".

In textbook is a discussion of the complex numbers $\mathbb{C} = \{a+bi \mid a,b \in \mathbb{R},\ i^2=-1\}$ a very important "field" containing $\mathbb{R}$ as well as "imaginary" numbers like $i=\sqrt{-1}$. Linear algebra can be done over any field using scalars from the field instead of from $\mathbb{R}$. This topic is developed in Advanced Linear Algebra, Math 404.

Review: For $A = [a_{ij}] \in \mathbb{R}^n_n$, define   3

$$\det(A) = |A| = \sum_{\sigma \in S_n} \mathrm{sgn}(\sigma)\, a_{1\sigma(1)} a_{2\sigma(2)} \cdots a_{n\sigma(n)}$$

where $\sigma = \begin{pmatrix} 1 & 2 & \cdots & i & \cdots & j & \cdots & n \\ \sigma(1) & \sigma(2) & & \sigma(i) & & \sigma(j) & & \sigma(n) \end{pmatrix} \in S_n$

is any bijection $\sigma : \{1, 2, \ldots, n\} \longrightarrow \{1, 2, \ldots, n\}$.

$\mathrm{sgn}(\sigma) = (-1)^{\mathrm{Inv}(\sigma)} \in \{\pm 1\}$ where

$\mathrm{Inv}(\sigma) = \#$ inversions in σ

An inversion in σ is a pair (i, j) such that

$1 \le i < j \le n$ and $\sigma(i) > \sigma(j)$.

Note: There are $n! = n(n-1)\cdots(2)(1)$ distinct bijections in S_n. Also called "permutations" since the values of σ are the numbers $1, \ldots, n$ in some order.

<u>Th</u>: If $A=[a_{ij}]$ has a zero row then $\det(A)=0$.

<u>Pf</u>: If row r of A is all zeros, then $a_{rj}=0$ for all $1\leq j\leq n$, so in the formula for $\det(A)$ every term has a factor $a_{r\sigma(r)}=0$, so every term is 0. $\square$

<u>Th</u>: If $A=[a_{ij}]$ is upper triangular, so if $i>j$ then $a_{ij}=0$, then $\det(A)=a_{11}a_{22}\cdots a_{nn}$

<u>Pf</u>: Let $\sigma\in S_n$. Either $\sigma(n)=n$ or $\sigma(n)<n$.
If $\sigma(n)<n$ then $a_{n\sigma(n)}=0$ so those terms contribute nothing to $\det(A)$.
Consider remaining $\sigma\in S_n$ s.t. $\sigma(n)=n$, so

$\sigma = \begin{pmatrix} 1 & 2 & \cdots & (n-1) & n \\ \sigma(1) & \sigma(2) & & \sigma(n-1) & n \end{pmatrix}$. Either $\sigma(n-1) = n-1$ or $\sigma(n-1) < n-1$.

If $\sigma(n-1) < n-1$ then $a_{(n-1)\sigma(n-1)} = 0$ $(i > j)$ so those terms contribute nothing to $|A|$.

Consider remaining $\sigma \in S_n$ s.t. $\sigma(n-1) = n-1$ and $\sigma(n) = n$. So

$\sigma = \begin{pmatrix} 1 & 2 & \cdots & (n-2) & (n-1) & n \\ \sigma(1) & \sigma(2) & & \sigma(n-2) & (n-1) & n \end{pmatrix}$. Either $\sigma(n-2) = n-2$ or $\sigma(n-2) < n-2$.

As before, $\sigma(n-2) < n-2$ gives $a_{(n-2)\sigma(n-2)} = 0$ so get no contributions to $|A|$. Can consider only $\sigma \in S_n$ s.t. $\sigma(n-2) = n-2$, $\sigma(n-1) = n-1$ and $\sigma(n) = n$. Continue same argument, get only contribution to $\det(A)$ is from $\sigma = I$,

$\sigma = I = \begin{pmatrix} 1 & 2 & \cdots & i & \cdots & n \\ 1 & 2 & \cdots & i & \cdots & n \end{pmatrix}$ and $sgn(I) = 1$

so $\det(A) = a_{11} a_{22} \cdots a_{ii} \cdots a_{nn}$. $\square$

<u>Th</u>: Let $A = [a_{ij}] \in \mathbb{R}^n_n$ and suppose $B = [b_{ij}]$ is obtained from A by doing an elementary row operation to A. Then we have:

(1) $\det(B) = -\det(A)$ if row op. is a <u>switcher</u>,

(2) $\det(B) = c \det(A)$ if row op. is <u>multip. by c</u>

(3) $\det(B) = \det(A)$ if row op. is an <u>adder</u>.

<u>Proof</u>: (2) is easiest from definition of $|A|$.

Suppose B is obtained from A by multiplying row r of A by $c \in \mathbb{R}$ (even if $c = 0$).

Then $b_{ij} = \begin{cases} a_{ij} & \text{if } i \neq r \\ c a_{ij} & \text{if } i = r \end{cases}$ so

$$\det(B) = \sum_{\sigma \in S_n} \operatorname{sgn}(\sigma)\, b_{1\sigma(1)} \cdots b_{r\sigma(r)} \cdots b_{n\sigma(n)}$$

$$= \sum_{\sigma \in S_n} \operatorname{sgn}(\sigma)\, a_{1\sigma(1)} \cdots c\, a_{r\sigma(r)} \cdots a_{n\sigma(n)}$$

$$= c \sum_{\sigma \in S_n} \operatorname{sgn}(\sigma)\, a_{1\sigma(1)} \cdots a_{r\sigma(r)} \cdots a_{n\sigma(n)}$$

$$= c \det(A).$$

Before doing proof of (1), need fact about sgn.

<u>Th</u>: For any $\sigma, \tau \in S_n$ we have

$$\operatorname{sgn}(\sigma \circ \tau) = \operatorname{sgn}(\sigma) \cdot \operatorname{sgn}(\tau)$$

Group Theory.

<u>Example</u>: For $\sigma = \begin{pmatrix} 1 & 2 & 3 \\ 2 & 3 & 1 \end{pmatrix}$ and $\tau = \begin{pmatrix} 1 & 2 & 3 \\ 2 & 1 & 3 \end{pmatrix}$

$\operatorname{sgn}(\sigma) = (-1)^2 = 1$ and $\operatorname{sgn}(\tau) = (-1)^1 = -1$

$$\tau = \begin{pmatrix} 1 & 2 & 3 \\ 2 & 1 & 3 \end{pmatrix}$$

$$\sigma = \begin{pmatrix} 1 & 2 & 3 \\ 2 & 3 & 1 \end{pmatrix}$$

$$\sigma \circ \tau = \begin{pmatrix} 1 & 2 & 3 \\ 3 & 2 & 1 \end{pmatrix}$$

$$\operatorname{sgn}(\sigma \circ \tau) = (-1)^3 = -1$$
$$= (1) \cdot (-1) = \operatorname{sgn}(\sigma) \cdot \operatorname{sgn}(\tau).$$

For $1 \le r < s \le n$ let $\tau \in S_n$ be the permutation that <u>just</u> switches r and s:
$\tau = \begin{pmatrix} 1 & 2 & \cdots & r & \cdots & s & \cdots & n \\ 1 & 2 & \cdots & s & \cdots & r & \cdots & n \end{pmatrix}$. Then for any

$\sigma = \begin{pmatrix} 1 & 2 & \cdots & r & \cdots & s & \cdots & n \\ \sigma(1) & \sigma(2) & \cdots & \sigma(r) & \cdots & \sigma(s) & \cdots & \sigma(n) \end{pmatrix}$ we have

$$\sigma\circ\tau = \begin{pmatrix} 1 & 2 & \cdots & r & \cdots & s & \cdots & n \\ \sigma(1) & \sigma(2) & \cdots & \sigma(s) & \cdots & \sigma(r) & \cdots & \sigma(n) \end{pmatrix}.$$

<u>Th</u>: $\mathrm{sgn}(\tau) = -1$. <u>Pf</u> Count Inversions.

<u>Pf</u> ((continued) (1) Suppose $1 \le r < s \le n$ and B is obtained from A by switching rows r and s. Then
$$b_{ij} = \begin{cases} a_{ij} & \text{if } i \ne r \text{ and } i \ne s \\ a_{sj} & \text{if } i = r \\ a_{rj} & \text{if } i = s \end{cases}$$
so

$$\det(B) = \sum_{\sigma\in S_n} \mathrm{sgn}(\sigma)\, b_{1\sigma(1)} \cdots b_{r\sigma(r)} \cdots b_{s\sigma(s)} \cdots b_{n\sigma(n)}$$

$$= \sum_{\sigma\in S_n} \mathrm{sgn}(\sigma)\, a_{1\sigma(1)} \cdots \underbrace{a_{s\sigma(r)} \cdots a_{r\sigma(s)}}_{\text{out of order}} \cdots a_{n\sigma(n)}$$

$$= \sum_{\sigma \in S_n} \operatorname{sgn}(\sigma)\, a_{1\sigma(1)} \cdots a_{r\sigma(s)} \cdots a_{s\sigma(r)} \cdots a_{n\sigma(n)}$$

$$= \sum_{\sigma \in S_n} \operatorname{sgn}(\sigma)\, a_{1(\sigma\circ\tau)(1)} \cdots a_{r(\sigma\circ\tau)(r)} \cdots a_{s(\sigma\circ\tau)(s)} \cdots a_{n(\sigma\circ\tau)(n)}$$

As σ varies over all elements of S_n, so does $\sigma\circ\tau$ because the function $f_\tau : S_n \to S_n$ defined by $f_\tau(\sigma) = \sigma\circ\tau$ is bijective! (Exercise in group theory.)

So, let $\mu = \sigma\circ\tau = f_\tau(\sigma)$ be a new index:

$$\det(B) = \sum_{\mu \in S_n} \operatorname{sgn}(\sigma)\, a_{1\mu(1)} \cdots a_{n\mu(n)}$$

$$= -\sum_{\mu \in S_n} \operatorname{sgn}(\mu)\, a_{1\mu(1)} \cdots a_{n\mu(n)} = -|A|.$$

$\operatorname{sgn}(\mu) = \operatorname{sgn}(\sigma\circ\tau) = \operatorname{sgn}(\sigma)\cdot\operatorname{sgn}(\tau) = -\operatorname{sgn}(\sigma)$

Corollary of (1): If A has two identical rows then $|A|=0$.

Pf: If B is obtained from A by switching those two identical rows, then $B=A$ and $\det(B)=-\det(A)$ so $\det(A)=-\det(A)$ so $2\det(A)=0$ so $\det(A)=0$.

Pf of (3): Suppose B is obtained from A by elementary adder row operation $cR_r+R_s \to R_s$ and then

so $b_{ij}=\begin{cases} a_{ij} & \text{if } i\neq s \\ a_{sj}+ca_{rj} & \text{if } i=s \end{cases}$

$$|B|=\sum_{\sigma\in S_n} \operatorname{sgn}(\sigma)\, \underset{\displaystyle a_{1\sigma(1)}}{\underset{\shortparallel}{b_{1\sigma(1)}}} \cdots \underset{\displaystyle a_{r\sigma(r)}}{\underset{\shortparallel}{b_{r\sigma(r)}}} \cdots \underset{\displaystyle (a_{s\sigma(s)}+ca_{r\sigma(s)})}{\underset{\shortparallel}{b_{s\sigma(s)}}} \cdots \underset{\displaystyle a_{n\sigma(n)}}{\underset{\shortparallel}{b_{n\sigma(n)}}}$$

$$|B| = \sum_{\sigma \in S_n} \operatorname{sgn}(\sigma)\, a_{1\sigma(1)} \cdots a_{r\sigma(r)} \cdots a_{s\sigma(s)} \cdots a_{n\sigma(n)}$$

$$+ c \sum_{\sigma \in S_n} \operatorname{sgn}(\sigma)\, a_{1\sigma(1)} \cdots a_{r\sigma(r)} \cdots a_{r\sigma(s)} \cdots a_{n\sigma(n)}$$

The first sum is just $|A|$ so why is the second sum zero?

Let $D = [d_{ij}]$ be the matrix obtained from A by replacing row s by row r, that is,

$$d_{ij} = \begin{cases} a_{ij} & \text{if } i \neq s \\ a_{rj} & \text{if } i = s. \end{cases}$$

Then

$$|D| = \sum_{\sigma \in S_n} \operatorname{sgn}(\sigma)\, d_{1\sigma(1)} \cdots d_{r\sigma(r)} \cdots d_{s\sigma(s)} \cdots d_{n\sigma(n)}$$

$$= \sum_{\sigma \in S_n} \operatorname{sgn}(\sigma)\, a_{1\sigma(1)} \cdots a_{r\sigma(r)} \cdots a_{r\sigma(s)} \cdots a_{n\sigma(n)}$$

is that second sum above. $\square$

How do we use these Theorems to efficiently calculate det(A)?

Ex:

$$\begin{vmatrix} 1 & 1 & -1 & -1 \\ 2 & 3 & 4 & 1 \\ 3 & 2 & 1 & 1 \\ 4 & 4 & -2 & -3 \end{vmatrix} = \begin{vmatrix} 1 & 1 & -1 & -1 \\ 0 & 1 & 6 & 3 \\ 0 & -1 & 4 & 4 \\ 0 & 0 & 2 & 1 \end{vmatrix} = \begin{vmatrix} 1 & 1 & -1 & -1 \\ 0 & 1 & 6 & 3 \\ 0 & 0 & 10 & 7 \\ 0 & 0 & 2 & 1 \end{vmatrix} =$$

$+$ -2 -2 2 2

$+$ -3 -3 3 3

$+$ -4 -4 4 4

$+$ 0 0 -10 -5

$$\begin{vmatrix} 1 & 1 & -1 & -1 \\ 0 & 1 & 6 & 3 \\ 0 & 0 & 0 & 2 \\ 0 & 0 & 2 & 1 \end{vmatrix} = -\begin{vmatrix} 1 & 1 & -1 & -1 \\ 0 & 1 & 6 & 3 \\ 0 & 0 & 2 & 1 \\ 0 & 0 & 0 & 2 \end{vmatrix} = -4$$

switch

Crosshatching was <u>NOT</u> an option!!
The definition would have involved adding 24 terms, not efficient or reasonable.

Ex: $\begin{vmatrix} 1 & 1 & 1 \\ 2 & 4 & 4 \\ 6 & 6 & 9 \end{vmatrix} = (2)(3)\begin{vmatrix} 1 & 1 & 1 \\ 1 & 2 & 2 \\ 2 & 2 & 3 \end{vmatrix} = 6\begin{vmatrix} 1 & 1 & 1 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{vmatrix} = 6$

is faster than crosshatching method for 3x3.

Th: Suppose E is an elementary matrix associated with an elem. row operation. Then

(1) $\det(E) = -1$ if E is a switcher,

(2) $\det(E) = c$ if E is a multiplier by c.

(3) $\det(E) = 1$ if E is an adder.

Pf: In each case, E is obtained from I_n by doing the row op. to I_n, and $\det(I_n) = 1$, so these follow from the theorem giving the effect on det of doing elem. row ops. □

<u>Cor</u>: Let E be the elem. matrix associated with an elem. row op, so that EA is the matrix obtained from A by doing that row op. to A. Then $\det(EA) = (\det E)(\det A)$.

<u>Pf</u>: This is the result of the last theorems.

<u>Th</u>: Suppose B is obtained from A by a sequence of elem. row ops corresponding to elem. matrices $E_1, E_2, \dots, E_r$. Then

$B = E_r \cdots E_2 E_1 A$ and

$\det(B) = \det(E_r)\det(E_{r-1}) \cdots \det(E_2)\det(E_1)\det A$

and for each $1 \le i \le r$, $\det(E_i) \neq 0$.

<u>Pf</u>: Follows from last theorem.

Th: A is invertible iff $\det(A) \neq 0$.

Pf: A is invertible iff A row reduces to I_n iff $I_n = E_r \cdots E_2 E_1 A$ for some elem. matrices $E_1, E_2, \ldots, E_r$, so

$1 = \det(I_n) = (\det E_r) \cdots (\det E_1)(\det A)$

and each $\det(E_i) \neq 0$ so A invertible implies $\det(A) \neq 0$.

If A is not invertible, it row reduces to a matrix C with a zero row, $C = E_r \cdots E_1 A$

$0 = \det C = (\det E_r) \cdots (\det E_1)(\det A)$ and each $\det(E_i) \neq 0$ so $\det(A) = 0$. $\square$

We only left out the proof that $\det(A^T) = \det(A)$.

Application of det to finding e-values:

<u>Th</u>: λ is an e-value of $A \in \mathbb{R}^n_n$ iff $A - \lambda I_n$ is not invertible iff $\det(A - \lambda I_n) = 0$.

<u>Ex</u>:
$$\begin{vmatrix} 1-\lambda & 1 & 1 \\ 1 & 1-\lambda & 1 \\ 1 & 1 & 1-\lambda \end{vmatrix} = \begin{vmatrix} -\lambda & 0 & \lambda \\ 0 & -\lambda & \lambda \\ 1 & 1 & (1-\lambda) \end{vmatrix} = \lambda^2 \begin{vmatrix} -1 & 0 & 1 \\ 0 & -1 & 1 \\ 1 & 1 & (1-\lambda) \end{vmatrix}$$

$-1 \quad -1 \quad \lambda - 1$

$$= \lambda^2 \begin{vmatrix} -1 & 0 & 1 \\ 0 & -1 & 1 \\ 0 & 0 & (3-\lambda) \end{vmatrix} = \lambda^2(3-\lambda)$$ is a polynomial of degree $n = 3$ whose roots are the e-values of $A = \begin{bmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{bmatrix}$.

Note: $-\lambda^2(\lambda - 3)$ is <u>factored</u> into 3 linear factors: $-(\lambda - 0)(\lambda - 0)(\lambda - 3) = -(\lambda - \lambda_1)^2(\lambda - \lambda_2)^1$

Th: For any $A, B \in \mathbb{R}^n_n$, $\det(AB) = (\det A)(\det B)$

Pf: If A is invertible, $A = E_r \cdots E_2 E_1$ is a product of elem. matrices, so

$$\det(AB) = \det(E_r \cdots E_1 B) = (\det E_r) \cdots (\det E_1)(\det B)$$
$$= \det(E_r \cdots E_1)(\det B) = (\det A)(\det B).$$

Suppose A is not invertible so $C = E_r \cdots E_1 A$ where C has a zero row, so $CB = E_r \cdots E_1 AB$ has a zero row so $\det(CB) = 0 = \det(E_r \cdots E_1 AB)$ $= (\det E_r) \cdots (\det E_1) \det(AB)$ so $\det(AB) = 0$ $= (\det A)(\det B)$ since $\det(A) = 0$. $\square$