Math 304-6 Linear Algebra Spring 2020 Practice Exam 3 Feingold

SHOW ALL NECESSARY WORK. Note: A^T means the transpose of A.

(1) (20 Pts) Let $L : \mathbf{R}^2_2 \to \mathbf{R}^2$ be given by

$$L \begin{bmatrix} a & b \\ c & d \end{bmatrix} = \begin{bmatrix} a + 2b + 3c + 4d \\ -a + b + 2c - 3d \end{bmatrix}.$$

Let $S = \{v_1, v_2, v_3, v_4\}$ be the standard basis of $\mathbf{R}^2_2$ and let $T = \{w_1, w_2\}$ be the standard basis of $\mathbf{R}^2$. Let other ordered bases be

$$S' = \left\{ \underbrace{\begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}}_{v'_1}, \underbrace{\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}}_{v'_2}, \underbrace{\begin{bmatrix} 0 & 0 \\ 1 & 1 \end{bmatrix}}_{v'_3}, \underbrace{\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}}_{v'_4} \right\} \quad \text{and} \quad T' = \left\{ w'_1 = \begin{bmatrix} 2 \\ 1 \end{bmatrix}, w'_2 = \begin{bmatrix} 3 \\ 2 \end{bmatrix} \right\}.$$

(a) (4 pts) Find the matrix ${}_T[L]_S$ representing L from S to T.

(b) (4 pts) Find the matrix ${}_{T'}[L]_{S'}$ representing L from S' to T' **without using transition matrices**. (Do it directly.)

(c) (12 pts) Find the transition matrices ${}_S P_{S'}$ and ${}_T Q_{T'}$ and verify that your answers satisfy ${}_{T'}[L]_{S'} = ({}_T Q_{T'})^{-1} \; {}_T[L]_S \; ({}_S P_{S'})$.

(2) (15 Points, 3 points each) Answer each question separately.

(a) If $A \in \mathbf{R}^n_n$ and the homogeneous linear system $AX = 0$ has only the trivial solution, then what is the most you can say about $det(A)$?

(b) Let $L : V \to V$ and suppose $v \in V$ is an eigenvector for L with eigenvalue λ. Show that v is an eigenvector for L^2 with eigenvalue λ^2. Do not assume L is diagonalizable.

(c) Suppose $A \in \mathbf{R}^n_n$ has characteristic polynomial $\det(\lambda I_n - A) = (\lambda - \lambda_1)^{k_1}(\lambda - \lambda_2)^{k_2} \cdots (\lambda - \lambda_r)^{k_r}$ with r distinct real eigenvalues $\lambda_1, \lambda_2, \cdots, \lambda_r$. For each $1 \le i \le r$, what is the most you can say about the relationship between the algebraic multiplicity k_i and the geometric multiplicity $g_i = \dim(A_{\lambda_i})$?

(d) In part (c), if you also know that A is diagonalizable, what else can you say about the relationship between k_i and g_i for each i?

(e) Let $E \in \mathbf{R}^n_n$ be an elementary matrix corresponding to an elementary row operation R, let $A \in \mathbf{R}^n_n$ and let B be the matrix obtained by doing the row operation R to A. What is the relationship between $\det(A)$, $\det(B)$ and $\det(E)$?

(3) (20 Points, 3 points each, (a) worth 8 pts) Answer each question separately.

(a) Find $\det \begin{bmatrix} 2 & 3 & 4 & 1 \\ 1 & 1 & -1 & -1 \\ 3 & 2 & 1 & 1 \\ 4 & 4 & -2 & -3 \end{bmatrix}$.

(b) If $\det(A) = 10$, $\det(B) = 4$ and $\det(C) = 3$, find $\det(A^{-1}B^2C^T)$.

(c) Suppose S is a basis of V, T is a basis of W, $\dim(V) = n$, $\dim(W) = m$. Then for any linear $L : V \to W$ we defined a map ${}_T\mathcal{M}_S : Lin(V, W) \to \mathbf{R}^m_n$ by ${}_T\mathcal{M}_S(L) = {}_T[L]_S$. What property of the map ${}_T\mathcal{M}_S$ implies $\dim(Lin(V, W)) = \dim(\mathbf{R}^m_n)$?

(d) Suppose $A, B \in \mathbf{R}^n_n$ are **similar**, that is, $B = P^{-1}AP$ for some invertible $P \in \mathbf{R}^n_n$. What is the relationship between the characteristic polynomials $Char_A(\lambda) = \det(\lambda I_n - A)$ and $Char_B(\lambda) = \det(\lambda I_n - B)$?

(e) For $A \in \mathbf{R}^n_n$ we know that $Char_A(\lambda)$ is a polynomial in the variable λ of degree n. What is the **constant term** of that polynomial?

(4) (20 Points) Let $A = \begin{bmatrix} 7 & -4 & 2 \\ 4 & -1 & 2 \\ 4 & -4 & 5 \end{bmatrix}$.

(a) (8 Pts) Find the **characteristic polynomial** of A, $\det(\lambda I_3 - A)$, find all **eigenvalues**, λ_i, of A and the corresponding **algebraic multiplicities**, k_i.

(b) (8 Pts) For each eigenvalue, λ_i, of A, find a **basis** for the **eigenspace**, A_{λ_i}, and the **geometric multiplicity** $g_i = \dim(A_{\lambda_i})$.

(c) (4 Pts)Determine whether or not A is **diagonalizable**. If it is, find D and P such that $D = P^{-1}AP$ is diagonal. If not, explain why.

1. (20 Points)

(a) (4 Pts) Find $L(S)$: $L(v_1) = \begin{bmatrix} 1 \\ -1 \end{bmatrix}$, $L(v_2) = \begin{bmatrix} 2 \\ 1 \end{bmatrix}$, $L(v_3) = \begin{bmatrix} 3 \\ 2 \end{bmatrix}$, $L(v_4) = \begin{bmatrix} 4 \\ -3 \end{bmatrix}$.

Then $[T \mid L(S)] = \left[\begin{array}{cc|cccc} 1 & 0 & 1 & 2 & 3 & 4 \\ 0 & 1 & -1 & 1 & 2 & -3 \\ \multicolumn{2}{c|}{T} & \multicolumn{4}{c}{L(S)} \end{array}\right]$ so ${}_T[L]_S = \begin{bmatrix} 1 & 2 & 3 & 4 \\ -1 & 1 & 2 & -3 \end{bmatrix}$.

(b) (4 Pts) Find $L(S')$: $L(v_1') = \begin{bmatrix} 7 \\ -3 \end{bmatrix}$, $L(v_2') = \begin{bmatrix} 5 \\ -4 \end{bmatrix}$, $L(v_3') = \begin{bmatrix} 7 \\ -1 \end{bmatrix}$, $L(v_4') = \begin{bmatrix} 5 \\ 3 \end{bmatrix}$.

Row reduce $\left[\begin{array}{cc|cccc} 2 & 3 & 7 & 5 & 7 & 5 \\ 1 & 2 & -3 & -4 & -1 & 3 \\ \multicolumn{2}{c|}{T'} & \multicolumn{4}{c}{L(S')} \end{array}\right]$ to $\left[\begin{array}{cc|cccc} 1 & 0 & 23 & 22 & 17 & 1 \\ 0 & 1 & -13 & -13 & -9 & 1 \\ \multicolumn{2}{c|}{I_2} & \multicolumn{4}{c}{{}_{T'}[L]_{S'}} \end{array}\right]$

(c) (12 Pts) ${}_SP_{S'} = \begin{bmatrix} 1 & 1 & 0 & 0 \\ 1 & 0 & 0 & 1 \\ 0 & 0 & 1 & 1 \\ 1 & 1 & 1 & 0 \end{bmatrix}$ and ${}_TQ_{T'} = \begin{bmatrix} 2 & 3 \\ 1 & 2 \end{bmatrix}$ since S and T are the standard bases.

To get ${}_{T'}Q_T = ({}_TQ_{T'})^{-1}$, reduce

$$\left[\begin{array}{cc|cc} 2 & 3 & 1 & 0 \\ 1 & 2 & 0 & 1 \\ \multicolumn{2}{c|}{T'} & \multicolumn{2}{c}{T} \end{array}\right] \text{ to } \left[\begin{array}{cc|cc} 1 & 0 & 2 & -3 \\ 0 & 1 & -1 & 2 \\ \multicolumn{2}{c|}{I_2} & \multicolumn{2}{c}{{}_{T'}Q_T} \end{array}\right]$$

$$({}_TQ_{T'})^{-1}{}_T[L]_S({}_SP_{S'}) = \begin{bmatrix} 2 & -3 \\ -1 & 2 \end{bmatrix} \begin{bmatrix} 1 & 2 & 3 & 4 \\ -1 & 1 & 2 & -3 \end{bmatrix} \begin{bmatrix} 1 & 1 & 0 & 0 \\ 1 & 0 & 0 & 1 \\ 0 & 0 & 1 & 1 \\ 1 & 1 & 1 & 0 \end{bmatrix} =$$

$\begin{bmatrix} 5 & 1 & 0 & 17 \\ -3 & 0 & 1 & -10 \end{bmatrix} \begin{bmatrix} 1 & 1 & 0 & 0 \\ 1 & 0 & 0 & 1 \\ 0 & 0 & 1 & 1 \\ 1 & 1 & 1 & 0 \end{bmatrix} = \begin{bmatrix} 23 & 22 & 17 & 1 \\ -13 & -13 & -9 & 1 \end{bmatrix} = {}_{T'}[L]_{S'}$ checks. Also,

$$\begin{bmatrix} 2 & -3 \\ -1 & 2 \end{bmatrix} \begin{bmatrix} 1 & 2 & 3 & 4 \\ -1 & 1 & 2 & -3 \end{bmatrix} \begin{bmatrix} 1 & 1 & 0 & 0 \\ 1 & 0 & 0 & 1 \\ 0 & 0 & 1 & 1 \\ 1 & 1 & 1 & 0 \end{bmatrix} =$$

$$\begin{bmatrix} 2 & -3 \\ -1 & 2 \end{bmatrix} \begin{bmatrix} 7 & 5 & 7 & 5 \\ -3 & -4 & -1 & 3 \end{bmatrix} = \begin{bmatrix} 23 & 22 & 17 & 1 \\ -13 & -13 & -9 & 1 \end{bmatrix}.$$

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(2) (15 Points, 3 points each) Answer each question separately.

(a) If $A \in \mathbf{R}^n_n$ and the homogeneous linear system $AX = 0$ has only the trivial solution, then A has rank n and is invertible so $det(A) \neq 0$.

(b) Let $L : V \to V$ and suppose $v \in V$ is an eigenvector for L with eigenvalue λ. Then $L(v) = \lambda v$ so $L^2(v) = L(L(v)) = L(\lambda v) = \lambda L(v) = \lambda\lambda v = \lambda^2 v$ so v is an eigenvector for L^2 with eigenvalue λ^2.

(c) Suppose $A \in \mathbf{R}^n_n$ has characteristic polynomial $\det(\lambda I_n - A) = (\lambda - \lambda_1)^{k_1}(\lambda - \lambda_2)^{k_2} \cdots (\lambda - \lambda_r)^{k_r}$ with r distinct real eigenvalues $\lambda_1, \lambda_2, \cdots, \lambda_r$. In general, for each $1 \leq i \leq r$, we know that $1 \leq g_i \leq k_i$.

(d) In part (c), if you also know that A is diagonalizable, then we know that $g_i = k_i$ for each i.

(e) Let $E \in \mathbf{R}^n_n$ be an elementary matrix corresponding to an elementary row operation R, let $A \in \mathbf{R}^n_n$ and let B be the matrix obtained by doing the row operation R to A. Then $B = EA$ so $\det(B) = \det(E)\det(A)$.

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(3) (20 Points, 3 points each, (a) worth 8 pts) Answer each question separately.

(a) $$\det\begin{bmatrix} 2 & 3 & 4 & 1 \\ 1 & 1 & -1 & -1 \\ 3 & 2 & 1 & 1 \\ 4 & 4 & -2 & -3 \end{bmatrix} = -\det\begin{bmatrix} 1 & 1 & -1 & -1 \\ 0 & 1 & 6 & 3 \\ 0 & -1 & 4 & 4 \\ 0 & 0 & 2 & 1 \end{bmatrix} = -\det\begin{bmatrix} 1 & 1 & -1 & -1 \\ 0 & 1 & 6 & 3 \\ 0 & 0 & 10 & 7 \\ 0 & 0 & 2 & 1 \end{bmatrix} =$$
$$-\det\begin{bmatrix} 1 & 1 & -1 & -1 \\ 0 & 1 & 6 & 3 \\ 0 & 0 & 0 & 2 \\ 0 & 0 & 2 & 1 \end{bmatrix} = \det\begin{bmatrix} 1 & 1 & -1 & -1 \\ 0 & 1 & 6 & 3 \\ 0 & 0 & 2 & 1 \\ 0 & 0 & 0 & 2 \end{bmatrix} = 4$$

(b) If $\det(A) = 10$, $\det(B) = 4$ and $\det(C) = 3$, then $\det(A^{-1}B^2C^T) = \frac{(4^2)(3)}{10} = \frac{24}{5}$.

(c) Suppose S is a basis of V, T is a basis of W, $\dim(V) = n$, $\dim(W) = m$. Then for any linear $L : V \to W$ we defined a map ${}_T\mathcal{M}_S : Lin(V, W) \to \mathbf{R}^m_n$ by ${}_T\mathcal{M}_S(L) = {}_T[L]_S$. What property of the map ${}_T\mathcal{M}_S$ implies $\dim(Lin(V, W)) = \dim(\mathbf{R}^m_n)$? Answer: The property that ${}_T\mathcal{M}_S$ is an **isomorphism**, that is, a **bijective linear map**.

(d) Suppose $A, B \in \mathbf{R}^n_n$ are **similar**, that is, $B = P^{-1}AP$ for some invertible $P \in \mathbf{R}^n_n$. The relationship between the characteristic polynomials is that they are equal, $Char_A(\lambda) = \det(\lambda I_n - A) = Char_B(\lambda) = \det(\lambda I_n - B)$.

(e) For $A \in \mathbf{R}^n_n$ we know that $Char_A(\lambda)$ is a polynomial in the variable λ of degree n. What is the **constant term** of that polynomial? Answer: The constant term is $Char_A(0) = \det(-A) = (-1)^n \det(A)$.

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(4) (20 Points) Let $A = \begin{bmatrix} 7 & -4 & 2 \\ 4 & -1 & 2 \\ 4 & -4 & 5 \end{bmatrix}$.

(a) (8 Points) The characteristic polynomial is $Char_A(t) = \det(\lambda I_3 - A) = -\det(A - \lambda I_3)$

$$-\det \begin{bmatrix} 7-\lambda & -4 & 2 \\ 4 & -1-\lambda & 2 \\ 4 & -4 & 5-\lambda \end{bmatrix} = -\det \begin{bmatrix} 3-\lambda & 0 & \lambda-3 \\ 0 & 3-\lambda & \lambda-3 \\ 4 & -4 & 5-\lambda \end{bmatrix}$$

$$= -(3-\lambda)^2 \det \begin{bmatrix} 1 & 0 & -1 \\ 0 & 1 & -1 \\ 4 & -4 & 5-\lambda \end{bmatrix} = -(3-\lambda)^2 \det \begin{bmatrix} 1 & 0 & -1 \\ 0 & 1 & -1 \\ 0 & 0 & 5-\lambda \end{bmatrix}$$

$$= (\lambda-3)^2 \ (\lambda-5)^1 = \lambda^3 - 11\lambda^2 + 39\lambda - 45.$$

So the eigenvalues are $\lambda_1 = 3$ and $\lambda_2 = 5$, the roots of $Char_A(\lambda)$, with algebraic multiplicities $k_1 = 2$ and $k_2 = 1$.

(b) (4 Points) For $\lambda_1 = 3$, the eigenspace, A_{λ_1}, is found by row reducing $[A - 3I_3|0]$:

$$\left[\begin{array}{ccc|c} 4 & -4 & 2 & 0 \\ 4 & -4 & 2 & 0 \\ 4 & -4 & 2 & 0 \end{array}\right] \text{ to } \left[\begin{array}{ccc|c} 1 & -1 & \frac{1}{2} & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right] \text{ so } \begin{array}{c} x_1 = r - \frac{1}{2}s \\ x_2 = r \in \mathbf{R} \\ x_3 = s \in \mathbf{R} \end{array}$$

so the $\lambda_1 = 3$ eigenspace

$$A_{\lambda_1} = \left\{ \begin{bmatrix} r - \frac{1}{2}s \\ r \\ s \end{bmatrix} \in \mathbf{R}^3 \;\middle|\; r, s \in \mathbf{R} \right\} \text{ has basis } \left\{ \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} -1 \\ 0 \\ 2 \end{bmatrix} \right\} \text{ and } g_1 = 2 = k_1.$$

(4 Points) For $\lambda_2 = 5$, the eigenspace, A_{λ_2}, is found by row reducing $[A - 5I_3|0]$:

$$\left[\begin{array}{ccc|c} 2 & -4 & 2 & 0 \\ 4 & -6 & 2 & 0 \\ 4 & -4 & 0 & 0 \end{array}\right] \text{ to } \left[\begin{array}{ccc|c} 1 & 0 & -1 & 0 \\ 0 & 1 & -1 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right] \text{ so } \begin{array}{c} x_1 = r \\ x_2 = r \\ x_3 = r \in \mathbf{R} \end{array}$$

so the $\lambda_1 = 5$ eigenspace

$$A_{\lambda_2} = \left\{ \begin{bmatrix} r \\ r \\ r \end{bmatrix} \in \mathbf{R}^3 \;\middle|\; r \in \mathbf{R} \right\} \text{ has basis } \left\{ \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} \right\} \text{ and } g_2 = 1 = k_2.$$

(c) (4 Points) A is diagonalizable since $g_1 + g_2 = 3$ and we found an eigenbasis for $\mathbf{R}^3$, $T = \left\{ \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} -1 \\ 0 \\ 2 \end{bmatrix}, \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} \right\}$. $D = \begin{bmatrix} 3 & 0 & 0 \\ 0 & 3 & 0 \\ 0 & 0 & 5 \end{bmatrix}$ and $P = {}_S P_T = \begin{bmatrix} 1 & -1 & 1 \\ 1 & 0 & 1 \\ 0 & 2 & 1 \end{bmatrix}$ is the transition matrix such that $D = P^{-1}AP$ is diagonal. As a numerical check:

$$PD = \begin{bmatrix} 1 & -1 & 1 \\ 1 & 0 & 1 \\ 0 & 2 & 1 \end{bmatrix} \begin{bmatrix} 3 & 0 & 0 \\ 0 & 3 & 0 \\ 0 & 0 & 5 \end{bmatrix} = \begin{bmatrix} 3 & -3 & 5 \\ 3 & 0 & 5 \\ 0 & 6 & 5 \end{bmatrix} = \begin{bmatrix} 7 & -4 & 2 \\ 4 & -1 & 2 \\ 4 & -4 & 5 \end{bmatrix} \begin{bmatrix} 1 & -1 & 1 \\ 1 & 0 & 1 \\ 0 & 2 & 1 \end{bmatrix} = AP.$$