Course Syllabus for all sections on webpage:
www2.math.binghamton.edu/p/math304/start
Read it for all details about how this course will run. Download main textbook (Heffron) as a free pdf. After class, use Webwork link to see homework system, reset your password. Email me if you are not already in the Webwork system for Math 304-6.
Class attendance is required and vital. Panopto recordings will be available for each lecture, as well as these written class notes. They will be on my separate Math 304-6 webpage.

Grades will be based on timed assessments:
Quizzes, Exams, Final Exam, and homework.
All work submitted for grading <u>must</u> be done by <u>you alone</u> with <u>no outside help</u>. Any cases of cheating will be severely penalized.
The order in which material is presented will be up to me, not strictly following the book, so these notes and my Panopto lecture recordings will be your best guides for learning and study.
The textbook will be good for examples, homework and other viewpoints, as a reference.
Feel free to ask questions at any time.
I encourage you to form study groups, and to help each other outside of class.

Linear algebra is a useful subject with many applications in physics, chemistry, biology, economics and other parts of mathematics. We will be fully occupied this semester just learning its basics, so not many applications will be covered. But you can read about applications in the textbook.

Basic topics we will cover include:

① Linear systems of equations and methods to solve them.

② Matrices and their uses.

③ Vector spaces and their relations to ① and ②.

④ Linear maps (functions) between vector spaces.

A detailed topics list is on my 304-6 webpage.

Concepts and calculations are both important.

We will use **numbers** (real or complex) as well as **variables**, and we will use standard symbols and notations from math. For example, $\mathbb{R} = \{\text{real numbers}\}$ and $\mathbb{C} = \{a+bi \mid a, b \in \mathbb{R}\} = \{\text{complex numbers}\}$ where $i = \sqrt{-1}$ in the context of $\mathbb{C}$. But in the context of matrices, $A = [a_{ij}] \in \mathbb{R}^m_n$ will mean an $m \times n$ array of real numbers with the entry in row i and column j denoted by a_{ij}, so $1 \leq i \leq m$, $1 \leq j \leq n$, for most of the time.

Set theory and logic will be reviewed when needed, as well as concepts from the theory of general functions $f : S \to T$ for sets S and T.

In most examples the numbers (scalars) used will be just integers. ($\mathbb{Z}$ is the set of all integers) Rational numbers (fractions, ratios of integers) may be needed to express solutions to problems.

$\mathbb{Q} = \left\{ \frac{m}{n} \,\middle|\, m, n \in \mathbb{Z},\ n \neq 0 \right\}$ is the set of all rational numbers, but where some ratios are equal to others by usual rules: $\frac{m}{n} = \frac{p}{q}$ iff $mq = np$ in $\mathbb{Z}$.

A set of numbers in which arithmetic can be done $(+, -, \cdot, /)$ with the usual algebra rules is called a <u>field</u>. $\mathbb{Q}$, $\mathbb{R}$ and $\mathbb{C}$ are the most commonly used examples of fields. $\mathbb{F}$ denotes a general field and some results hold for any $\mathbb{F}$.

The two most basic and important objects we will study are <u>linear systems</u> and <u>matrices</u>.

Let $\mathbb{F}$ be a field, which you can think of as $\mathbb{Q}$ or $\mathbb{R}$ or $\mathbb{C}$. Fix a choice of $1 \leq m, n \in \mathbb{Z}$.

<u>Def</u>. A linear system of m equations in n variables is a list of equations:

$$\begin{aligned} a_{11}x_1 + a_{12}x_2 + \cdots + a_{1n}x_n &= b_1 \\ a_{21}x_1 + a_{22}x_2 + \cdots + a_{2n}x_n &= b_2 \\ &\vdots \\ a_{m1}x_1 + a_{m2}x_2 + \cdots + a_{mn}x_n &= b_m \end{aligned}$$

where $a_{ij} \in \mathbb{F}$, $b_i \in \mathbb{F}$ for $1 \leq i \leq m$, $1 \leq j \leq n$, and the x_j are variables whose values in $\mathbb{F}$ we seek such that all m equations are simultaneously true.

Def. For $1 \le m, n \in \mathbb{Z}$, let

$\mathbb{F}_n^m = \{A = [a_{ij}] \mid 1 \le i \le m,\ 1 \le j \le n,\ a_{ij} \in \mathbb{F}\}$ be the set of all $m \times n$ matrices "over" $\mathbb{F}$, so

$$A = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & & \vdots \\ a_{m1} & a_{m2} & & a_{mn} \end{bmatrix}$$

is a rectangular array of numbers from $\mathbb{F}$, where a_{ij} is the "entry" in row i and column j of matrix A.

Notations: We write $\mathbb{F}_1^m = \mathbb{F}^m$ and $\mathbb{F}_n^1 = \mathbb{F}_n$

so $\mathbb{F}^m = \left\{ \begin{bmatrix} a_1 \\ a_2 \\ \vdots \\ a_m \end{bmatrix} \middle| \ a_i \in \mathbb{F}, 1 \le i \le m \right\}$ "column vectors" and

$\mathbb{F}_n = \{ [a_1\ a_2 \dots a_n] \mid a_j \in \mathbb{F},\ 1 \le j \le n \}$ "row vectors"

since double subscripts are not needed for these.

Given all the "coefficients" a_{ij} and the "constant targets" b_i of a linear system (lin. sys.) the values $x_j \in \mathbb{F}$ for which all equations are true are called the "solutions". If there are no solutions, we say the system is <u>inconsistent</u>. If there is at least one solution, the system is called <u>consistent</u>, and we write the <u>solution set</u> as $\left\{ X = \begin{bmatrix} x_1 \\ \vdots \\ x_n \end{bmatrix} \in \mathbb{F}^n \right\}$ as a subset of $\mathbb{F}^n$.

<u>Def.</u> Let $A = [a_{ij}] \in \mathbb{F}_n^m$ be called the <u>coefficient matrix</u> of the lin. sys. and let $B = \begin{bmatrix} b_1 \\ \vdots \\ b_m \end{bmatrix} \in \mathbb{F}^m$ be the <u>constant matrix</u> (column) of the lin. sys.

We can do some arithmetic with matrices.

<u>Def</u>. For $A=[a_{ij}]$, $B=[b_{ij}]\in\mathbb{F}^m_n$ define $A+B=C=[c_{ij}]\in\mathbb{F}^m_n$ by $c_{ij}=a_{ij}+b_{ij}$ and for $\alpha\in\mathbb{F}$ define $\alpha\cdot A=[\alpha\cdot a_{ij}]\in\mathbb{F}^m_n$.

Let $O^m_n=[0]\in\mathbb{F}^m_n$ be the $m\times n$ "zero" matrix with all entries $0\in\mathbb{F}$.

These operations with matrices are called <u>matrix addition</u> and <u>scalar multiplication</u>.

They obey basic laws of algebra which we will discuss later.

<u>Def</u>. For $A = [a_{ij}] \in \mathbb{F}^m_n$ let

$\mathrm{Row}_i(A) = [a_{i1}\ a_{i2} \cdots a_{in}] \in \mathbb{F}_n$ for $1 \le i \le m$,

$\mathrm{Col}_j(A) = \begin{bmatrix} a_{1j} \\ a_{2j} \\ \vdots \\ a_{mj} \end{bmatrix} \in \mathbb{F}^m$ for $1 \le j \le n$.

We can use these matrix concepts to write a lin. sys. in a more compact form.

<u>Define</u> $AX = \begin{bmatrix} \sum_{j=1}^{n} a_{1j} x_j \\ \sum_{j=1}^{n} a_{2j} x_j \\ \vdots \\ \sum_{j=1}^{n} a_{mj} x_j \end{bmatrix} = \sum_{j=1}^{n} x_j \mathrm{Col}_j(A) \in \mathbb{F}^m$

as the column matrix made from the left sides of the m equations.

is a <u>linear</u> <u>combination</u> of the columns of A.

Then the lin. sys. is a single matrix equation

$AX = B$ and the solution set is $\{X \in F^n \mid AX = B\}$.

<u>Def.</u> Say that lin. sys. $AX = B$ is <u>homogeneous</u> when $B = 0_1^m$, that is, all $b_i = 0$ for $1 \le i \le m$. If any $b_i \ne 0$ we call the system <u>inhomogeneous</u>.

<u>Note</u>: $AX = 0_1^m$ is <u>always consistent</u> since it has the "trivial" solution $X = 0_1^n$.

So the question for $AX = 0_1^m$ is whether it has any "non-trivial" solutions $X \ne 0_1^n$.

Several important properties of the solution set $W = \{X \in F^n \mid AX = 0_1^m\}$ of the homog. lin. sys. $AX = 0$ follow from the next theorem.

<u>Th</u>: For any $A=[a_{ij}]\in\mathbb{F}^m_n$, $X, Y\in\mathbb{F}^n_1$, $\alpha\in\mathbb{F}$,
we have:

① $A(X+Y)=(AX)+(AY)$ ③ $A0^n_1=0^m_1$

② $A(\alpha\cdot X)=\alpha\cdot(AX)$

<u>Pf</u>: ① $A(X+Y)=\begin{bmatrix}\sum_{j=1}^{n} a_{1j}(x_j+y_j)\\ \vdots\\ \sum_{j=1}^{n} a_{mj}(x_j+y_j)\end{bmatrix}=\begin{bmatrix}\sum_{j=1}^{n}(a_{1j}x_j+a_{1j}y_j)\\ \vdots\\ \sum_{j=1}^{n}(a_{mj}x_j+a_{mj}y_j)\end{bmatrix}$

$=\begin{bmatrix}\sum_{j=1}^{n} a_{1j}x_j\\ \vdots\\ \sum_{j=1}^{n} a_{mj}x_j\end{bmatrix}+\begin{bmatrix}\sum_{j=1}^{n} a_{1j}y_j\\ \vdots\\ \sum_{j=1}^{n} a_{mj}y_j\end{bmatrix}=AX+AY.$

What algebra laws were used in each step?

② $A(\alpha \cdot X) = \begin{bmatrix} \sum a_{1j}(\alpha x_j) \\ \vdots \\ \sum a_{mj}(\alpha x_j) \end{bmatrix} = \begin{bmatrix} \sum (a_{1j}\alpha) x_j \\ \vdots \\ \sum (a_{mj}\alpha) x_j \end{bmatrix}$

$= \begin{bmatrix} \sum (\alpha a_{1j}) x_j \\ \vdots \\ \sum (\alpha a_{mj}) x_j \end{bmatrix} = \begin{bmatrix} \sum \alpha (a_{1j} x_j) \\ \vdots \\ \sum \alpha (a_{mj} x_j) \end{bmatrix} = \begin{bmatrix} \alpha \sum a_{1j} x_j \\ \vdots \\ \alpha \sum a_{mj} x_j \end{bmatrix}$

$= \alpha \begin{bmatrix} \sum a_{1j} x_j \\ \vdots \\ \sum a_{mj} x_j \end{bmatrix} = \alpha(AX).$

③ $A O_1^n = \begin{bmatrix} \sum a_{1j} \cdot 0 \\ \vdots \\ \sum a_{mj} \cdot 0 \end{bmatrix} = \begin{bmatrix} 0 \\ \vdots \\ 0 \end{bmatrix} = O_1^m.$ $\square$

Th. Let $A = [a_{ij}] \in F^m_n$. Then $W = \{X \in F^n \mid AX = 0^m_1\}$ satisfies these properties:

① If $X, Y \in W$ then $X+Y \in W$ (W is closed under +)

② If $X \in W$, $\alpha \in F$, then $\alpha \cdot X \in W$ (W is closed under $\cdot$)

③ $0^n_1 \in W$.

Pf. ① If $X, Y \in W$ then $AX = 0^m_1$ and $AY = 0^m_1$ so $A(X+Y) = AX + AY = 0^m_1 + 0^m_1 = 0^m_1$ so $X+Y \in W$.

② If $X \in W$ and $\alpha \in F$ then $AX = 0^m_1$ so $A(\alpha \cdot X) = \alpha \cdot (AX) = \alpha \cdot 0^m_1 = 0^m_1$ so $\alpha \cdot X \in W$.

③ $0^n_1 \in W$ since $A0^n_1 = 0^m_1$. □

Goal: Find an efficient algorithm to solve linear systems.

Solution: Row reduction to Reduced Row Echelon Form.

Ex. Let $A=\begin{bmatrix}1&2&3\\2&3&4\end{bmatrix}$. Solve $AX=0_1^2$, that is, solve

$$\left.\begin{aligned}1x_1+2x_2+3x_3&=0\\2x_1+3x_2+4x_3&=0\end{aligned}\right\}.$$

Option 1 (high school method): Equation manipulations.

Option 2 (Math 304 method): Matrix row operations and interpretation. Encode the lin. sys. as $[A|0_1^2]$

$$\left[\begin{array}{ccc|c}1&2&3&0\\2&3&4&0\end{array}\right]$$

so each row corresponds to an equation. Legal equation manipulations correspond to legal "elementary row operations" (row ops.)

$$\left[\begin{array}{ccc|c}1&2&3&0\\2&3&4&0\end{array}\right]\xrightarrow{-2R_1+R_2\to R_2}\left[\begin{array}{ccc|c}1&2&3&0\\0&-1&-2&0\end{array}\right]\xrightarrow{2R_2+R_1\to R_1}$$

$-2\ \ -4\ \ -6\ \ 0$

$0\ \ -2\ \ -4\ \ 0$

$$\left[\begin{array}{ccc|c}1&0&-1&0\\0&-1&-2&0\end{array}\right]\xrightarrow{-R_2\to R_2}\left[\begin{array}{ccc|c}1&0&-1&0\\0&1&2&0\end{array}\right]$$

Interpretation

$x_1=x_3=r$

$x_2=-2x_3=-2r$

$x_3=r\in F$ free var.

So $W = \{X \in F^3 \mid AX = 0^2_1\} = \left\{X = \begin{bmatrix} r \\ -2r \\ r \end{bmatrix} \in F^3 \;\middle|\; r \in F\right\}$

$= \left\{X = r\begin{bmatrix} 1 \\ -2 \\ 1 \end{bmatrix} \in F^3 \;\middle|\; r \in F\right\}$ describes the solution set in terms of one "parameter" $r \in F$.

Since the entries of A were all integers, F could have been $\mathbb{Q}$ or $\mathbb{R}$ or $\mathbb{C}$. So there were infinitely many solutions and they formed a <u>line</u> in space F^3.

<u>Question:</u> If the lin. sys. had been $AX = B$ for an arbitrary $B \in F^2$, what would be the solution?