Recall the standard dot product on $\mathbb{R}^n$:

For $v = \begin{bmatrix} a_1 \\ \vdots \\ a_n \end{bmatrix}$, $w = \begin{bmatrix} b_1 \\ \vdots \\ b_n \end{bmatrix} \in \mathbb{R}^n$ define $v \cdot w = \sum_{i=1}^{n} a_i b_i \in \mathbb{R}$.
$= v^T w$

Properties: Symmetric: $v \cdot w = w \cdot v$

Bilinear: $(c_1 v_1 + c_2 v_2) \cdot w = c_1(v_1 \cdot w) + c_2(v_2 \cdot w)$

and $v \cdot (c_1 w_1 + c_2 w_2) = c_1(v \cdot w_1) + c_2(v \cdot w_2)$

Positive Definite: $v \cdot v = \sum_{i=1}^{n} a_i^2 \geq 0$ and

$v \cdot v = 0$ implies $v = 0_1^n$.

Def. Length of $v \in \mathbb{R}^n$ is $\|v\| = \sqrt{v \cdot v}$

Distance between $v, w \in \mathbb{R}^n$ is $\|v - w\|$.

Say v is a unit vector when $\|v\| = 1$.

Th: (Cauchy-Schwarz Inequality)
For any $v, w \in \mathbb{R}^n$, $|v \cdot w| \leq \|v\| \cdot \|w\|$ so

$$-1 \leq \frac{v \cdot w}{\|v\| \cdot \|w\|} \leq 1.$$

Def: For $v, w \in \mathbb{R}^n$ the angle between v and w, $\theta_{v,w}$ is the unique angle between 0 and π such that $\cos(\theta_{v,w}) = \dfrac{v \cdot w}{\|v\| \, \|w\|}$.

Def. Say $v \perp w$ (perpendicular, orthogonal) when $\theta_{v,w} = \pi/2$, Same as $\cos(\theta_{v,w}) = 0$, iff $v \cdot w = 0$.

Def: Say $S = \{v_1, \ldots, v_m\} \subseteq \mathbb{R}^n$ is orthogonal when $v_i \perp v_j$ for all $1 \leq i \neq j \leq m$.

Def. Say $S = \{v_1, \dots, v_m\} \subseteq \mathbb{R}^n$ is **orthonormal** when $v_i \cdot v_j = \delta_{ij} = \begin{cases} 1 & \text{if } i = j \\ 0 & \text{if } i \neq j. \end{cases}$

So S consists of unit vectors which are mutually perpendicular.

Th: If $S = \{v_1, \dots, v_m\} \subseteq \mathbb{R}^n$ is an orthogonal set of non-zero vectors then S is **indep.**

Pf: Suppose $\sum_{i=1}^{m} c_i v_i = \theta$ then for any $1 \leq j \leq m$

$$\left(\sum_{i=1}^{m} c_i v_i\right) \cdot v_j = \theta \cdot v_j = 0.$$

By bilinearity, get

$$\sum_{i=1}^{m} c_i (v_i \cdot v_j) = 0$$

but for $i \neq j$, $v_i \cdot v_j = 0$ so

$$c_j (v_j \cdot v_j) = 0.$$

Since $v_j \neq \theta$, $v_j \cdot v_j > 0$ (Pos. Definite property) so $c_j = 0$, true for all $1 \leq j \leq m$. □

Th. Suppose $S=\{v_1,\dots,v_n\}$ is an orthogonal basis of $\mathbb{R}^n$. For any $v\in\mathbb{R}^n$, $v=\sum_{i=1}^{n} c_i v_i$ and $c_j=\dfrac{v\cdot v_j}{v_j\cdot v_j}$ for each $1\le j\le n$ gives the coordinates of v with respect to S, $[v]_S$. 4

Pf. For each $1\le j\le n$, $v\cdot v_j=\sum_{i=1}^{n} c_i(v_i\cdot v_j)$ $=c_j(v_j\cdot v_j)$ since $v_i\cdot v_j=0$ for $i\ne j$. Since $v_j\cdot v_j\ne 0$ (pos. def.) get $c_j=\dfrac{v\cdot v_j}{v_j\cdot v_j}$. $\square$

Cor: If $S=\{v_1,\dots,v_n\}$ is an orthonormal basis of $\mathbb{R}^n$ then $\forall v\in\mathbb{R}^n$, $v=\sum_{i=1}^{n}(v\cdot v_i)\,v_i$.

Ex: Std. basis $S=\{e_1,\dots,e_n\}$ of $\mathbb{R}^n$ is an orthonormal basis of $\mathbb{R}^n$.

Ex: For any angle ϕ let $S = \left\{ \begin{bmatrix} \cos\phi \\ \sin\phi \end{bmatrix}, \begin{bmatrix} -\sin\phi \\ \cos\phi \end{bmatrix} \right\}$ 5
$\qquad\qquad v_1 \qquad\qquad v_2$

Then $v_1 \cdot v_1 = \cos^2\phi + \sin^2\phi = 1$

$v_2 \cdot v_2 = (-\sin\phi)^2 + (\cos^2\phi) = 1$

$v_1 \cdot v_2 = -\cos\phi\sin\phi + \sin\phi\cos\phi = 0$

so S is an orthonormal set in $\mathbb{R}^2$, indep, basis of $\mathbb{R}^2$. $\forall v = \begin{bmatrix} a \\ b \end{bmatrix} \in \mathbb{R}^2$

$$\begin{bmatrix} v \cdot v_1 \\ v \cdot v_2 \end{bmatrix} = \begin{bmatrix} a\cos\phi + b\sin\phi \\ -a\sin\phi + b\cos\phi \end{bmatrix} = [v]_S \text{ since}$$

$$(a\cos\phi + b\sin\phi)\begin{bmatrix} \cos\phi \\ \sin\phi \end{bmatrix} + (-a\sin\phi + b\cos\phi)\begin{bmatrix} -\sin\phi \\ \cos\phi \end{bmatrix}$$

$$= \begin{bmatrix} a\cos^2\phi + b\sin\phi\cos\phi + a\sin^2\phi - b\cos\phi\sin\phi \\ a\cos\phi\sin\phi + b\sin^2\phi - a\sin\phi\cos\phi + b\cos^2\phi \end{bmatrix} = \begin{bmatrix} a \\ b \end{bmatrix}$$

Let $S=\{v_1,\dots,v_n\}$ be an orthonormal (o.n.) basis of $\mathbb{R}^n$ and let $A\in\mathbb{R}^n_n$ be the matrix whose columns are the vectors from S, so $Col_j(A)=v_j$ for $1\le j\le n$. Then

$$\underset{n\times 1}{Col_i(A)}\cdot\underset{n\times 1}{Col_j(A)}=\underset{1\times n}{(Col_i(A))^T}\;\underset{n\times 1}{Col_j(A)}=\delta_{ij}$$

But $(Col_i(A))^T = Row_i(A^T)$ so

$$\underset{1\times n}{Row_i(A^T)}\;\underset{n\times 1}{Col_j(A)}=\delta_{ij}$$ is the (i,j)-entry of $\underset{(n\times n)(n\times n)}{A^TA}=[\delta_{ij}]=I_n$ = the identity matr

so $A^T=A^{-1}$.

Def. We say $A \in \mathbb{R}^n_n$ is an <u>orthogonal</u> matrix when $A^T = A^{-1}$.

Th: $A \in \mathbb{R}^n_n$ is orthogonal iff $S = \{Col_1(A), \ldots, Col_n(A)\}$ is an <u>orthonormal</u> set in $\mathbb{R}^n$ iff $T = \{Row_1(A), \ldots, Row_n(A)\}$ is an o.n. set in $\mathbb{R}_n$ (with respect to the std dot product in $\mathbb{R}_n$).

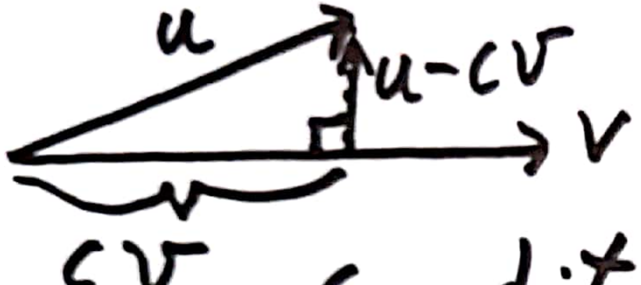
EX: For each $\phi \in \mathbb{R}$, $A = \begin{bmatrix} \cos\phi & -\sin\phi \\ \sin\phi & \cos\phi \end{bmatrix}$ is orthogonal.

$$A^T A = \begin{bmatrix} \cos\phi & \sin\phi \\ -\sin\phi & \cos\phi \end{bmatrix} \begin{bmatrix} \cos\phi & -\sin\phi \\ \sin\phi & \cos\phi \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

and $AA^T = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$ so $A^T = A^{-1}$.

Applications of dot product to geometry:

In $\mathbb{R}^2$ can find the projection of one vector onto another as follows:



Find $c \in \mathbb{R}$ s.t. $(u-cv)\perp v$

Call $cv = \mathrm{Proj}_v(u)$.

Condition on c is that $(u-cv)\cdot v = 0$

that is, $u\cdot v - c(v\cdot v) = 0$ so $u\cdot v = c(v\cdot v)$

so $c = \dfrac{u\cdot v}{v\cdot v}$ if $v\cdot v \neq 0$. Can't do this if $v = 0_1^n = \theta$.

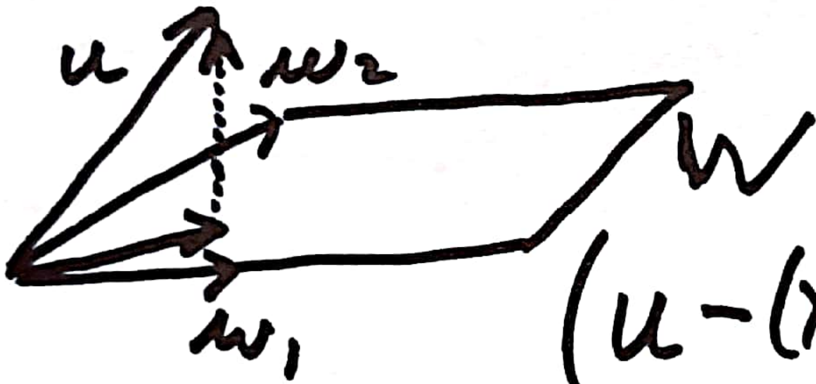
Note: If $u = xv \in \langle v \rangle$ then $\dfrac{u\cdot v}{v\cdot v} = \dfrac{(xv)\cdot v}{v\cdot v} = x$

so $\mathrm{Proj}_v(xv) = xv = u$.

In $\mathbb{R}^3$ can we find projection of $u \in \mathbb{R}^3$ onto a subspace $W = \langle w_1, w_2 \rangle$, a plane with basis $T = \{w_1, w_2\}$?

Find $\mathrm{Proj}_W(u) = x_1 w_1 + x_2 w_2 \in W$ such that $(u - (x_1 w_1 + x_2 w_2)) \perp W$

<u>Picture:</u>



Equivalent conditions: $(u - x_1 w_1 - x_2 w_2) \cdot w_j = 0$ for $j = 1, 2$, iff $u \cdot w_j = x_1 (w_1 \cdot w_j) + x_2 (w_2 \cdot w_j)$ for $j = 1, 2$. This is a <u>linear</u> <u>system</u>

$$\begin{bmatrix} w_1 \cdot w_1 & w_2 \cdot w_1 \\ w_1 \cdot w_2 & w_2 \cdot w_2 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} u \cdot w_1 \\ u \cdot w_2 \end{bmatrix}$$

with coefficient matrix $A = [w_i \cdot w_j]$ (symmetric)

Claim: rank$(A) = 2$ so A is invertible and this lin. sys. can be solved for any $u \in \mathbb{R}^3$. 10

Pf. If rank$(A) = 1$, would have $\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} \neq \begin{bmatrix} 0 \\ 0 \end{bmatrix}$ s.t.

$A \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$ that means $(x_1 w_1 + x_2 w_2) \cdot w_1 = 0$

and $(x_1 w_1 + x_2 w_2) \cdot w_2 = 0$

So $(x_1 w_1 + x_2 w_2) \cdot (x_1 w_1 + x_2 w_2) =$

$x_1 (x_1 w_1 + x_2 w_2) \cdot w_1 + x_2 (x_1 w_1 + x_2 w_2) \cdot w_2$

$= x_1 \cdot 0 + x_2 \cdot 0 = 0$ so by pos. def. property

$x_1 w_1 + x_2 w_2 = \Theta = 0^3_1 \in \mathbb{R}^3$ (zero vector in W)

But $T = \{w_1, w_2\}$ is a basis of W, indep, so

$x_1 = 0 = x_2$, contradiction. Rank$(A) = 2$ $\square$

Better way: If $T=\{w_1, w_2\}$ were an orthogonal basis of W, so $w_1 \cdot w_2 = 0$, then can easily solve $\begin{bmatrix} w_1\cdot w_1 & 0 \\ 0 & w_2 \cdot w_2 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} u\cdot w_1 \\ u \cdot w_2 \end{bmatrix}$ so

$\begin{bmatrix} (w_1\cdot w_1)x_1 \\ (w_2 \cdot w_2) x_2 \end{bmatrix} = \begin{bmatrix} u\cdot w_1 \\ u\cdot w_2 \end{bmatrix}$ so $\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} (u\cdot w_1)/(w_1\cdot w_1) \\ (u\cdot w_2)/(w_2 \cdot w_2) \end{bmatrix}$

$$\mathrm{Proj}_W(u) = \left(\frac{u\cdot w_1}{w_1\cdot w_1}\right) w_1 + \left(\frac{u\cdot w_2}{w_2\cdot w_2}\right) w_2$$
$$= \mathrm{Proj}_{w_1}(u) + \mathrm{Proj}_{w_2}(u)$$

This projection problem can be solved for any subspace $W \leq \mathbb{R}^n$.

Th: Let $T=\{w_1,\dots,w_m\}$ be a basis of subspace W in $\mathbb{R}^n$. For any $u\in\mathbb{R}^n$ can find

$$\operatorname{Proj}_W(u)=\sum_{i=1}^m x_i w_i \in W$$ such that

$$\left(u-\sum_{i=1}^m x_i w_i\right)\perp W,$$ that is, for each $1\le j\le m$,

$$\sum_{i=1}^m x_i(w_i\cdot w_j)=u\cdot w_j.$$ This is the lin. sys.

$AX=B$ where $A=[w_i\cdot w_j]\in\mathbb{R}^m_m$, $X=\begin{bmatrix}x_1\\ \vdots\\ x_m\end{bmatrix}$ (is invertible)

$B=\begin{bmatrix}u\cdot w_1\\ \vdots\\ u\cdot w_m\end{bmatrix}$. If T is orthogonal basis then A is diagonal and $x_i=\dfrac{u\cdot w_i}{w_i\cdot w_i}$

so $$\operatorname{Proj}_W(u)=\sum_{i=1}^m\left(\frac{u\cdot w_i}{w_i\cdot w_i}\right)w_i.$$

<u>Example</u>: In $\mathbb{R}^3$ let $W = \{X \in \mathbb{R}^3 \mid x_1 + x_2 + x_3 = 0\}$ $= v^\perp$ for $v = \begin{bmatrix}1\\1\\1\end{bmatrix}$. Let $u = \begin{bmatrix}a\\b\\c\end{bmatrix} \in \mathbb{R}^3$.

Find $\mathrm{Proj}_W(u)$.

Step ①: Pick a basis for W. Solve $[1\ 1\ 1 \mid 0]$

$$\left.\begin{aligned} x_1 &= -r-s \\ x_2 &= r \in \mathbb{R} \\ x_3 &= s \in \mathbb{R}\end{aligned}\right\} \text{ so } W = \left\{\begin{bmatrix}-r-s\\r\\s\end{bmatrix} \in \mathbb{R}^3 \,\middle|\, r,s \in \mathbb{R}\right\} = \left\langle \underbrace{\begin{bmatrix}-1\\1\\0\end{bmatrix}}_{w_1}, \underbrace{\begin{bmatrix}-1\\0\\1\end{bmatrix}}_{w_2} \right\rangle$$

$\mathrm{Proj}_W(u) = x_1 w_1 + x_2 w_2$ with x_1 and x_2 determined by conditions: $\big(u - (x_1 w_1 + x_2 w_2)\big) \cdot w_j = 0$

$\Leftrightarrow x_1(w_1 \cdot w_j) + x_2(w_2 \cdot w_j) = u \cdot w_j$ for $j = 1, 2$.

$$\Leftrightarrow \begin{bmatrix} w_1 \cdot w_1 & w_2 \cdot w_1 \\ w_1 \cdot w_2 & w_2 \cdot w_2 \end{bmatrix} \begin{bmatrix}x_1\\x_2\end{bmatrix} = \begin{bmatrix} u \cdot w_1 \\ u \cdot w_2 \end{bmatrix}$$

$A = [w_i \cdot w_j] = \begin{bmatrix} 2 & 1 \\ 1 & 2 \end{bmatrix}$ and $\begin{bmatrix} u \cdot w_1 \\ u \cdot w_2 \end{bmatrix} = \begin{bmatrix} -a+b \\ -a+c \end{bmatrix}$ so

solve $\begin{bmatrix} 2 & 1 \\ 1 & 2 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} b-a \\ c-a \end{bmatrix}$. Either use A^{-1}:

$$\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 2 & 1 \\ 1 & 2 \end{bmatrix}^{-1} \begin{bmatrix} b-a \\ c-a \end{bmatrix} = \frac{1}{3} \begin{bmatrix} 2 & -1 \\ -1 & 2 \end{bmatrix} \begin{bmatrix} b-a \\ c-a \end{bmatrix}$$ or

row reduce $\left[\begin{array}{cc|c} 2 & 1 & b-a \\ 1 & 2 & c-a \end{array}\right]$ to $\left[\begin{array}{cc|c} 1 & 0 & (-a+2b-c)/3 \\ 0 & 1 & (-a-b+2c)/3 \end{array}\right]$

$$\mathrm{Proj}_W(u) = \frac{1}{3}(-a+2b-c) \begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix} + \frac{1}{3}(-a-b+2c) \begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix}$$

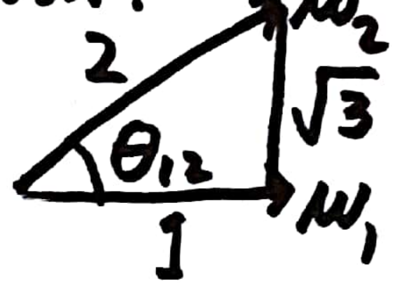
$$= \frac{1}{3} \begin{bmatrix} 2a-b-c \\ -a+2b-c \\ -a-b+2c \end{bmatrix} \in W.$$

Note: $\mathrm{Proj}_W\left(\begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}\right) = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$ and $\mathrm{Proj}_W(u) = u$ when $u \in W$.

Can we find an orthogonal basis of W?

$T = \left\{ w_1 = \begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix}, w_2 = \begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix} \right\}$ but $w_1 \cdot w_2 = 1$ so not orthogonal.

$\cos(\theta_{w_1, w_2}) = \dfrac{w_1 \cdot w_2}{\|w_1\| \, \|w_2\|} = \dfrac{1}{\sqrt{2}\sqrt{2}} = \dfrac{1}{2}$



But $(w_2 - \mathrm{Proj}_{w_1}(w_2)) \perp w_1$

Let $w_2' = w_2 - \left(\dfrac{w_2 \cdot w_1}{w_1 \cdot w_1}\right) w_1 = w_2 - \dfrac{1}{2} w_1 = \begin{bmatrix} -1/2 \\ -1/2 \\ 1 \end{bmatrix}$

Then $T' = \left\{ w_1' = w_1 = \begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix}, w_2' = \begin{bmatrix} -1/2 \\ -1/2 \\ 1 \end{bmatrix} \right\}$ is an orthogonal basis of W.

$\mathrm{Proj}_W(u) = x_1 w_1' + x_2 w_2'$ is solved easily:

$x_1 = (u \cdot w_1')/(w_1' \cdot w_1') = (b-a)/2$

$x_2 = (u \cdot w_2')/(w_2' \cdot w_2') = \left(c - \frac{a}{2} - \frac{b}{2}\right) / \left(3/2\right)$

$$\text{Proj}_W(u) = \frac{(b-a)}{2}\begin{bmatrix}-1\\1\\0\end{bmatrix} + \frac{2}{3}\left(\frac{2c-a-b}{2}\right)\begin{bmatrix}-1/2\\-1/2\\1\end{bmatrix}$$

$$= \frac{(3b-3a)}{6}\begin{bmatrix}-1\\1\\0\end{bmatrix} + \frac{(2c-a-b)}{6}\begin{bmatrix}-1\\-1\\2\end{bmatrix} = \frac{1}{6}\begin{bmatrix}3a-3b+a+b-2c\\-3a+3b+a+b-2c\\-2a-2b+4c\end{bmatrix}$$

$$= \frac{1}{3}\begin{bmatrix}2a-b-c\\-a+2b-c\\-a-b+2c\end{bmatrix}$$ is same answer as before.