Th (Pythagorean Thm in $\mathbb{R}^n$). For $X, Y \in \mathbb{R}^n$, if $X \cdot Y = 0$ (so $X \perp Y$) then $\|X+Y\|^2 = \|X\|^2 + \|Y\|^2$. For $v_1, v_2, \dots, v_m \in \mathbb{R}^n$, if $\{v_1, \dots, v_m\}$ is orthogonal (so $v_i \cdot v_j = 0$ for $i \neq j$) then $\|\sum_{i=1}^m v_i\|^2 = \sum_{i=1}^m \|v_i\|^2$.

Pf. $\|X+Y\|^2 = (X+Y)\cdot(X+Y) = X\cdot X + X\cdot Y + Y\cdot X + Y\cdot Y$ $= X\cdot X + Y\cdot Y = \|X\|^2 + \|Y\|^2$. The general case of m orthogonal vectors follows by induction, using $X = v_1 + \cdots + v_{m-1}$, $Y = v_m$. $\square$

Th (Triangle Inequality in $\mathbb{R}^n$)

For any $X, Y \in \mathbb{R}^n$, $\|X+Y\| \le \|X\| + \|Y\|$.

Pf. $\|X+Y\|^2 = \|X\|^2 + 2(X \cdot Y) + \|Y\|^2$

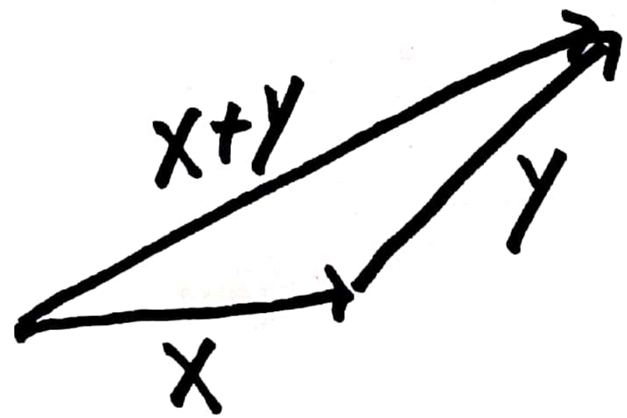
$\le \|X\|^2 + 2|X \cdot Y| + \|Y\|^2$ so by

Cauch-Schw. Ineq. $\le \|X\|^2 + 2(\|X\|)(\|Y\|) + \|Y\|^2$

$= (\|X\| + \|Y\|)^2$. This gives

$0 \le \|X+Y\| \le \|X\| + \|Y\|$. $\square$

Geometrical Picture:



Complex Numbers: $\mathbb{C} = \{a+bi \mid a,b \in \mathbb{R},\ i^2=-1\}$

The "imaginary" number $i \in \mathbb{C}$, $i \notin \mathbb{R}$, is a special number such that $i^2=-1$.

$\mathbb{C}$ is a "field", like $\mathbb{R}$ = real numbers and $\mathbb{Q}$ = rational numbers, where we can do arithmetic, use for scalars in Lin. Alg.

Addition: $(a+bi)+(c+di) = (a+c)+(b+d)i$

Mult: $(a+bi)\cdot(c+di) = ac+adi+bci+bdi^2$

(commutative) $= (ac-bd)+(ad+bc)i$

Def. For $z=a+bi$ let "complex conjugate" of z be $\bar{z} = a-bi$, so $z\bar{z} = a^2+b^2 \geq 0$ and $z\bar{z}=0$ iff $z=0+0i=0$.

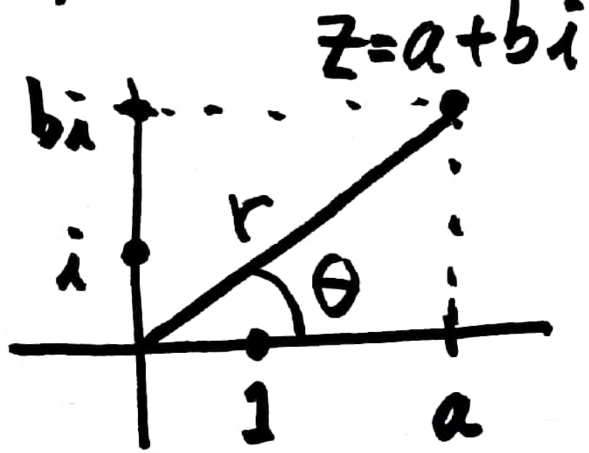
Note: For $0 \neq z = a+bi \in \mathbb{C}$, $z\bar{z} > 0$
and $z^{-1} = \dfrac{\bar{z}}{a^2+b^2} \in \mathbb{C}$ is mult. inverse of z.

Ex: If $z = 3+4i$ then $z\bar{z} = 3^2+4^2 = 25$
so $z\left(\dfrac{\bar{z}}{25}\right) = 1$, $z^{-1} = \dfrac{3-4i}{25}$.

Graphical Picture of $\mathbb{C}$:

looks like $\mathbb{R}^2$ with $1 \leftrightarrow \begin{bmatrix}1\\0\end{bmatrix}$
$i \leftrightarrow \begin{bmatrix}0\\1\end{bmatrix}$

but $\mathbb{C}$ has a <u>mult</u>
while $\mathbb{R}^2$ does <u>not</u>.



Related to "polar coordinates"

$$z = (r\cos\theta) + (r\sin\theta)\,i = a + b\,i$$

Complex vector spaces: Definition

Say $(V, +, \cdot, \theta)$ is a <u>complex</u> vector space (or V is a vector space over $\mathbb{C}$) when V obeys all the usual vector space axioms where scalars are in $\mathbb{C}$ (instead of in $\mathbb{R}$).

<u>Ex 1</u>: $\mathbb{C}^n = \left\{ \begin{bmatrix} z_1 \\ \vdots \\ z_n \end{bmatrix} \middle| \, z_j \in \mathbb{C}, 1 \le j \le n \right\}$ with the usual $+$ and $\cdot$

<u>Ex 2:</u> $\mathbb{C}^m_n = \{ A = [a_{ij}] \mid a_{ij} \in \mathbb{C}, 1 \le i \le m, 1 \le j \le n \}$

= $m \times n$ complex matrices. As before, For $A \in \mathbb{C}^m_n$, $L_A : \mathbb{C}^n \to \mathbb{C}^m$ is $L_A(X) = AX$.

Can do <u>any</u> linear algebra problem for complex vector spaces:

Solve linear system $AX = B$,

Find $Ker(L)$, $Range(L)$ for any linear map $L : V \to W$ for complex v. spaces V and W,

Find a basis for a subspace $U \leq V$,

For bases S in V, T in W, $L : V \to W$, find ${}_T[L]_S \in \mathbb{C}^m_n$ if $\dim(V) = n$, $\dim(W) = m$.

Have <u>std</u> basis $S = \left\{ e_1 = \begin{bmatrix} 1 \\ 0 \\ \vdots \\ 0 \end{bmatrix}, \ldots, e_n = \begin{bmatrix} 0 \\ \vdots \\ 0 \\ 1 \end{bmatrix} \right\}$ of $\mathbb{C}^n$

since $\mathbb{C}^n = \left\{ Z = \begin{bmatrix} z_1 \\ \vdots \\ z_n \end{bmatrix} = \sum_{j=1}^{n} z_j e_j \;\middle|\; z_j \in \mathbb{C},\ 1 \leq j \leq n \right\}$.

<u>Ex</u>: For $A=\begin{bmatrix} i & 1+i & 1-i \\ 1+i & 1-i & 1 \end{bmatrix}$ Solve $AX=0$; $X\in\mathbb{C}^3$

$$\left[\begin{array}{ccc|c} i & 1+i & 1-i & 0 \\ 1+i & 1-i & 1 & 0 \end{array}\right] \quad \begin{array}{l} -iR_1\to R_1 \\ (1-i)R_2\to R_2 \end{array}$$

using $(1-i)(1-i)=1-1-2i=-2i$

$$\to\left[\begin{array}{ccc|c} 1 & 1-i & -1-i & 0 \\ 2 & -2i & 1-i & 0 \end{array}\right] \quad -2R_1+R_2\to R_2$$

$$+\ \ -2 \quad -2+2i \quad 2+2i$$

$$\boxed{\begin{array}{l} -\tfrac{1}{2}(3+i)(-1+i)= \\ -\tfrac{1}{2}(-4+2i)=2-i \end{array}}$$

$$+\ \ 0 \quad -1+i \quad 2-i$$

$$\to\left[\begin{array}{ccc|c} 1 & 1-i & -1-i & 0 \\ 0 & -2 & 3+i & 0 \end{array}\right]\to\left[\begin{array}{ccc|c} 1 & 1-i & -1-i & 0 \\ 0 & 1 & -\tfrac{1}{2}(3+i) & 0 \end{array}\right] \quad (-1+i)R_3+R_1\to R_1$$

$$\to\left[\begin{array}{ccc|c} 1 & 0 & 1-2i & 0 \\ 0 & 1 & -\tfrac{1}{2}(3+i) & 0 \end{array}\right] \quad \begin{array}{l} x_1=(-1+2i)z \\ x_2=\tfrac{1}{2}(3+i)z \\ x_3=z\in\mathbb{C} \text{ free} \end{array}$$

$\dim(Nul(A))=1$

$$Nul(A)=\left\{z\begin{bmatrix} -1+2i \\ \tfrac{1}{2}(3+i) \\ 1 \end{bmatrix}\in\mathbb{C}^3 \;\middle|\; z\in\mathbb{C}\right\}=\left\langle\begin{bmatrix} -1+2i \\ \tfrac{3}{2}+\tfrac{1}{2}i \\ 1 \end{bmatrix}\right\rangle$$

Std. dot product in $\mathbb{C}^n$:

For $Z = \begin{bmatrix} z_1 \\ \vdots \\ z_n \end{bmatrix}$, $W = \begin{bmatrix} w_1 \\ \vdots \\ w_n \end{bmatrix} \in \mathbb{C}^n$ define

$Z \cdot W = \sum_{j=1}^{n} z_j \overline{w_j}$ (note complex conjugate on W coordinates)

$= Z^T \overline{W}$ where $\overline{W} = \begin{bmatrix} \overline{w_1} \\ \vdots \\ \overline{w_n} \end{bmatrix}$. $\forall a, b \in \mathbb{C}$,

<u>Then</u>: $(aZ + bZ') \cdot W = a(Z \cdot W) + b(Z' \cdot W)$ but

$Z \cdot (aW + bW') = \bar{a}(Z \cdot W) + \bar{b}(Z \cdot W')$

called "sesquilinear", linear in first input, conjugate linear in second input.

$\cdot : \mathbb{C}^n \times \mathbb{C}^n \longrightarrow \mathbb{C}$

Also; $Z \cdot W = \overline{W \cdot Z}$ (conjugate symm.)

and $Z \cdot Z = \sum_{j=1}^{n} z_j \overline{z}_j = \sum_{j=1}^{n} (a_j^2 + b_j^2) \geq 0$ (real)

where $z_j = a_j + b_j i$, and $Z \cdot Z = 0$ iff $Z = 0^n$,

called "positive definite".

This dot product gives geometry on $\mathbb{C}^n$:

$\|Z\| = \sqrt{Z \cdot Z} \geq 0$ length, etc.

Important advantage working over $\mathbb{C}$ is all polynomials factor into linear factors.

Ex: $X^2 + 1 = (X + i)(X - i)$

$ax^2+bx+c=0$ has roots $\dfrac{-b\pm\sqrt{b^2-4ac}}{2a}$

in $\mathbb{R}$ when $b^2-4ac \geq 0$

in $\mathbb{C}$ when $b^2-4ac<0$.

Application to diagonalization:

Ex. $A=\begin{bmatrix}0 & 1\\ -1 & 0\end{bmatrix}$ has $\mathrm{Char}_A(\lambda)=\begin{vmatrix}\lambda & -1\\ 1 & \lambda\end{vmatrix}=\lambda^2+1$

has two distinct complex $=(\lambda+i)(\lambda-i)$

e-values, $\lambda_1=-i$, $\lambda_2=i$.

Espaces: A_{λ_1}: $\left[\begin{array}{cc|c}-i & -1 & 0\\ 1 & -i & 0\end{array}\right]\to\left[\begin{array}{cc|c}1 & -i & 0\\ 0 & 0 & 0\end{array}\right]$ $x_1=iz$, $x_2=z\in\mathbb{C}$ free

$A_{\lambda_1}=\left\langle\begin{bmatrix}i\\1\end{bmatrix}\right\rangle$. A_{λ_2}: $\left[\begin{array}{cc|c}i & -1 & 0\\ 1 & i & 0\end{array}\right]\to\left[\begin{array}{cc|c}1 & i & 0\\ 0 & 0 & 0\end{array}\right]$ $x_1=-iz$, $x_2=z\in\mathbb{C}$ free

$A_{\lambda_2}=\left\langle\begin{bmatrix}-i\\1\end{bmatrix}\right\rangle$ Get e-basis $T=\left\{\begin{bmatrix}i\\1\end{bmatrix},\begin{bmatrix}-i\\1\end{bmatrix}\right\}$ for $\mathbb{C}^2$

If $S = \left\{ \begin{bmatrix} 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \end{bmatrix} \right\}$ is std basis of $\mathbb{C}^2$, [11]

transition matrix $P = {}_S P_T = \begin{bmatrix} i & -i \\ 1 & 1 \end{bmatrix}$

has inverse $P^{-1} = {}_T P_S = \frac{1}{2i} \begin{bmatrix} 1 & i \\ -1 & i \end{bmatrix} = \frac{1}{2} \begin{bmatrix} -i & 1 \\ i & 1 \end{bmatrix}$

and
$$P^{-1} A P = \frac{1}{2} \begin{bmatrix} -i & 1 \\ i & 1 \end{bmatrix} \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix} \begin{bmatrix} i & -i \\ 1 & 1 \end{bmatrix}$$
$$= \frac{1}{2} \begin{bmatrix} -i & 1 \\ i & 1 \end{bmatrix} \begin{bmatrix} 1 & 1 \\ -i & i \end{bmatrix} = \frac{1}{2} \begin{bmatrix} -2i & 0 \\ 0 & 2i \end{bmatrix} = \begin{bmatrix} -i & 0 \\ 0 & i \end{bmatrix}$$
$$= \begin{bmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{bmatrix} = D \text{ is diagonal.}$$

So working over $\mathbb{C}$ allows more matrices to be diagonalizable, but still not all.

Ex: $A = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}$ $\mathrm{Char}_A(\lambda) = \begin{vmatrix} (\lambda-1) & -1 \\ 0 & (\lambda-1) \end{vmatrix} = (\lambda-1)^2$

has only e-value $\lambda_1 = 1$, $k_1 = 2$

A_{λ_1}: $\left[\begin{array}{cc|c} 0 & -1 & 0 \\ 0 & 0 & 0 \end{array}\right]$ $x_1 = z \in \mathbb{C}$ free, $x_2 = 0$

$A_{\lambda_1} = \left\{ \begin{bmatrix} z \\ 0 \end{bmatrix} = z \begin{bmatrix} 1 \\ 0 \end{bmatrix} \middle| z \in \mathbb{C} \right\} = \left\langle \begin{bmatrix} 1 \\ 0 \end{bmatrix} \right\rangle$

$g_1 = 1 < 2 = k_1$

Cannot find a basis of $\mathbb{C}^2$ consisting of e-vectors for A.

A is not diag-able over $\mathbb{C}$.