

Wed. Apr. 22, Math 304-6 Feingold 1

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Homework Question (Z. Wang): (5a)

$b_1 = \begin{bmatrix} 1 \\ 3 \end{bmatrix}$ ,  $b_2 = \begin{bmatrix} -1 \\ -2 \end{bmatrix}$ ,  $B = \{b_1, b_2\}$  is a basis of  $\mathbb{R}^2$ .  
 $T: \mathbb{R}^2 \rightarrow \mathbb{R}^2$  is lin. map s.t.  $T(b_1) = 2b_1 + 4b_2$  and  
 $T(b_2) = 6b_1 + 5b_2$ . Then the matrix of  $T$   
relative to basis  $B$  is

$${}_B [T]_B = \begin{bmatrix} 2 & 6 \\ 4 & 5 \end{bmatrix} \text{ since its columns are the coordinates } [T(b_1)]_B \text{ and } [T(b_2)]_B$$

Follow-up question: If  $S = \{\begin{bmatrix} 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \end{bmatrix}\}$  std  $\mathbb{R}^2$  basis,  
find  ${}_S [T]_S$ . Solution: Transition matrix

$${}_S P_B = \begin{bmatrix} 1 & -1 \\ 3 & -2 \end{bmatrix} \text{ and } {}_S [T]_S = {}_S P_B ({}_B [T]_B) P_B^T {}_S$$

Question 7: Let  $U^{2 \times 2} = \left\{ \begin{bmatrix} a & b \\ 0 & c \end{bmatrix} \in \mathbb{R}^2 \mid a, b, c \in \mathbb{R} \right\} = U \quad \underline{2}$   
be the subspace of all upper triangular  $2 \times 2$  matrices.

Let  $T: U \rightarrow U$  be defined by  $T(M) = \begin{bmatrix} 1 & 8 \\ 0 & 6 \end{bmatrix} M$  so

$$T\left(\begin{bmatrix} a & b \\ 0 & c \end{bmatrix}\right) = \begin{bmatrix} 1 & 8 \\ 0 & 6 \end{bmatrix} \begin{bmatrix} a & b \\ 0 & c \end{bmatrix} = \begin{bmatrix} a & (b+8c) \\ 0 & 6c \end{bmatrix}.$$

Given basis  $B = \left\{ \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 1 & 1 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix} \right\}$  of  $U$

find  ${}_B[T]_B$ . Solution: First find  $T(B)$ :

$$T(b_1) = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} = b_1, \quad T(b_2) = \begin{bmatrix} 1 & 1 \\ 0 & 0 \end{bmatrix} = b_2, \quad T(b_3) = \begin{bmatrix} 0 & 8 \\ 0 & 6 \end{bmatrix}$$

$= -8b_1 + 8b_2 + 6b_3$  so coordinates of these w.r.t.  $B$

form the columns of  ${}_B[T]_B = \begin{bmatrix} 1 & 0 & -8 \\ 0 & 1 & 8 \\ 0 & 0 & 6 \end{bmatrix}$ .

Follow-up: For  $S = \left\{ \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix} \right\}$  find  ${}_S[T]_S$ .

Question: Why is char. poly of  $A \in \mathbb{R}^n$  L3

$|\lambda I_n - A|$  rather than  $|A - \lambda I_n|$ ?

Choice was necessary,  $\text{Char}_A(\lambda) = |\lambda I_n - A|$  has advantage that top term is  $\lambda^n$ , not  $(-1)^n \lambda^n$ .

$$|\lambda I_n - A| = (-1)^n |A - \lambda I_n|$$

$$|cB| = \det [cb_{ij}] = \sum_{\sigma \in S_n} \text{sgn}(\sigma) (cb_{1\sigma(1)}) \cdots (cb_{n\sigma(n)})$$
$$= c^n \det(B) = c^n |B|$$

Some roots, some e-values.

$[A - \lambda I_n | 0]$  is easier to write  $\left[ \begin{array}{ccc|c} a_{11} - \lambda & & & a_{1n} \\ \vdots & \ddots & & \vdots \\ a_{n1} & \cdots & a_{nn} - \lambda & 0 \end{array} \right]$  to get e-spaces.

Def.  $\text{Lin}(V, W) = \{L: V \rightarrow W \mid L \text{ is linear}\}$ . 19

Have  $+$  and  $\cdot$  on  $\text{Lin}(V, W)$  defined by:

$$(L_1 + L_2)(v) = L_1(v) + L_2(v) \quad \forall v \in V$$

$$(cL)(v) = c \cdot L(v) \quad \forall v \in V, \forall c \in \mathbb{R}.$$

$O_w^V: V \rightarrow W$  is def'd by  $O_w^V(v) = \theta_w, \forall v \in V$ .

Th:  $\text{Lin}(V, W)$  is a vector space under this  $+$  and  $\cdot$  with "zero vector"  $O_w^V$ .

Th  $T \mathcal{M}_S: \text{Lin}(V, W) \rightarrow \mathbb{R}^m$  where

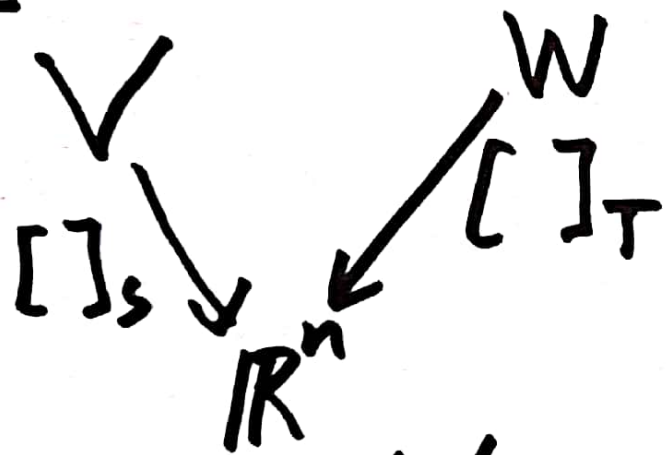
$S = \{v_1, \dots, v_n\}$  is a basis of  $V$ ,

$T = \{w_1, \dots, w_m\}$  is a basis of  $W$ , Then  $T \mathcal{M}_S$  is a bijective linear map

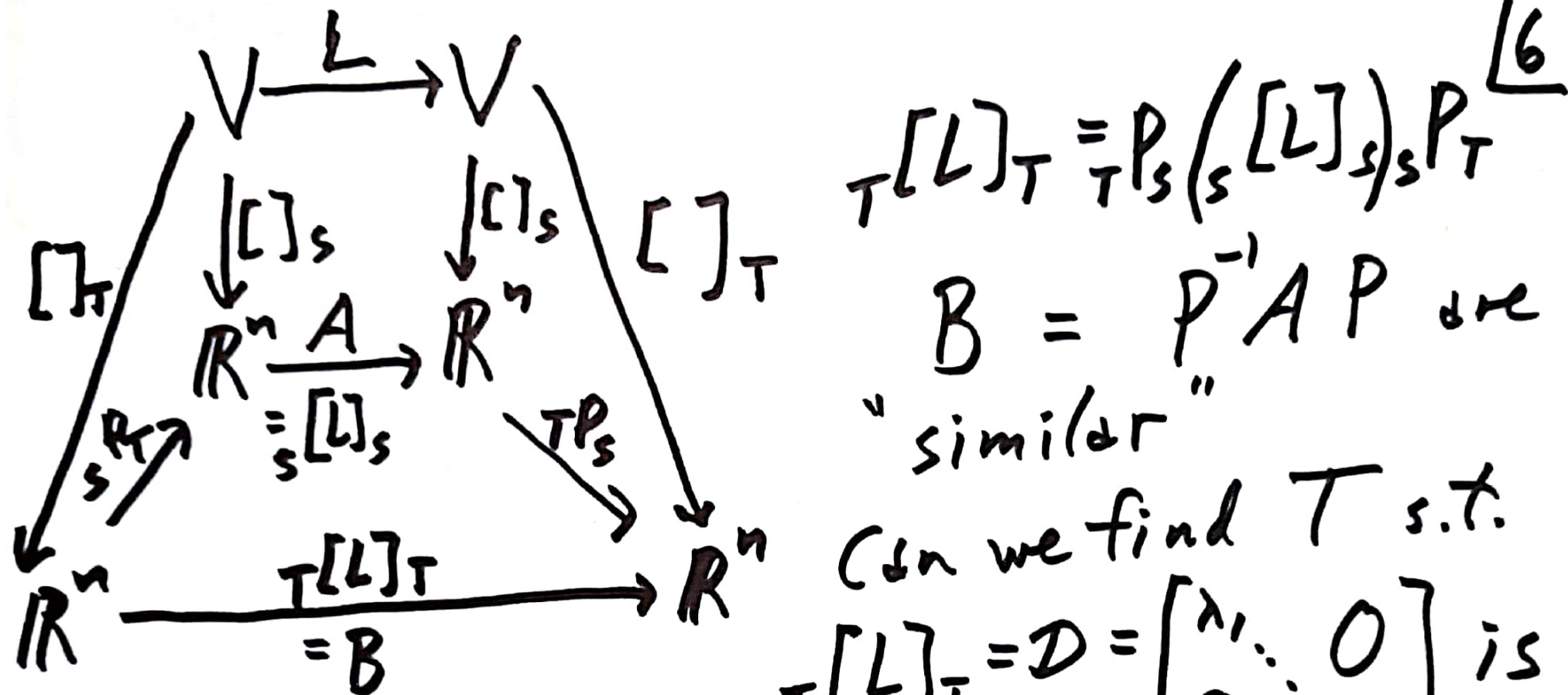
Review: Say  $L: V \rightarrow W$  is an isomorphism [5]  
when  $L$  is bijective linear map.

Say  $V$  and  $W$  are isomorphic when  
 $\exists L: V \rightarrow W$  s.t.  $L$  is an isomorphism. Notation:  
Th:  $V \cong W$  iff  $\dim(V) = \dim(W)$ .  $V \cong W$

$V \cong \mathbb{R}^n$  if  $\dim(V) = n$



Th: The relation  $\cong$  is an equivalence rel.  
(reflexive, symmetric, transitive).



$$T[L]_T = P_S^{-1}([L]_S)P_S$$

$$B = P^{-1}AP$$

"similar"

Can we find  $T$  s.t.

$$T[L]_T = D = \begin{bmatrix} \lambda_1 & & 0 \\ & \ddots & \\ 0 & & \lambda_n \end{bmatrix}$$

is diagonal?

Can also just start with  $A$ .

Answer: Can find such a  $T$  and  $D$  iff  $T$  is a basis on e-vectors for  $V$  with corresponding e-values  $\lambda_1, \dots, \lambda_n \in \mathbb{R}$ .

Need:  $T = \{\omega_1, \dots, \omega_n\}$  basis of  $V$  s.t.  $\square$

$$L(\omega_j) = \lambda_j \omega_j \text{ for } 1 \leq j \leq n.$$

Can only find e-basis after know e-values:  
 $\lambda_j \in \mathbb{R}$  is an e-value of  $L$ , of  $A = {}_S[L]_S$ , iff

$$\ker(L - \lambda_j I_V) \neq \{\theta_V\} \text{ iff}$$

$$\text{Nul}(A - \lambda_j I_n) \neq \{0_n\} \text{ iff}$$

$$\det(A - \lambda_j I_n) = 0 \text{ iff } \text{rank}(A - \lambda_j I_n) < n$$

$$\dim(L_{\lambda_j}) = g_j \text{ geom. mult. of e.value } \lambda_j$$

$$\text{Look for roots of } \text{Char}_A(\lambda) = |\lambda I_n - A| = \prod_{i=1}^r (\lambda - \lambda_i)^{k_i}$$

Step ①: Find distinct roots of  
 $\text{Char}_A(\lambda) = \prod_{i=1}^r (\lambda - \lambda_i)^{k_i}$  distinct  $\lambda_i$   
 $\lambda_1, \dots, \lambda_r \in \mathbb{R}$

Step ②: For each  $\lambda_i$  root, find e-space

$A_{\lambda_i} = \text{Nul}(A - \lambda_i I_n)$  has basis  $T_i = \{w_{i1}, \dots, w_{ig_i}\}$   
 $1 \leq i \leq r$   $g_i = \dim(A_{\lambda_i})$

Step ③: Is  $T = T_1 \cup T_2 \cup \dots \cup T_r$  a basis of  $\mathbb{R}^n$ ?

Iff  $n = g_1 + g_2 + \dots + g_r$

Th:  $1 \leq g_i \leq k_i$  for  $1 \leq i \leq r$ .

Know  $n = k_1 + k_2 + \dots + k_r$ .

Then  
 $D = \begin{bmatrix} \lambda_1 & & \\ & \ddots & \\ & & \lambda_1 \\ & & & \ddots \\ & & & & \lambda_r \\ & & & & & \ddots \\ & & & & & & \lambda_r \end{bmatrix}$



$$v \in V \xrightarrow{L} W$$

$$\begin{array}{ccc} \downarrow [ ]_S & & \downarrow [ ]_T \\ \mathbb{R}^n & \xrightarrow{LA} & \mathbb{R}^m \\ A = [ ]_T [L]_S & & \end{array}$$

$m \times n$

$S = \{v_1, \dots, v_n\}$  basis of  $V$

$T = \{w_1, \dots, w_m\}$  " "  $W$

$$\forall v \in V, [ ]_T [L]_S [v]_S = [L(v)]_T$$

$m \times n \quad n \times 1$

$$[v]_S = \begin{bmatrix} a_1 \\ \vdots \\ a_n \end{bmatrix}$$

method

$$v = \sum_{j=1}^n a_j v_j$$

Find  $[ ]_T [L]_S$  by

$$[T | L(S)] \xrightarrow{\text{r.r.}} [I_m | [ ]_T [L]_S]$$

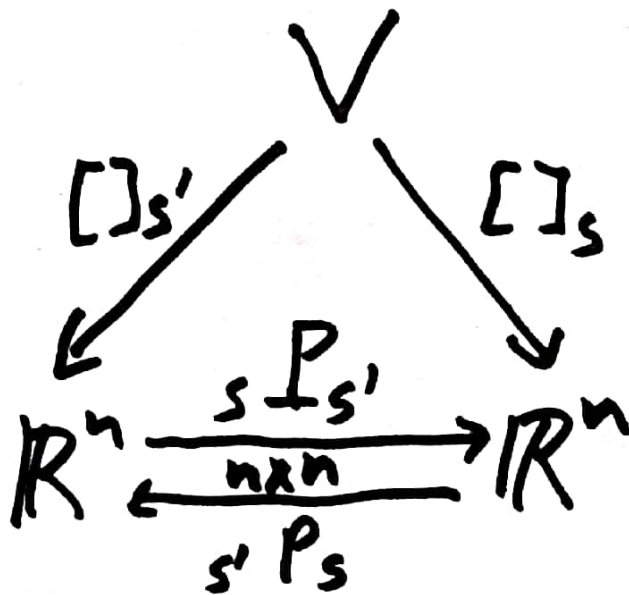
algorithm

"as columns"

Trans. Mat.

$$[S | S'] \xrightarrow{\text{r.r.}} [I_n | {}_S P_{S'}]$$

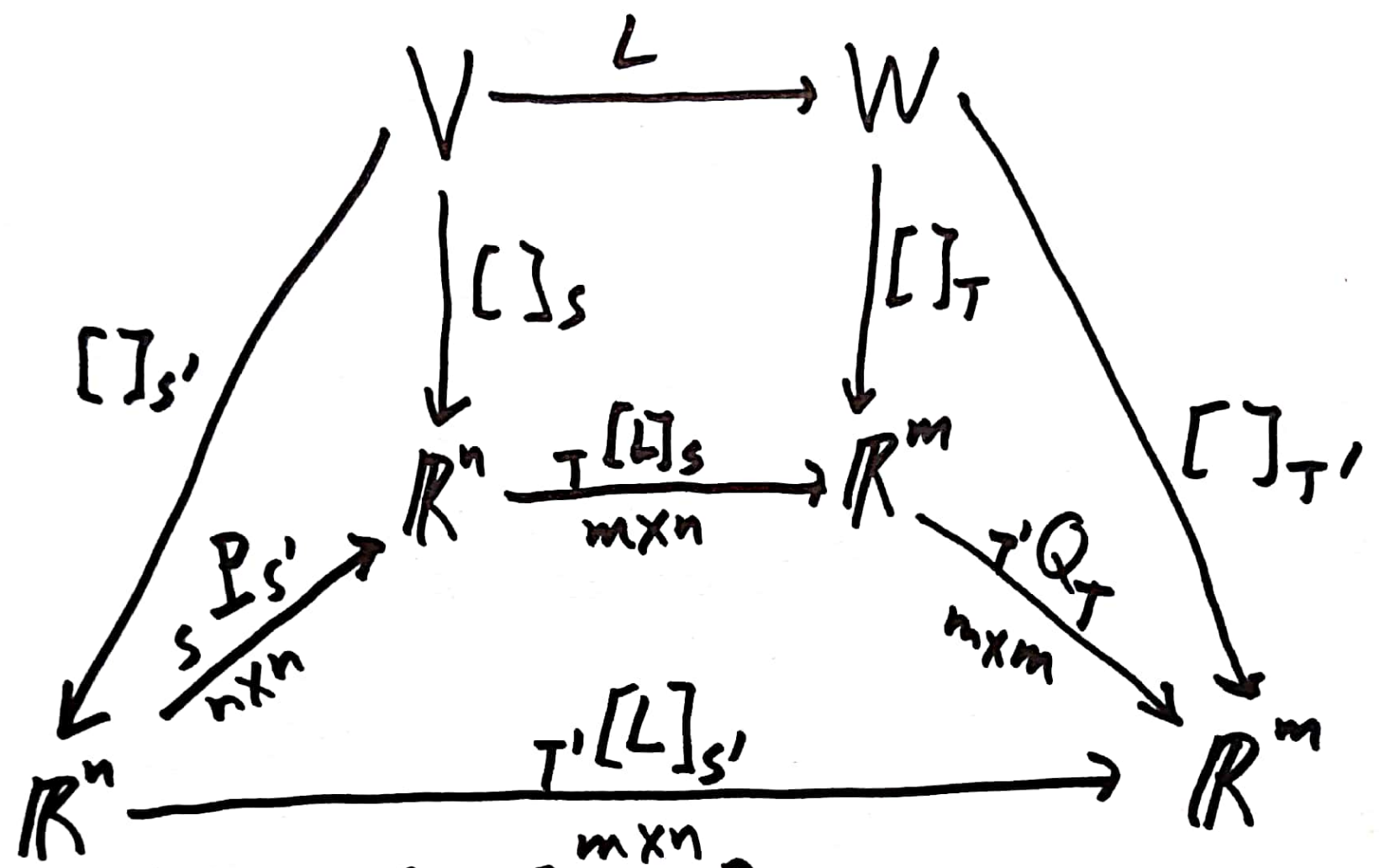
$$[S' | S] \rightarrow [I_n | {}_{S'} P_S]$$



$${}_S P_{S'} [v]_{S'} = [v]_S$$

$${}_{S'} P_S [v]_S = [v]_{S'}$$

$${}_S P_{S'} = ({}_{S'} P_S)^{-1}$$



Th  $T'[L]_{S'} = T'Q_T T[L]_S sP_{S'}$

$$[T | L(S)] \xrightarrow{\text{r.r.}} [I_m | T[L]_S]$$

$$[T' | L(S')] \xrightarrow{\text{r.r.}} [I_m | T'[L]_{S'}]$$