Math 304-6 Linear Algebra Spring 2020 Practice Exam 2 Feingold

SHOW ALL NECESSARY WORK FOR EACH PROBLEM

(1) (15 Points) $A = \begin{bmatrix} 1 & 2 & 1 & 4 & 1 \\ 1 & 2 & 2 & 5 & 0 \\ 1 & 2 & 3 & 6 & 1 \\ 1 & 2 & 4 & 7 & 0 \end{bmatrix}$ row reduces to $C = \begin{bmatrix} 1 & 2 & 0 & 3 & 0 \\ 0 & 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$.

Let $L_A : \mathbf{R}^5 \to \mathbf{R}^4$ be the linear map $L_A(X) = AX$.

(a) (3 Points) Find a **basis** for $Row(A)$, the row space of A.
(b) (3 Points) Find a **basis** for $Ker(L_A) = Nul(A)$, the kernel of L_A.
(c) (3 Points) Find a **basis** for $Range(L_A) = Col(A)$.
(d) (6 Points) Use your answers to find the **dependence relations** among the columns of A.

(2) (15 Points, 3 Points Each) Answer each question **separately**.

(a) What elementary matrix corresponds to the elementary row operation $3Row_1(A) + Row_2(A) \to Row_2(A)$ done to a $2 \times n$ matrix A in $\mathbf{R}^2_n$?
(b) If $S = \{v_1, v_2, \dots, v_n\}$ is a **basis** of vector space V, and $L : V \to W$ is linear, what can you be sure is true about $L(S) = \{L(v_1), L(v_2), \dots, L(v_n)\}$ in W?
(c) If $S = \{v_1, v_2, \dots, v_m\}$ is **dependent** in V, and linear map $L : V \to W$ is **surjective**, what can you be sure is true about $L(S) = \{L(v_1), \dots, L(v_m)\}$?
(d) Suppose a list of **nonzero** vectors $T = \{w_1, w_2, \dots, w_m\}$ spans W, and no vector w_j in T is a linear combination of the previous vectors, then what can you be sure is true about T?
(e) If A is a square **invertible** $n \times n$ matrix, what can you say about $dim(Col(A))$?

(3) (15 points, 3 points each) Answer each question separately.

(a) If $L : \mathbf{R}^4_4 \to \mathbf{R}^{16}$ is **surjective**, what is the most you can say about L?
(b) If $L : \mathbf{R}^9 \to \mathbf{R}^5$ what are all the possibilities for $\dim(Ker(L))$?
(c) If $L : \mathbf{R}^2_2 \to \mathbf{R}^7$ what are all the possibilities for $dim(Range(L))$?
(d) If $A \in \mathbf{R}^n_n$ and the homogeneous linear system $AX = 0$ has only the trivial solution, then what can you be sure is true about $Rank(A)$?
(e) Let $L : V \to V$ and suppose $v \in V$ satisfies $L(v) = \lambda v$ for $\lambda \in \mathbf{R}$. Compute $(L \circ L)(v)$, where $\circ$ means composition.

(4) (15 Pts)

(a) (7 Pts) $S = \left\{ v_1 = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, v_2 = \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}, v_3 = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}, \right\}$ is a basis of $\mathbf{R}^3$. For $v = \begin{bmatrix} a \\ b \\ c \end{bmatrix}$ write $v = x_1 v_1 + x_2 v_2 + x_3 v_3$ as a linear combination from S by solving for x_1, x_2, x_3.
(b) (8 Pts) $T = \{t^2 + 2t + 3, t^2 + 2, t\}$ is a basis of $\mathbf{P}_2$ (polynomials of degree at most 2). Write $p = 3t^2 + 5t + 7$ as a linear combination from T by solving a linear system.

(5) (15 Points, 3 points each) Answer each question separately.

(a) Determine whether $A = \begin{bmatrix} 1 & 1 & 0 \\ 2 & 0 & 1 \\ 3 & 2 & 0 \end{bmatrix}$ is invertible, and find A^{-1} if it is.

(b) If $A, B, C \in \mathbf{R}_n^n$ are invertible, write a formula for $(ABC)^{-1}$ in terms of A^{-1}, B^{-1} and C^{-1}.

(c) Determine whether or not the set of all 2×2 invertible matrices is a **subspace** of $\mathbf{R}_2^2$. Justify your answer!

(d) Show that $L : \mathbf{R}_2^2 \to \mathbf{R}$ defined by $L\left(\begin{bmatrix} a & b \\ c & d \end{bmatrix}\right) = b - c$ is a **linear** map.

(e) How does the result from problem 5(d) prove that the set of all **symmetric** matrices in $\mathbf{R}_2^2$ is a subspace?

1. (15 Points) (a) (3 Points)

$$A = \begin{bmatrix} 1 & 2 & 1 & 4 & 1 \\ 1 & 2 & 2 & 5 & 0 \\ 1 & 2 & 3 & 6 & 1 \\ 1 & 2 & 4 & 7 & 0 \end{bmatrix} \text{ row reduces to } C = \begin{bmatrix} 1 & 2 & 0 & 3 & 0 \\ 0 & 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix} \text{ so}$$

$\{[1 \quad 2 \quad 0 \quad 3 \quad 0], [0 \quad 0 \quad 1 \quad 1 \quad 0], [0 \quad 0 \quad 0 \quad 0 \quad 1]\}$ is a basis for $Row(A)$.

(b) (3 Points) A basis for $Ker(L_A) = Nul(A)$ is found by row reducing $[A|0_1^4]$ to $[C|0_1^4]$, interpreting the solutions in $\mathbf{R}^5$, and separating the free variables to get two independent vectors which span it:

$$\left\{ \begin{bmatrix} -2r-3s \\ r \\ -s \\ s \\ 0 \end{bmatrix} = r \begin{bmatrix} -2 \\ 1 \\ 0 \\ 0 \\ 0 \end{bmatrix} + s \begin{bmatrix} -3 \\ 0 \\ -1 \\ 1 \\ 0 \end{bmatrix} \in \mathbf{R}^5 \;\middle|\; r, s \in \mathbf{R} \right\} \text{ so } \left\{ \begin{bmatrix} -2 \\ 1 \\ 0 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} -3 \\ 0 \\ -1 \\ 1 \\ 0 \end{bmatrix} \right\}$$

is a basis for $Ker(L_A)$.

(c) (3 Points) A basis for $Col(A)$ consists of the three pivot columns of A, the columns with leading ones in the RREF C, that is, $\{Col_1(A), Col_3(A), Col_5(A)\}$. Other correct answers can be obtained by linear combinations of those three columns, for example,

$$\left\{ \begin{bmatrix} 1 \\ 0 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 0 \\ 1 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 1 \\ 1 \end{bmatrix} \right\}$$

(d) (6 Points) Each basis vector in $Ker(L_A)$ gives a dependence relation among the columns of A. The two dependence relations obtained that way are:

$$-2Col_1(A) + 1Col_2(A) = 0_1^4 \quad \text{and} \quad -3Col_1(A) - 1Col_3(A) + 1Col_4(A) = 0_1^4$$

so

$$Col_2(A) = 2Col_1(A) \quad \text{(obviously!)} \quad \text{and} \quad Col_4(A) = 3Col_1(A) + Col_3(A).$$

---

2. (15 Points, 3 points each)

(a) The elementary matrix that corresponds to the row operation $3Row_1(A)+Row_2(A) \to Row_2(A)$ is $\begin{bmatrix} 1 & 0 \\ 3 & 1 \end{bmatrix}$ obtained by doing the operation to I_2.

(b) You can be sure that $L(S)$ spans $Range(L) = \{\sum_{j=1}^{n} x_j L(v_j) \in W \mid x_j \in \mathbf{R}\}$. $L(S)$ will be dependent when $Ker(L)$ is non-trivial. $L(S)$ is a basis of $Range(L)$ only when L is injective.

(c) You can be sure that $L(S)$ is dependent in W since a linear map L takes a dependent set to a dependent set. The fact that L is given to be surjective tells us nothing about $L(S)$ since we are not told that S spans W.

(d) T has no redundant vectors, so it is independent and is a basis of W since it also spans W.

(e) A invertible means $Rank(A) = dim(Col(A)) = n$.

---

3. (15 Points, 3 points each)

(a) Use $\dim(V) = \dim(Ker(L)) + \dim(Range(L))$. L surjective means $\dim(Range(L)) = 16$ and $\dim(V) = \dim(\mathbf{R}_4^4) = 16$, so $\dim(Ker(L)) = 0$ so L is injective. L is thus bijective, invertible and an isomorphism.

(b) $L : \mathbf{R}^9 \to \mathbf{R}^5$ so $9 = \dim(Ker(L)) + \dim(Range(L))$ and $0 \leq dim(Range(L)) \leq 5$ so $4 \leq dim(Ker(L)) \leq 9$.

(c) $L : \mathbf{R}_2^2 \to \mathbf{R}^7$ so $4 = \dim(\mathbf{R}_2^2) = \dim(Ker(L)) + \dim(Range(L))$ and $0 \leq dim(Ker(L)) \leq 4$, so $0 \leq \dim(Range(L)) \leq 4$.

(d) If $A \in \mathbf{R}_n^n$ and the homogeneous linear system $AX = 0$ has only the trivial solution, then $Rank(A) = n$

(e) Let $L : V \to V$ and suppose $v \in V$ satisfies $L(v) = \lambda v$ for $\lambda \in \mathbf{R}$. Then $(L \circ L)(v) = L(L(v)) = L(\lambda v) = \lambda L(v) = \lambda(\lambda v) = \lambda^2 v$.

---

4. (15 Points) (a) (7 Pts) To solve $x_1v_1 + x_2v_2 + x_3v_3 = v$ row reduce

$$\left[\begin{array}{ccc|c} 1 & 1 & 1 & a \\ 0 & 1 & 1 & b \\ 0 & 0 & 1 & c \\ & S & & v \end{array}\right] \text{ to } \left[\begin{array}{ccc|c} 1 & 0 & 0 & a-b \\ 0 & 1 & 0 & b-c \\ 0 & 0 & 1 & c \\ & I_3 & & X \end{array}\right] \text{ so } v = (a-b)v_1 + (b-c)v_2 + cv_3.$$

(b) (8 Pts) To write p as a linear combination from T, solve the equation $x_1(t^2 + 2t + 3) + x_2(t^2 + 2) + x_3 t = 3t^2 + 5t + 7$.
Compare coefficients of each power of t on both sides to get the linear system $x_1 + x_2 = 3$, $2x_1 + x_3 = 5$, $3x_1 + 2x_2 = 7$.
To solve that, row reduce

$$\left[\begin{array}{ccc|c} 1 & 1 & 0 & 3 \\ 2 & 0 & 1 & 5 \\ 3 & 2 & 0 & 7 \\ & T & & p \end{array}\right] \text{ to } \left[\begin{array}{ccc|c} 1 & 0 & 0 & 1 \\ 0 & 1 & 0 & 2 \\ 0 & 0 & 1 & 3 \\ & I_3 & & X \end{array}\right].$$

Check that $1(t^2 + 2t + 3) + 2(t^2 + 2) + 3t = 3t^2 + 5t + 7$.

---

(5) (15 Points, 3 points each) Answer each question separately.

(a) $A = \begin{bmatrix} 1 & 1 & 0 \\ 2 & 0 & 1 \\ 3 & 2 & 0 \end{bmatrix}$ is invertible and $A^{-1} = \begin{bmatrix} -2 & 0 & 1 \\ 3 & 0 & -1 \\ 4 & 1 & -2 \end{bmatrix}$ because

$$\left[\begin{array}{ccc|ccc} 1 & 1 & 0 & 1 & 0 & 0 \\ 2 & 0 & 1 & 0 & 1 & 0 \\ 3 & 2 & 0 & 0 & 0 & 1 \\ & A & & & I_3 & \end{array}\right] \text{ reduces to } \left[\begin{array}{ccc|ccc} 1 & 0 & 0 & -2 & 0 & 1 \\ 0 & 1 & 0 & 3 & 0 & -1 \\ 0 & 0 & 1 & 4 & 1 & -2 \\ & I_3 & & & A^{-1} & \end{array}\right].$$

(b) If $A, B, C \in \mathbf{R}_n^n$ are invertible, the formula is $(ABC)^{-1} = C^{-1}B^{-1}A^{-1}$.

(c) The set of all 2×2 invertible matrices is **not a subspace** of $\mathbf{R}_2^2$ for many reasons: (1) The zero matrix $\mathbf{0}_2^2$ is not invertible, (2) The sum of two invertible matrices need not be invertible, for example $I_2 - I_2 = \mathbf{0}_2^2$ is not invertible, (3) Not all multiples of an invertible matrix are invertible, for example, $0I_2 = \mathbf{0}_2^2$ is not invertible.

(d) We can show $L : \mathbf{R}_2^2 \to \mathbf{R}$ defined by $L\left(\begin{bmatrix} a & b \\ c & d \end{bmatrix}\right) = b - c$ is a linear map in two steps:

(1) If $A_1 = \begin{bmatrix} a_1 & b_1 \\ c_1 & d_1 \end{bmatrix}$ and $A_2 = \begin{bmatrix} a_2 & b_2 \\ c_2 & d_2 \end{bmatrix}$ then $L(A_1 + A_2) = L\left(\begin{bmatrix} a_1 + a_2 & b_1 + b_2 \\ c_1 + c_2 & d_1 + d_2 \end{bmatrix}\right) =$
$(b_1 + b_2) - (c_1 + c_2) = (b_1 - c_1) + (b_2 - c_2) = L(A_1) + L(A_2)$

(2) For any $r \in \mathbf{R}$ and $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$, we have

$$L(rA) = L\left(\begin{bmatrix} ra & rb \\ rc & rd \end{bmatrix}\right) = rb - rc = r(b - c) = rL(A).$$

(e) The result from problem 5(d) proves that the set of all symmetric matrices in $\mathbf{R}_2^2$ is a subspace because that set is $Ker(L) = \{A \in \mathbf{R}_2^2 \mid b - c = 0\}$ and the kernel of any linear map is a subspace of the domain.
