Another method to compute $\det(A)$:

Cofactor Expansion:

Def: For $A = [a_{ij}] \in \mathbb{R}^n_n$, $1 \le r, s \le n$, let $M_{rs} \in \mathbb{R}^{n-1}_{n-1}$ be the matrix obtained from A by deleting row r and column s.

Ex: For $A = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix}$, $M_{11} = [a_{22}]$, $M_{12} = [a_{21}]$,
$M_{21} = [a_{12}]$, $M_{22} = [a_{11}]$

For $A = \begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{bmatrix}$, $M_{11} = \begin{bmatrix} 5 & 6 \\ 8 & 9 \end{bmatrix}$, $M_{12} = \begin{bmatrix} 4 & 6 \\ 7 & 9 \end{bmatrix}$

$M_{13} = \begin{bmatrix} 4 & 5 \\ 7 & 8 \end{bmatrix}$, $M_{21} = \begin{bmatrix} 2 & 3 \\ 8 & 9 \end{bmatrix}$, $M_{22} = \begin{bmatrix} 1 & 3 \\ 7 & 9 \end{bmatrix}$, etc.

Def: With notation as above, let

$$A_{rs} = (-1)^{r+s}|M_{rs}| = (-1)^{r+s}\det(M_{rs}).$$

Th (Cofactor Expansion) For each $1 \le r, s \le n$, we have

(a) $\det(A) = \sum_{j=1}^{n} a_{rj} A_{rj}$ (expansion along row r)

(b) $\det(A) = \sum_{i=1}^{n} a_{is} A_{is}$ (expansion along column s)

Ex: $n=2$: $|A| = a_{11}A_{11} + a_{12}A_{12}$ | row 1 cofactor expansion.

$= a_{11}(-1)^{1+1}|M_{11}| + a_{12}(-1)^{1+3}|M_{12}|$

$= a_{11}a_{22} - a_{12}a_{21}$

$$|A| = a_{21}A_{21} + a_{22}A_{22} \quad \text{(row 2 expansion)}$$

$$= a_{21}(-1)^{2+1}|M_{21}| + a_{22}(-1)^{2+2}|M_{22}|$$

$$= -a_{21}a_{12} + a_{22}a_{11}$$

$$|A| = a_{11}A_{11} + a_{21}A_{21} \quad \text{(column 1 expansion)}$$

$$= a_{11}(-1)^{1+1}|M_{11}| + a_{21}(-1)^{2+1}|M_{21}|$$

$$= a_{11}a_{22} - a_{21}a_{12}$$

<u>Ex</u>: $n=3$: $|A| = a_{11}A_{11} + a_{12}A_{12} + a_{13}A_{13}$

$$= a_{11}(-1)^{1+1}|M_{11}| + a_{12}(-1)^{1+2}|M_{12}| + a_{13}(-1)^{1+3}|M_{13}|$$

$$= a_{11}\begin{vmatrix} a_{22} & a_{23} \\ a_{32} & a_{33} \end{vmatrix} - a_{12}\begin{vmatrix} a_{21} & a_{23} \\ a_{31} & a_{33} \end{vmatrix} + a_{13}\begin{vmatrix} a_{21} & a_{22} \\ a_{31} & a_{32} \end{vmatrix}$$

Value of cofactor expansion is best when used on a row (or column) with most number of 0 entries.

Ex:

$$\begin{vmatrix} 1 & 2 & 3 \\ 0 & 5 & 0 \\ 7 & 8 & 9 \end{vmatrix} = 5(-1)^{2+2}\begin{vmatrix} 1 & 3 \\ 7 & 9 \end{vmatrix} = 5(9-21) = -60$$

(using cofactor expansion along row 2)

$$\begin{vmatrix} 1 & 0 & -1 & 1 \\ 2 & 0 & 3 & 4 \\ 3 & 2 & 1 & 0 \\ 1 & 0 & -1 & 2 \end{vmatrix} = 2(-1)^{3+2}\begin{vmatrix} 1 & -1 & 1 \\ 2 & 3 & 4 \\ 1 & -1 & 2 \end{vmatrix} = -2\begin{vmatrix} 1 & -1 & 1 \\ 0 & 5 & 2 \\ 0 & 0 & 1 \end{vmatrix} = -10$$

(cof. exp along col. 2) (row ops.)

shows efficient use of a combination of methods.

Recall: $\det(AB) = (\det A)(\det B)$.

Cor: If A is invertible then

$$\det(A^{-1}) = \frac{1}{\det(A)} = (\det A)^{-1}.$$

Pf: $I_n = AA^{-1}$ so $1 = \det(I_n) = (\det A)(\det(A^{-1}))$

so $\det(A^{-1}) = \frac{1}{(\det A)}$. $\square$

Application: If $|A| = 5$, $|B| = 6$, $|C| = 7$

then $|A^2 B^T C^{-1}| = \frac{|A|^2 |B|}{|C|} = \frac{(25)(6)}{7} = \frac{150}{7}$

since $|A^2| = |A \cdot A| = |A| \cdot |A| = |A|^2$ and $|B^T| = |B|$.
(transpose)

<u>Th</u>: For $A \in \mathbb{R}^n_n$, $|A - \lambda I_n| = (-1)^n |\lambda I_n - A|$ is a polynomial in λ of degree n whose highest term is $(-1)^n \lambda^n$ and whose lowest constant term is $|A|$.

<u>Pf</u>: By definition of $|A - \lambda I_n|$, it is a sum of products of the entries of $A - \lambda I_n = [a_{ij} - \lambda \delta_{ij}]$ so each factor in each term is either a constant a_{ij} or a linear polynomial $a_{ii} - \lambda$. The sum of such products is a polynomial in λ with top term coming from $(a_{11} - \lambda)(a_{22} - \lambda) \cdots (a_{nn} - \lambda)$.

$\square$

Def: The <u>characteristic</u> polynomial of A is $|\lambda I_n - A| = \lambda^n + \cdots + (-1)^n |A|$.

Def: If char. poly. of A factors as

$$|\lambda I_n - A| = \prod_{i=1}^{r} (\lambda - \lambda_i)^{k_i} = (\lambda - \lambda_1)^{k_1} \cdots (\lambda - \lambda_r)^{k_r}$$

for <u>distinct</u> roots $\lambda_1, \ldots, \lambda_r \in \mathbb{R}$, the powers $k_1, \ldots, k_r$ are called the <u>algebraic multiplicities</u> of those roots, which are the <u>e-values</u> of A, say

k_i = algeb. mult. of λ_i for A.

Note: $n = k_1 + k_2 + \cdots + k_r$

Ex: We did $A=\begin{bmatrix}1&1&1\\1&1&1\\1&1&1\end{bmatrix}$ and found

$|A-\lambda I_3| = -\lambda^2(\lambda-3) = (-1)^3(\lambda-0)^2(\lambda-3)$ so

char. poly. of A is $|\lambda I_3 - A| = \lambda^2(\lambda-3)^1$

$\lambda_1 = 0$, $k_1 = 2$ and $\lambda_2 = 3$, $k_2 = 1$.

Note: $g_1 = 2$ and $g_2 = 1$ were found.

Ex: For $A=\begin{bmatrix}a_{11} & \cdots & & a_{1n}\\ & a_{22} & & \vdots\\ & & \ddots & \\ O & & & a_{nn}\end{bmatrix}$ upper triangular, char poly is

$$|\lambda I_n - A| = \begin{vmatrix}\lambda-a_{11} & \cdots & & -a_{1n}\\ & \lambda-a_{22} & & \vdots\\ & & \ddots & \\ O & & & \lambda-a_{nn}\end{vmatrix} = \prod_{i=1}^{n}(\lambda-a_{ii})$$

So, for example;

$$\begin{vmatrix} \lambda-2 & & & & * \\ & \lambda-2 & & & \\ & & \lambda+5 & & \\ & & & \lambda+5 & \\ & 0 & & & \lambda-6 \end{vmatrix} = (\lambda-2)^2(\lambda+5)^2(\lambda-6)$$

has degree $n=5$ but only 3 distinct roots; $\lambda_1 = 2$, $\lambda_2 = -5$, $\lambda_3 = 6$ with alg. mults. $k_1 = 2$, $k_2 = 2$, $k_3 = 1$.

To find e-values of A, must factor char. poly. to get its roots. Some polys. don't factor over $\mathbb{R}$ into all linear factors.

<u>Th</u>: If A is similar to B, so $B = P^{-1}AP$ for some invertible P, then $|B| = |A|$.

<u>Pf</u>: $|P^{-1}AP| = |P^{-1}| \cdot |A| \cdot |P| = \frac{1}{|P|} |A| \cdot |P| = |A|$ since this is a product of real numbers. $\square$

<u>Th</u>: If A is similar to B, then they have the same char. poly., $|\lambda I_n - A| = |\lambda I_n - B|$.

<u>Pf</u>: $|\lambda I_n - B| = |\lambda I_n - P^{-1}AP| = |P^{-1}\lambda I_n P - P^{-1}AP|$
$= |P^{-1}(\lambda I_n - A)P| = |P^{-1}| \cdot |\lambda I_n - A| \cdot |P| = |\lambda I_n - A|$,
since this is a product of two real numbers with a polynomial in λ. $\square$

Note: Suppose A is diagonalizable and $\lfloor 11 \rfloor$

$$P^{-1}AP = D = \left[\begin{array}{ccc|ccc|c|ccc} \lambda_1 & & & & & & & & & \\ & \ddots & & & & & & & & \\ & & \lambda_1 & & & & & & & \\ \hline & & & \lambda_2 & & & & & & \\ & & & & \ddots & & & & & \\ & & & & & \lambda_2 & & & & \\ \hline & & & & & & \ddots & & & \\ \hline & & & & & & & \lambda_r & & \\ & & & & & & & & \ddots & \\ & & & & & & & & & \lambda_r \end{array}\right]$$

for distinct e-values $\lambda_1, \ldots, \lambda_r$ with λ_i repeated g_i times, $1 \leq i \leq r$. Then

$$|\lambda I_n - A| = |\lambda I_n - D| = \prod_{i=1}^{r} (\lambda - \lambda_i)^{g_i}$$

so $g_i = k_i$ for all $1 \leq i \leq r$,

geom. mult. = alg. mult. for all e-values.

<u>Th.</u> Suppose $|\lambda I_n - A| = \prod_{i=1}^{r} (\lambda - \lambda_i)^{k_i}$ for distinct $\lambda_1, \dots, \lambda_r \in \mathbb{R}$, and recall that $g_i = \dim(A_{\lambda_i})$ = geom. mult. of λ_i for A. Then $1 \le g_i \le k_i$ for each $1 \le i \le r$.

<u>Cor</u>: A is diag-able iff each $g_i = k_i$.

<u>Practical Application</u>: If $k_i = 1$ then $g_i = 1$, but if $1 < k_i$ there is a chance that $1 \le g_i < k_i$. Check <u>largest</u> k_i first. If any $g_i < k_i$ then A <u>not</u> diag-able, can stop process. Don't waste time on finding other e-vectors if A not diag-able.

Def: For $L: V \longrightarrow V$, S any basis of V / 13
$A = {}_S[L]_S$, let char. poly. of L be $|\lambda I_n - A|$.
If T is any other basis of V, let
$B = {}_T[L]_T$ so $B = P^{-1}AP$ for $P = {}_S P_T$
(transition matrix). Then we know
$|\lambda I_n - B| = |\lambda I_n - A|$ is the same char. poly.
giving a consistent definition of char. poly.
for L.

Possible Notations: $\mathrm{Char}_A(\lambda) = |\lambda I_n - A|$
$= \mathrm{Char}_L(\lambda)$

Some books use
$\Delta_A(\lambda)$ or $p_A(\lambda)$ for char. poly of A.

When does a quadratic poly factor into linear factors?

Let poly. be $a\lambda^2+b\lambda+c$. Quadratic formula for roots is $\dfrac{-b \pm \sqrt{b^2-4ac}}{2a}$ which is <u>real</u> iff $b^2-4ac \geq 0$.

<u>discriminant</u> of poly is b^2-4ac.

If $b^2-4ac<0$ then get only a pair of <u>Complex</u> roots, not real roots, so poly does not factor over $\mathbb{R}$.

If $b^2-4ac=0$ get one real root, repeated like $\lambda^2-4\lambda+4=(\lambda-2)^2$

$b^2-4ac=16-4(1)(4)=0$

<u>EX</u>: $A = \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 0 \\ -1 & 0 & 1 \end{bmatrix}$ so $A - \lambda I_3 = \begin{bmatrix} 1-\lambda & 0 & 1 \\ 0 & 1-\lambda & 0 \\ -1 & 0 & 1-\lambda \end{bmatrix}$

$|A - \lambda I_3| = (1-\lambda) \begin{vmatrix} 1-\lambda & 1 \\ -1 & 1-\lambda \end{vmatrix} = -(\lambda - 1)\left((1-\lambda)^2 + 1\right)$

(cofactor expansion along row 2) $= -(\lambda-1)(\lambda^2 - 2\lambda + 2)$

but $b^2 - 4ac = 4 - 4(1)(2) = -4 < 0$

so $\lambda^2 - 2\lambda + 2$ has no real roots.

only got one real e-value, $\lambda_1 = 1$, $k_1 = 1$, $g_1 = 1$

could not get an e-basis of $\mathbb{R}^3$ for A.

This A is not diag-able over $\mathbb{R}$.

Ex: $A = \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 1 \end{bmatrix}$ so $A-\lambda I_3 = \begin{bmatrix} 1-\lambda & 0 & 1 \\ 0 & 1-\lambda & 0 \\ 1 & 0 & 1-\lambda \end{bmatrix}$

$|A-\lambda I_3| = (1-\lambda)\begin{vmatrix} 1-\lambda & 1 \\ 1 & 1-\lambda \end{vmatrix} = -(\lambda-1)\left((1-\lambda)^2-1\right)$

$= -(\lambda-1)(\lambda^2-2\lambda) = -(\lambda-1)(\lambda)(\lambda-2)$ has three distinct roots: order by inc. size:

$\lambda_1 = 0$, $\lambda_2 = 1$, $\lambda_3 = 2$, alg. mult.s: $m_1 = 1$, $m_2 = 1$, $m_3 = 1$, so geom. mults: $g_1 = 1$, $g_2 = 1$, $g_3 = 1$. Guaranteed to get one e-basis vector for each e-value, indep, e-basis of $\mathbb{R}^3$ for A. Find P s.t.

$P^{-1}AP = D = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 2 \end{bmatrix}$ is diagonal.

$\boxed{\lambda_1 = 0}$: Get A_0: Solve $\left[\begin{array}{ccc|c} 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 1 & 0 & 1 & 0 \end{array}\right] \to \left[\begin{array}{ccc|c} 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right]$

$x_1 = -r$
$x_2 = 0$
$x_3 = r \in \mathbb{R}$

$A_0 = \left\{ \begin{bmatrix} -r \\ 0 \\ r \end{bmatrix} \in \mathbb{R}^3 \,\middle|\, r \in \mathbb{R} \right\}$ has basis $T_1 = \left\{ \begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix} \right\}$

$\boxed{\lambda_2 = 1}$ Get A_1: Solve $\left[\begin{array}{ccc|c} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{array}\right] \to \left[\begin{array}{ccc|c} 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right]$ $x_1 = 0$, $x_2 = r \in \mathbb{R}$, $x_3 = 0$

$A_1 = \left\{ \begin{bmatrix} 0 \\ r \\ 0 \end{bmatrix} \in \mathbb{R}^3 \,\middle|\, r \in \mathbb{R} \right\}$ basis $T_2 = \left\{ \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} \right\}$

$\boxed{\lambda_3 = 2}$: Get A_2: Solve $\left[\begin{array}{ccc|c} -1 & 0 & 1 & 0 \\ 0 & -1 & 0 & 0 \\ 1 & 0 & -1 & 0 \end{array}\right] \to \left[\begin{array}{ccc|c} 1 & 0 & -1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right]$ $x_1 = r$, $x_2 = 0$, $x_3 = r \in \mathbb{R}$

$A_2 = \left\{ \begin{bmatrix} r \\ 0 \\ r \end{bmatrix} \in \mathbb{R}^3 \,\middle|\, r \in \mathbb{R} \right\}$ has basis $T_3 = \left\{ \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix} \right\}$.

So e-basis $T = \left\{ \begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix} \right\}$ and

$P = {}_S P_T = \begin{bmatrix} -1 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 1 \end{bmatrix}$ should make $P^{-1}AP = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 2 \end{bmatrix} = D$

Easier to check that $\boxed{AP = PD}$

$$\underset{A}{\begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 1 \end{bmatrix}} \underset{P}{\begin{bmatrix} -1 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 1 \end{bmatrix}} \overset{\checkmark}{=} \begin{bmatrix} 0 & 0 & 2 \\ 0 & 1 & 0 \\ 0 & 0 & 2 \end{bmatrix} \overset{\checkmark}{=} \underset{P}{\begin{bmatrix} -1 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 1 \end{bmatrix}} \underset{D}{\begin{bmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 2 \end{bmatrix}}.$$