Math 304 Linear Algebra Spring 2020 Practice Exam 1

SHOW ALL NECESSARY WORK FOR EACH PROBLEM

Notations: $\mathbf{R}^m_n$ is the set of all $m \times n$ real matrices, $\mathbf{R}^m = \mathbf{R}^m_1$ and $\mathbf{0} \in \mathbf{R}^m$ is the zero matrix. The transpose of matrix A is denoted by A^T.

(1) (15 Points) Let $L_A : \mathbf{R}^4 \to \mathbf{R}^4$ be the linear function $L_A(X) = AX$ associated with the matrix

$$A = \begin{bmatrix} 1 & 1 & -1 & -1 \\ 1 & 2 & 0 & 1 \\ -1 & 1 & 3 & 5 \\ 2 & 3 & -1 & 0 \end{bmatrix}.$$

(a) Find all the vectors in $\text{Ker}(L_A) = \{X \in \mathbf{R}^4 \mid L_A(X) = 0\}$ in terms of some free variables.
(b) Find the consistency conditions on the entries of $Y = [y_i]$ required for Y to be in $\text{Range}(L_A) = \{Y = L_A(X) \in \mathbf{R}^4 \mid X \in \mathbf{R}^4\}$.
(c) Determine whether L_A is injective and whether L_A is surjective. Explain why.

(2) (15 Points) Let $V = \mathbf{R}^4$ with standard basis $S = \{\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3, \mathbf{e}_4\}$ and with the standard dot product. Let $T = \{u_1 = \mathbf{e}_1 - \mathbf{e}_2, u_2 = \mathbf{e}_2 - \mathbf{e}_3, u_3 = \mathbf{e}_3 - \mathbf{e}_4\}$. For $u, v \in V$ let $\theta_{u,v}$ be the angle between u and v, so $\cos(\theta_{u,v}) = \frac{u \cdot v}{||u||\ ||v||}$.
(a) Find $\cos(\theta_{u_1,u_2})$, $\cos(\theta_{u_2,u_3})$ and $\cos(\theta_{u_1,u_3})$.
(b) Find $T^\perp = \{v \in V \mid v \cdot u_i = 0, i = 1, 2, 3\}$.

(3) (15 Points) Answer each question separately

(a) If $A \in \mathbf{R}^m_n$, $B \in \mathbf{R}^n_p$, and $C \in \mathbf{R}^m_p$ are matrices such that the associated functions $L_A : \mathbf{R}^n \to \mathbf{R}^m$, $L_B : \mathbf{R}^p \to \mathbf{R}^n$, and $L_C : \mathbf{R}^p \to \mathbf{R}^m$ satisfy $L_A \circ L_B = L_C$, then what is the relationship between the matrices A, B and C?
(b) If a nonzero matrix $A \in \mathbf{R}^7_4$ row reduces to B in RREF having r leading ones, what are the possible values of r?
(c) For $A \in \mathbf{R}^m_n$ what relation between m and n would guarantee that the homogeneous linear system $AX = \mathbf{0}$ has nontrivial solutions?
(d) What condition on the rank of $A \in \mathbf{R}^m_n$ is equivalent to the non-homogeneous linear system $AX = B$ being consistent for any choice of B?
(e) For $u, v \in \mathbf{R}^n$ write the statement of the Cauchy-Schwarz Inequality.

(4) (15 Points) For $A \in \mathbf{R}^m_n$, let $L_A : \mathbf{R}^n \to \mathbf{R}^m$ be the linear function $L_A(X) = AX$. For $1 \le j \le n$ let $\mathbf{e}_j \in \mathbf{R}^n$ be the matrix with 1 in row j and 0 in all other rows.
(a) Write $\text{Range}(L_A)$ as the set of all linear combinations of some specific vectors.
(b) If L_A is injective then what is the most you can say about the relation between m and n?
(c) If L_A is surjective then what is the most you can say about the relation between m and n?
(d) If $\text{rank}(A) = m$ what does that tell you about L_A?
(e) If $\text{rank}(A) = n$ what does that tell you about L_A?

(5) (15 Points) Let $L : \mathbf{R}^2 \to \mathbf{R}^2$ be linear, so $L(aX + bY) = aL(X) + bL(Y)$, with $L(\mathbf{e}_1) = \begin{bmatrix} 1 \\ 2 \end{bmatrix}$ and $L(\mathbf{e}_2) = \begin{bmatrix} -3 \\ 4 \end{bmatrix}$. Find $A \in \mathbf{R}^2_2$ such that $L(X) = L_A(X) = AX$.

1. (a) (6 Points) To find $\text{Ker}(L_A)$ we must solve a linear system by row reducing

$$\left[\begin{array}{cccc|c} 1 & 1 & -1 & -1 & 0 \\ 1 & 2 & 0 & 1 & 0 \\ -1 & 1 & 3 & 5 & 0 \\ 2 & 3 & -1 & 0 & 0 \end{array}\right] \text{ to } \left[\begin{array}{cccc|c} 1 & 0 & -2 & -3 & 0 \\ 0 & 1 & 1 & 2 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right] \text{ so } \begin{array}{c} x_1 = 2r+3s \\ x_2 = -r-2s \\ x_3 = r \in \mathbf{R} \\ x_4 = s \in \mathbf{R} \end{array}.$$

$$\text{Ker}(L) = \left\{ \begin{bmatrix} 2r+3s \\ -r-2s \\ r \\ s \end{bmatrix} \in \mathbf{R}^4 \;\middle|\; r, s \in \mathbf{R} \right\}.$$

(b) (6 Points) $Y = [y_i] \in \text{Range}(L_A)$ iff the following system is consistent:

$$\left[\begin{array}{cccc|c} 1 & 1 & -1 & -1 & y_1 \\ 1 & 2 & 0 & 1 & y_2 \\ -1 & 1 & 3 & 5 & y_3 \\ 2 & 3 & -1 & 0 & y_4 \end{array}\right] \to \left[\begin{array}{cccc|c} 1 & 0 & -2 & -3 & 2y_1 - y_2 \\ 0 & 1 & 1 & 2 & -y_1 + y_2 \\ 0 & 0 & 0 & 0 & 3y_1 - 2y_2 + y_3 \\ 0 & 0 & 0 & 0 & y_1 + y_2 - y_4 \end{array}\right] \begin{array}{c} \text{is consistent iff} \\ 0 = 3y_1 - 2y_2 + y_3 \\ \text{and} \\ 0 = y_1 + y_2 - y_4 \end{array}.$$

(c) (3 Points) L_A is not injective since by (a) more than one vector is sent to the zero vector, and L_A is not surjective since by (b) not all vectors of $\mathbf{R}^4$ are in $\text{Range}(L_A)$.

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2. (15 Points) Let $V = \mathbf{R}^4$ with standard basis $S = \{\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3, \mathbf{e}_4\}$ and with the standard dot product. Let $T = \{u_1 = \mathbf{e}_1 - \mathbf{e}_2, u_2 = \mathbf{e}_2 - \mathbf{e}_3, u_3 = \mathbf{e}_3 - \mathbf{e}_4\}$. For $u, v \in V$ let $\theta_{u,v}$ be the angle between u and v, so $\cos(\theta_{u,v}) = \frac{u \cdot v}{||u||\ ||v||}$.

(a) (6 Pts) Find $\cos(\theta_{u_1,u_2})$, $\cos(\theta_{u_2,u_3})$ and $\cos(\theta_{u_1,u_3})$.

Solution: Since $\mathbf{e}_i \cdot \mathbf{e}_j = \delta_{i,j}$ equals 1 when $i = j$ and 0 when $i \neq j$, we see that $u_1 \cdot u_2 = (\mathbf{e}_1 - \mathbf{e}_2) \cdot (\mathbf{e}_2 - \mathbf{e}_3) = -1$ and $u_2 \cdot u_3 = (\mathbf{e}_2 - \mathbf{e}_3) \cdot (\mathbf{e}_3 - \mathbf{e}_4) = -1$ but $u_1 \cdot u_3 = (\mathbf{e}_1 - \mathbf{e}_2) \cdot (\mathbf{e}_3 - \mathbf{e}_4) = 0$. We also see that $||u_i|| = \sqrt{u_i \cdot u_i} = \sqrt{2}$ for $i = 1, 2, 3$, so $\cos(\theta_{u_1,u_2}) = \frac{-1}{2} = \cos(\theta_{u_2,u_3})$ but $\cos(\theta_{u_1,u_3}) = 0$.

(b) (9 Pts) Find $T^\perp = \{v \in V \mid v \cdot u_i = 0, i = 1, 2, 3\}$.

Solution: If $v = [\,a \quad b \quad c \quad d\,]^T$ then $v \cdot u_1 = a - b$, $v \cdot u_2 = b - c$ and $v \cdot u_3 = c - d$ so the conditions for $v \in T^\perp$ are that $a = b = c = d$. We get that
$T^\perp = \{[\,a \quad a \quad a \quad a\,]^T \in \mathbf{R}^4 \mid a \in \mathbf{R}\} = \langle [\,1 \quad 1 \quad 1 \quad 1\,]^T \rangle$.

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3. (15 Points, 3 points each)
   (a) The relationship is $AB = C$, so C is the matrix product of A and B.
   (b) If $A \in \mathbf{R}^7_4$ is not the zero matrix, the number of leading ones in its RREF could only be 1, 2, 3 or 4 since each leading one occupies a column, and there is at least one.
   (c) The relation $n > m$ guarantees that the homogeneous linear system $AX = \mathbf{0}$ has nontrivial solutions since there must be at least 1 free variable.
   (d) The non-homogeneous linear system $AX = B$ is consistent iff $rank(A) = m$.
   (e) For $u, v \in \mathbf{R}^n$ the statement of the Cauchy-Schwarz Inequality is $|u \cdot v| \leq (||u||)(||v||)$.

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4. (15 Points, 3 points each)
   (a) Range(L_A) consists of all vectors

$$AX = A\left(\sum_{j=1}^{n} x_j \mathbf{e}_j\right) = \sum_{j=1}^{n} x_j A\mathbf{e}_j = \sum_{j=1}^{n} x_j L_A(\mathbf{e}_j)$$

   for all $x_j \in \mathbf{R}$. So Range(L_A) is the set of all linear combinations of the n vectors $L_A(\mathbf{e}_j) = A\mathbf{e}_j = Col_j(A)$ for $1 \leq j \leq n$, the column vectors of matrix A.
   (b) If L_A is injective then $n \leq m$ since more variables than equations would guarantee free variables.
   (c) If L_A is surjective then $m \leq n$ since more equations than variables would guarantee a row of zeros in the RREF of A, giving a consistency condition for $AX = B$.
   (d) If rank$(A) = m$ then L_A is surjective since m leading ones in the RREF means no zero rows so $AX = B$ is always consistent.
   (e) If rank$(A) = n$ then L_A is injective since n leading ones in the RREF means a leading one in each column and there are no free variables.

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5. (15 Points) Let $L : \mathbf{R}^2 \to \mathbf{R}^2$ be linear, so $L(aX + bY) = aL(X) + bL(Y)$, with $L(\mathbf{e}_1) = \begin{bmatrix} 1 \\ 2 \end{bmatrix}$ and $L(\mathbf{e}_2) = \begin{bmatrix} -3 \\ 4 \end{bmatrix}$. The $A \in \mathbf{R}^2_2$ such that $L(X) = L_A(X) = AX$ for all $X \in \mathbf{R}^2$ would have to satisfy
$A\mathbf{e}_1 = \begin{bmatrix} 1 \\ 2 \end{bmatrix}$ and $A\mathbf{e}_2 = \begin{bmatrix} -3 \\ 4 \end{bmatrix}$, but $A\mathbf{e}_1 = Col_1(A)$ and $A\mathbf{e}_2 = Col_2(A)$, so
$A = \begin{bmatrix} 1 & -3 \\ 2 & 4 \end{bmatrix}$. Then for all $X \in \mathbf{R}^2$ we have

$$\begin{aligned} L(X) = L\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} &= L(x_1\mathbf{e}_1 + x_2\mathbf{e}_2) = x_1 L(\mathbf{e}_1) + x_2 L(\mathbf{e}_2) \\ &= x_1 \begin{bmatrix} 1 \\ 2 \end{bmatrix} + x_2 \begin{bmatrix} -3 \\ 4 \end{bmatrix} = \begin{bmatrix} 1 & -3 \\ 2 & 4 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = AX. \end{aligned}$$
