<u>Given:</u> $L : V \to V$ basis S of V.

$$\begin{tikzcd}
 & V \arrow[r, "L"] \arrow[d, "{[\;]_S}"] \arrow[ddl, "{[\;]_T}"'] & V \arrow[d, "{[\;]_S}"] \arrow[ddr, "{[\;]_T}"] & \\
 & \mathbb{R}^n \arrow[r, "A"', "{{}_S[L]_S}"] & \mathbb{R}^n \arrow[dr, "{{}_T P_S}"] & \\
\mathbb{R}^n \arrow[ur, "{{}_S P_T}"] \arrow[rrr, "B", "{{}_T[L]_T}"'] & & & \mathbb{R}^n
\end{tikzcd}$$

<u>Goal:</u> Can we find a "better" basis T such that $B = {}_T[L]_T$ is diagonal?

$${}_T[L]_T = {}_T P_S \left({}_S[L]_S\right) {}_S P_T$$

$$B = P^{-1} A P$$ means B is similar to A

notation: $B \underset{\text{sim}}{\sim} A$

<u>Th</u> $\underset{\text{sim}}{\sim}$ is an equivalence relation on $\mathbb{R}^n_n$

What would it mean for $T = \{w_1, \dots, w_n\}$ (basis of V) if ${}_T[L]_T =$

$$D = \begin{bmatrix} \lambda_1 & 0 & \cdots & 0 \\ 0 & \lambda_2 & & \vdots \\ \vdots & & \ddots & 0 \\ 0 & & 0 & \lambda_n \end{bmatrix}$$

is diagonal?

Recall: $\mathrm{Col}_j({}_T[L]_T) = [L(w_j)]_T$

so it means

$$[L(w_j)]_T = \lambda_j e_j = \begin{bmatrix} 0 \\ \vdots \\ \lambda_j \\ \vdots \\ 0 \end{bmatrix} \leftarrow \text{row } j$$

so

$$L(w_j) = 0w_1 + \cdots + \lambda_j w_j + \cdots + 0w_n = \lambda_j w_j$$

<u>Def</u>: Say $\theta \neq w \in V$ is an <u>eigenvector</u> for L with <u>eigenvalue</u> $\lambda \in \mathbb{R}$ when $L(w) = \lambda w$.

Def: Say $L : V \to V$ is *diagonalizable* when V has a basis T such that ${}_T[L]_T$ is diagonal.

Th: L is diag-able iff V has a *basis* $T = \{w_1, \dots, w_n\}$ consisting of *e-vectors* s.t. $L(w_j) = \lambda_j w_j$ for some $\lambda_j \in \mathbb{R}$. Then λ_j is called the *e-value* of w_j for L. If such an *e-basis* T can be found, then

$$
{}_T[L]_T = D = \begin{bmatrix} \lambda_1 & & & \\ & \lambda_2 & & 0 \\ & & \ddots & \\ 0 & & & \lambda_n \end{bmatrix}
$$

<u>Example:</u> Let $L: \mathbb{R}^n \longrightarrow \mathbb{R}^n$ be

$$L\begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix} = \begin{bmatrix} \lambda_1 x_1 \\ \lambda_2 x_2 \\ \vdots \\ \lambda_n x_n \end{bmatrix} = \begin{bmatrix} \lambda_1 & & & 0 \\ & \lambda_2 & & \\ & & \ddots & \\ 0 & & & \lambda_n \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix}$$

then for $S = \{e_1, e_2, \ldots, e_n\}$ std basis of $\mathbb{R}^n$,

${}_S[L]_S = \begin{bmatrix} \lambda_1 & & 0 \\ & \ddots & \\ 0 & & \lambda_n \end{bmatrix} = D$ is diagonal so L is diag-able. $L(e_j) = \lambda_j e_j$ for $1 \le j \le n$, works for <u>any</u> choices of $\lambda_1, \ldots, \lambda_n \in \mathbb{R}$.

Special case: $\lambda_1 = \cdots = \lambda_n = c$, $D = cI_n$ is a <u>scalar</u> matrix, $L = c\, I_{\mathbb{R}^n}$ is a scalar operator on $V = \mathbb{R}^n$.

Example: $L : \mathbb{R}^2 \to \mathbb{R}^2$ by $L\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} y \\ x \end{bmatrix}$

$= \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix}$ so ${}_S[L]_S = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$ (for S std basis) is not diagonal. $T = \left\{ \begin{bmatrix} 1 \\ 1 \end{bmatrix} = w_1, \begin{bmatrix} 1 \\ -1 \end{bmatrix} = w_2 \right\}$

$L\begin{bmatrix} 1 \\ 1 \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \end{bmatrix} = 1 \cdot \begin{bmatrix} 1 \\ 1 \end{bmatrix}$ means w_1 is an e-vector for this L with e-value $\lambda_1 = 1$.

$L\begin{bmatrix} 1 \\ -1 \end{bmatrix} = \begin{bmatrix} -1 \\ 1 \end{bmatrix} = -1\begin{bmatrix} 1 \\ -1 \end{bmatrix}$ means w_2 is an e-vector for this L with e-value $\lambda_2 = -1$.

$${}_T[L]_T = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} = \begin{bmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{bmatrix} = D \text{ is diagonal}$$

so this L is diag-able.

Problem: Let $A = \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix} \in \mathbb{R}^2_2$, and let 6

$L = L_A : \mathbb{R}^2 \to \mathbb{R}^2$ so $L\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} x+y \\ x+y \end{bmatrix}$

Is L diag-able? What would it mean for $w = \begin{bmatrix} x \\ y \end{bmatrix}$ to be an e-vector for L with e-value $\lambda \in \mathbb{R}$? It would mean that

$$\begin{bmatrix} x+y \\ x+y \end{bmatrix} = \lambda\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} \lambda x \\ \lambda y \end{bmatrix} \text{ iff } \begin{matrix} x+y = \lambda x \ \underline{\text{and}} \\ x+y = \lambda y \end{matrix}$$

so $\begin{matrix} (1-\lambda)x + y = 0 \text{ and} \\ x + (1-\lambda)y = 0 \end{matrix}$ so $\left[\begin{array}{cc|c} (1-\lambda) & 1 & 0 \\ 1 & (1-\lambda) & 0 \end{array}\right]$

would have to have a <u>non-trivial</u> solution.

Have two approaches:

Row reduction:

$$\left[\begin{array}{cc|c} (1-\lambda) & 1 & 0 \\ 1 & (1-\lambda) & 0 \end{array}\right] \to \left[\begin{array}{cc|c} -\lambda & \lambda & 0 \\ 1 & 1-\lambda & 0 \end{array}\right] \to \left[\begin{array}{cc|c} 1 & 1 & 0 \\ 0 & 0 & 0 \end{array}\right] \text{ if } \lambda = 0$$

$$-1 \qquad \lambda - 1$$

but if $\lambda \neq 0$, get $\left[\begin{array}{cc|c} 1 & -1 & 0 \\ 1 & 1-\lambda & 0 \end{array}\right] \to \left[\begin{array}{cc|c} 1 & -1 & 0 \\ 0 & 2-\lambda & 0 \end{array}\right]$

for $\lambda \neq 2$ this has only the trivial solution.

For $\lambda = 2$ get $\left[\begin{array}{cc|c} 1 & -1 & 0 \\ 0 & 0 & 0 \end{array}\right]$ $x_1 = x_2 = r$, $x_2 = r \in \mathbb{R}$ free

For $\lambda = 0$ get $\left[\begin{array}{cc|c} 1 & 1 & 0 \\ 0 & 0 & 0 \end{array}\right]$ $x_1 = -x_2$, $x_2 = r \in \mathbb{R}$ free

Let $w_1 = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$ so $L\begin{bmatrix} 1 \\ 1 \end{bmatrix} = \begin{bmatrix} 2 \\ 2 \end{bmatrix} = 2\begin{bmatrix} 1 \\ 1 \end{bmatrix}$

$w_2 = \begin{bmatrix} 1 \\ -1 \end{bmatrix}$ so $L\begin{bmatrix} 1 \\ -1 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} = 0\begin{bmatrix} 1 \\ -1 \end{bmatrix}$

For $T=\left\{\begin{bmatrix}1\\1\end{bmatrix}=w_1, \begin{bmatrix}1\\-1\end{bmatrix}=w_2\right\}$ basis of $\mathbb{R}^2$ L8

$L(w_1)=2w_1$ and $L(w_2)=\theta=0\cdot w_2$

$${}_T[L]_T=\begin{bmatrix}2&0\\0&0\end{bmatrix}=D=\begin{bmatrix}\lambda_1&0\\0&\lambda_2\end{bmatrix}$$ is <u>diagonal</u>.

<u>Second approach</u>: $\begin{bmatrix}(1-\lambda)&1\\1&(1-\lambda)\end{bmatrix}$ not invertible when its determinant is zero:

$(1-\lambda)^2-1=0$ iff $(1-2\lambda+\lambda^2)-1=\lambda^2-2\lambda=$

$\lambda(\lambda-2)=0$. Only <u>roots</u> of this polynomial are $\lambda_1=0$ and $\lambda_2=2$. Solve two lin. systems:

For $\lambda_1=0$: $\left[\begin{array}{cc|c}1&1&0\\1&1&0\end{array}\right]$ and for $\lambda_2=2$: $\left[\begin{array}{cc|c}-1&1&0\\1&-1&0\end{array}\right]$.

<u>Def</u>: For $L: V \to V$, $\lambda \in \mathbb{R}$, let

$L_\lambda = \{v \in V \mid L(v) = \lambda v\}$. Note: $L(\theta) = \theta = \lambda \cdot \theta$

so $\theta \in L_\lambda$.

<u>Th</u>: $L_\lambda \leq V$, called the λ <u>eigenspace</u> of

L if $L_\lambda \neq \{\theta\}$ non-trivial subspace

whose non zero vector are all e-vectors for

L with e-value λ.

<u>Pf</u>: $\theta \in L_\lambda$ done. If $u, v \in L_\lambda$ then

$L(u) = \lambda u$ and $L(v) = \lambda v$ so

$L(u+v) = L(u) + L(v) = \lambda u + \lambda v = \lambda(u+v)$ so

$u+v \in L_\lambda$.

If $u \in L_\lambda$ and $c \in \mathbb{R}$, $L(u) = \lambda u$ and $L(cu) = cL(u) = c(\lambda u) = (c\lambda)u = (\lambda c)u = \lambda(cu)$ so $cu \in L_\lambda$. $\square$

<u>General Procedure</u>: Given $L: V \to V$

① Find all $\lambda \in \mathbb{R}$ s.t. $L_\lambda \neq \{\theta\}$

② List them $\lambda_1, \lambda_2, \dots, \lambda_r$.

③ For each λ_i find a <u>basis</u> of L_{λ_i}, $T_i = \{w_{i1}, w_{i2}, \dots, w_{ig_i}\}$, $g_i = \dim(L_{\lambda_i})$ called <u>geometric multiplicity</u> of λ_i for L.

④ Is $T = T_1 \cup T_2 \cup \dots \cup T_r$ a basis for V?

⑤ If T is a basis for V, it is an e-basis, L is diag-able, and

$$
{}_T[L]_T = \begin{bmatrix} \lambda_1 I_{g_1} & & & O \\ & \lambda_2 I_{g_2} & & \\ & & \ddots & \\ O & & & \lambda_r I_{g_r} \end{bmatrix} = D \text{ is a diagonal matrix}
$$

representing L w.r.t. T.

If $A = {}_S[L]_S$, $\boxed{P = {}_S I_T}$ = transition matrix from T to S

then $D = P^{-1}AP$ has "diagonalized" A.

We can just start with $A \in \mathbb{R}^n_n$ and try to "diagonalize" it, try to find invertible $P \in \mathbb{R}^n_n$ s.t. $D = P^{-1}AP$ is diagonal. 12

<u>Def</u> For $\lambda \in \mathbb{R}$, let $A_\lambda = \{X \in \mathbb{R}^n \mid AX = \lambda X\}$

<u>Th</u>: $A_\lambda \leq \mathbb{R}^n$. Called λ e-space of A if $A_\lambda \neq \{0^n_1\}$ nontrivial.

<u>Special Cases</u>: For $\lambda = 0$, $A_0 = Nul(A)$ $L_0 = Ker(L)$.

<u>Generally</u>: $L_\lambda = Ker(L - \lambda I_V)$, $A_\lambda = Nul(A - \lambda I_n)$

$L(v)=\lambda v$ iff $L(v)-\lambda v=\theta$

$I_V: V \to V$ is lin. map s.t. $I_V(v)=v$ so

$\lambda v = \lambda I_V(v)$ and $L(v)-\lambda v = L(v)-\lambda I_V(v)$

$\overset{if}{=} (L-\lambda I_V)(v)$ then $L_\lambda = \operatorname{Ker}(L-\lambda I_V)$

Why is $(L-\lambda I_V): V \longrightarrow V$ a linear map?

Can we make $\{L : V \to V \mid L \text{ is lin.}\}$ a vector space?

<u>Def</u>: Let $Lin(V, W) = \{L : V \to W \mid L \text{ is linear}\}$

with $+$ and $\cdot$ defined by

$(L_1+L_2)(v) = L_1(v)+L_2(v)$ and $(cL_1)(v) = c(L_1(v))$

<u>Th</u>: $Lin(V,W)$ with the above $+$ and $\cdot$ is a vector space and its "zero vector" is the "zero lin. map" $O_W^V : V \longrightarrow W$ defined by $O_W^V(v) = \theta_W$, $\forall v \in V$.

<u>Proof</u>: Long an tedious but straight forward.

<u>EX</u>: Let $L_1, L_2 : \mathbb{R}^2 \longrightarrow \mathbb{R}^2$ be in $Lin(\mathbb{R}^2, \mathbb{R}^2)$ with $L_1\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 2x+y \\ -x+3y \end{bmatrix}$ and $L_2\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} -5x+2y \\ 3x-4y \end{bmatrix}$

so $(L_1+L_2)\begin{bmatrix} x \\ y \end{bmatrix} = L_1\begin{bmatrix} x \\ y \end{bmatrix} + L_2\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} -3x+3y \\ 2x-y \end{bmatrix}$

$${}_S[L_1]_S + {}_S[L_2]_S = \begin{bmatrix} 2 & 1 \\ -1 & 3 \end{bmatrix} + \begin{bmatrix} -5 & 2 \\ 3 & -4 \end{bmatrix} = \begin{bmatrix} -3 & 3 \\ 2 & -1 \end{bmatrix} = {}_S[L_1+L_2]_S$$

Also $(2L_1)\begin{bmatrix} x \\ y \end{bmatrix} = 2\begin{bmatrix} 2x+y \\ -x+3y \end{bmatrix} = \begin{bmatrix} 4x+2y \\ -2x+6y \end{bmatrix}$

$${}_S[2L_1]_S = \begin{bmatrix} 4 & 2 \\ -2 & 6 \end{bmatrix} = 2\begin{bmatrix} 2 & 1 \\ -1 & 3 \end{bmatrix} = 2\,{}_S[L_1]_S$$

<u>Th</u>: Let V have basis S, W have basis T so $\forall L \in Lin(V,W)$ have ${}_T[L]_S \in \mathbb{R}^m_n$ when $\dim(V) = n$ and $\dim(W) = m$. This defines a map ${}_T\mathcal{M}_S : Lin(V,W) \longrightarrow \mathbb{R}^m_n$ by ${}_T\mathcal{M}_S(L) = {}_T[L]_S$. Then ${}_T\mathcal{M}_S$ is <u>linear</u>, and bijective, so is an isomorphism.

$\underline{\text{Cor}}$: $\dim(\mathrm{Lin}(V,W)) = (\dim V)(\dim W) = n\cdot m$ //16

$\underline{\text{Ex}}$: $\mathrm{Lin}(\mathbb{R}^2_2, \mathbb{R}^3) = \{L : \underbrace{\mathbb{R}^2_2}_{\text{bases } S} \to \underbrace{\mathbb{R}^3}_{T} \mid L \text{ is lin.}\}$

$${}_T m_S(L) \in \mathbb{R}^3_4$$

$$\dim(\mathrm{Lin}(\mathbb{R}^2_2, \mathbb{R}^3)) = \dim(\mathbb{R}^2_2)\dim(\mathbb{R}^3) = (4)(3) = 12$$

$$\begin{tikzcd}
\mathbb{R}^2_2 \arrow[r, "L"] \arrow[d, "{[\ ]_S}"] & \mathbb{R}^3 \arrow[d, "{[\ ]_T}"] \\
\mathbb{R}^4 \arrow[r, "{{}_T[L]_S}", "\substack{3\times 4 \\ m\times n}"'] & \mathbb{R}^3
\end{tikzcd}$$