Homework Question (Z. Wang): (5a)

$b_1 = \begin{bmatrix}1\\3\end{bmatrix}$, $b_2 = \begin{bmatrix}-1\\-2\end{bmatrix}$. $B = \{b_1, b_2\}$ is a basis of $\mathbb{R}^2$. $T: \mathbb{R}^2 \to \mathbb{R}^2$ is lin. map s.t. $T(b_1) = 2b_1 + 4b_2$ and $T(b_2) = 6b_1 + 5b_2$. Then the matrix of T relative to basis B is

$${}_B[T]_B = \begin{bmatrix}2 & 6\\4 & 5\end{bmatrix}$$ since its columns are the coordinates $[T(b_1)]_B$ and $[T(b_2)]_B$

Follow-up question: If $S = \left\{\begin{bmatrix}1\\0\end{bmatrix}, \begin{bmatrix}0\\1\end{bmatrix}\right\}$ std $\mathbb{R}^2$ basis, find ${}_S[T]_S$. Solution: Transition matrix

${}_SP_B = \begin{bmatrix}1 & -1\\3 & -2\end{bmatrix}$ and ${}_S[T]_S = {}_SP_B\left({}_B[T]_B\right){}_BP_S$.

Question 7: Let $U^{2\times 2} = \left\{ \begin{bmatrix} a & b \\ 0 & c \end{bmatrix} \in \mathbb{R}^2_2 \,\middle|\, a,b,c \in \mathbb{R} \right\} = \underline{U|2}$ be the subspace of all upper triangular 2x2 matrices. so

Let $T: U \to U$ be defined by $T(M) = \begin{bmatrix} 1 & 8 \\ 0 & 6 \end{bmatrix} M$

$$T\left(\begin{bmatrix} a & b \\ 0 & c \end{bmatrix}\right) = \begin{bmatrix} 1 & 8 \\ 0 & 6 \end{bmatrix}\begin{bmatrix} a & b \\ 0 & c \end{bmatrix} = \begin{bmatrix} a & (b+8c) \\ 0 & 6c \end{bmatrix}.$$

Given basis $B = \left\{ \underset{b_1}{\begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}}, \underset{b_2}{\begin{bmatrix} 1 & 1 \\ 0 & 0 \end{bmatrix}}, \underset{b_3}{\begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}} \right\}$ of U

find ${}_B[T]_B$. Solution: First find $T(B)$:

$T(b_1) = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} = b_1$, $T(b_2) = \begin{bmatrix} 1 & 1 \\ 0 & 0 \end{bmatrix} = b_2$, $T(b_3) = \begin{bmatrix} 0 & 8 \\ 0 & 6 \end{bmatrix}$

$= -8b_1 + 8b_2 + 6b_3$ so coordinates of these w.r.t. B

form the columns of ${}_B[T]_B = \begin{bmatrix} 1 & 0 & -8 \\ 0 & 1 & 8 \\ 0 & 0 & 6 \end{bmatrix}.$

Follow-up: For $S = \left\{ \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix} \right\}$ find ${}_S[T]_S$.

Question: Why is char. poly of $A \in \mathbb{R}^n_n$ $|\lambda I_n - A|$ rather than $|A - \lambda I_n|$?

Choice was necessary, $Char_A(\lambda) = |\lambda I_n - A|$ has advantage that top term is λ^n, not $(-1)^n \lambda^n$.

$$|\lambda I_n - A| = (-1)^n |A - \lambda I_n|$$

$$|cB| = \det [c b_{ij}] = \sum_{\sigma \in S_n} \mathrm{sgn}(\sigma)(c b_{1\sigma(1)}) \cdots (c b_{n\sigma(n)})$$

$$= c^n \det(B) = c^n |B|$$

Same roots, same e-values.

$[A - \lambda I_n | 0]$ is easier to write to get e-spaces. $\left[\begin{array}{ccc|c} a_{11}-\lambda & & a_{1n} & 0 \\ \vdots & \ddots & \vdots & \\ a_{n1} & \cdots & a_{nn}-\lambda & 0 \end{array}\right]$

Def. $Lin(V,W) = \{L : V \to W \mid L \text{ is linear}\}$.

Have $+$ and $\cdot$ on $Lin(V,W)$ defined by:

$(L_1 + L_2)(v) = L_1(v) + L_2(v) \quad \forall v \in V$

$(cL)(v) = c \cdot L(v) \quad \forall v \in V, \forall c \in \mathbb{R}$.

$O^V_W : V \to W$ is def'd by $O^V_W(v) = \theta_W, \forall v \in V$.

Th: $Lin(V,W)$ is a vector space under this $+$ and $\cdot$ with "zero vector" O^V_W.

Th ${}_T\mathcal{M}_S : Lin(V,W) \to \mathbb{R}^m_n$ where

$S = \{v_1, \dots, v_n\}$ is a basis of V,

$T = \{w_1, \dots, w_m\}$ is a basis of W, Then ${}_T\mathcal{M}_S$ is a bijective linear map

Review: Say $L: V \to W$ is an <u>isomorphism</u> when L is bijective linear map.

Say V and W are <u>isomorphic</u> when $\exists L: V \to W$ s.t. L is an isomorphism. Notation: $V \cong W$

<u>Th</u>: $V \cong W$ iff $\dim(V) = \dim(W)$.

$V \cong \mathbb{R}^n$ if $\dim(V) = n$

$$\begin{tikzcd}
V \arrow[dr, "{[\;]_S}"'] & & W \arrow[dl, "{[\;]_T}"] \\
& \mathbb{R}^n &
\end{tikzcd}$$

<u>Th</u>: The relation $\cong$ is an equivalence rel. (reflexive, symmetric, transitive).

$$\begin{tikzcd}
 & V \arrow[r, "L"] \arrow[d, "{[\,]_S}"] \arrow[ddl, "{[\,]_T}"'] & V \arrow[d, "{[\,]_S}"] \arrow[ddr, "{[\,]_T}"] & \\
 & \mathbb{R}^n \arrow[r, "A"', "{={}_S[L]_S}"'] & \mathbb{R}^n \arrow[dr, "{{}_TP_S}"] & \\
\mathbb{R}^n \arrow[ur, "{{}_SP_T}"] \arrow[rrr, "{{}_T[L]_T}", "{=B}"'] & & & \mathbb{R}^n
\end{tikzcd}$$

$${}_T[L]_T = {}_TP_S \left({}_S[L]_S\right) {}_SP_T$$

$B = P^{-1} A P$ are "similar"

Can we find T s.t. ${}_T[L]_T = D = \begin{bmatrix} \lambda_1 & & 0 \\ & \ddots & \\ 0 & & \lambda_n \end{bmatrix}$ is is diagonal?

Can also just start with A.

Answer: Can find such a T and D iff T is a basis on e-vectors for V with corresponding e-values $\lambda_1, \dots, \lambda_n \in \mathbb{R}$.

Need: $T = \{w_1, \dots, w_n\}$ basis of V s.t.
$L(w_j) = \lambda_j w_j$ for $1 \le j \le n$.
Can only find e-basis after know e-values:
$\lambda_j \in \mathbb{R}$ is an e-value of L, of $A = {}_S[L]_S$, iff
$$Ker(L - \lambda_j I_V) \neq \{\theta_V\} \text{ iff}$$
$$Nul(A - \lambda_j I_n) \neq \{0^n_1\} \text{ iff}$$
$$\det(A - \lambda_j I_n) = 0 \text{ iff } rank(A - \lambda_j I_n) < n$$
$\dim(L_{\lambda_j}) = g_j$ geom. mult. of e.value λ_j
Look for roots of $Char_A(\lambda) = |\lambda I_n - A| = \prod_{i=1}^{r} (\lambda - \lambda_i)^{k_i}$

Step ①: Find distinct roots of

$$Char_A(\lambda) = \prod_{i=1}^{r} (\lambda-\lambda_1)^{k_1}\cdots(\lambda-\lambda_r)^{k_i}$$

distinct $\lambda_1,\dots,\lambda_r \in \mathbb{R}$

step ②: For each λ_i root, find e-space

$A_{\lambda_i} = Nul(A-\lambda_i I_n)$ has basis $T_i = \{w_{i1},\dots,w_{ig_i}\}$

$1 \le i \le r$ $\quad g_i = \dim(A_{\lambda_i})$

Step ③: Is $T = T_1 \cup T_2 \cup \cdots \cup T_r$ a basis of $\mathbb{R}^n$?

Iff $\quad n = g_1 + g_2 + \cdots + g_r$

Then

$$D = \begin{bmatrix} \begin{matrix} \lambda_1 & & \\ & \ddots & \\ & & \lambda_1 \end{matrix} & & \\ & \ddots & \\ & & \begin{matrix} \lambda_r & & \\ & \ddots & \\ & & \lambda_r \end{matrix} \end{bmatrix}$$

__Th__: $1 \le g_i \le k_i$ for $1 \le i \le r$.

Know $n = k_1 + k_2 + \cdots + k_n$.

$$\begin{tikzcd} v\in V \arrow[r, "L"] \arrow[d, "{[\ ]_S}"] & W \arrow[d, "{[\ ]_T}"] \\ \mathbb{R}^n \arrow[r, "L_A"] & \mathbb{R}^m \end{tikzcd}$$

$A = {}_T[L]_S$

$m\times n$

$S=\{v_1,\dots,v_n\}$ basis of V

$T=\{w_1,\dots,w_m\}$ " " W

$\forall v\in V,\ \underset{m\times n}{{}_T[L]_S}\ \underset{n\times 1}{[v]_S} = [L(v)]_T$

$[v]_S = \begin{bmatrix} a_1 \\ \vdots \\ a_n \end{bmatrix}$ means $v=\sum_{j=1}^{n} a_j v_j$

Find ${}_T[L]_S$ by

$$[T \mid L(S)] \xrightarrow{\text{r.r.}} [I_m \mid {}_T[L]_S]$$ algorithm

"as columns"

Trans. Mat.

$[S \mid S'] \xrightarrow{\text{r.r.}} [I_n \mid {}_SP_{S'}]$

$[S' \mid S] \rightarrow [I_n \mid {}_{S'}P_S]$

$$\begin{tikzcd} & V \arrow[dl, "{[\ ]_{S'}}"'] \arrow[dr, "{[\ ]_S}"] & \\ \mathbb{R}^n \arrow[rr, shift left, "{{}_SP_{S'}}", "n\times n"'] & & \mathbb{R}^n \arrow[ll, shift left, "{{}_{S'}P_S}"] \end{tikzcd}$$

${}_SP_{S'}[v]_{S'} = [v]_S$

${}_{S'}P_S[v]_S = [v]_{S'}$

${}_SP_{S'} = ({}_{S'}P_S)^{-1}$

$$
\begin{tikzcd}
 & V \arrow[r, "L"] \arrow[d, "{[\ ]_S}"] \arrow[ddl, bend right, "{[\ ]_{S'}}"'] & W \arrow[d, "{[\ ]_T}"] \arrow[ddr, bend left, "{[\ ]_{T'}}"] & \\
 & \mathbb{R}^n \arrow[r, "{{}_T[L]_S}", "{m \times n}"'] & \mathbb{R}^m \arrow[dr, "{{}_{T'}Q_T}", "{m \times m}"'] & \\
\mathbb{R}^n \arrow[ur, "{{}_S P_{S'}}", "{n \times n}"'] \arrow[rrr, "{{}_{T'}[L]_{S'}}", "{m \times n}"'] & & & \mathbb{R}^m
\end{tikzcd}
$$

Th $\quad {}_{T'}[L]_{S'} = {}_{T'}Q_T \; {}_T[L]_S \; {}_S P_{S'}$

$$[T \mid L(S)] \xrightarrow{\text{r.r.}} [I_m \mid {}_T[L]_S]$$

$$[T' \mid L(S')] \xrightarrow{\text{r.r.}} [I_m \mid {}_{T'}[L]_{S'}]$$