Questions for discussion:

If $A \in F_4^2$ is non-zero, what are possible values of $\text{rank}(A)$? $\text{rank}(A)$ is 1 or 2.

How many free variables could there be in the solution set for $AX = 0_1^2$? $n - r = 4 - r$ is 2 or 3.

If $K, L : \mathbb{R}^2 \to \mathbb{R}^2$ are given by formulas $K\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} x - y \\ 2x + y \end{bmatrix}$ and $L\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 3x + y \\ -x + y \end{bmatrix}$ find the formulas for $K \circ L$ and $L \circ K$.

$$(K \circ L)\begin{bmatrix} x \\ y \end{bmatrix} = K\left(L\begin{bmatrix} x \\ y \end{bmatrix}\right) = K\begin{bmatrix} 3x+y \\ -x+y \end{bmatrix} = \begin{bmatrix} (3x+y) - (-x+y) \\ 2(3x+y) + (-x+y) \end{bmatrix}$$

$$= \begin{bmatrix} 4x \\ 5x+3y \end{bmatrix} = \underbrace{\begin{bmatrix} 4 & 0 \\ 5 & 3 \end{bmatrix}}_{C_{2\times 2}} \begin{bmatrix} x \\ y \end{bmatrix} = L_C(\mathbb{X}) = (K \circ L)(\mathbb{X})$$

$$K\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} x-y \\ 2x+y \end{bmatrix} = \underbrace{\begin{bmatrix} 1 & -1 \\ 2 & 1 \end{bmatrix}}_{A} \underbrace{\begin{bmatrix} x \\ y \end{bmatrix}}_{\mathbb{X}} = L_A(\mathbb{X}) = K(\mathbb{X})$$

$$L\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 3x+y \\ -x+y \end{bmatrix} = \underbrace{\begin{bmatrix} 3 & 1 \\ -1 & 1 \end{bmatrix}}_{B} \underbrace{\begin{bmatrix} x \\ y \end{bmatrix}}_{X} = L_B(\mathbb{X}) = L(\mathbb{X})$$

Recall: $L_A \circ L_B = L_{AB}$ so $K \circ L = L_A \circ L_B = L_C$

says $C = AB$. $\underbrace{\begin{bmatrix} 1 & -1 \\ 2 & 1 \end{bmatrix}}_{A} \underbrace{\begin{bmatrix} 3 & 1 \\ -1 & 1 \end{bmatrix}}_{B} = \underbrace{\begin{bmatrix} 4 & 0 \\ 5 & 3 \end{bmatrix}}_{C}$

$$(L \circ K)\begin{bmatrix} x \\ y \end{bmatrix} = L\left(K\begin{bmatrix} x \\ y \end{bmatrix}\right) = L\begin{bmatrix} x-y \\ 2x+y \end{bmatrix} = \begin{bmatrix} 3(x-y)+(2x+y) \\ -(x-y)+(2x+y) \end{bmatrix}$$

$$= \begin{bmatrix} 5x-2y \\ x+2y \end{bmatrix} = \underbrace{\begin{bmatrix} 5 & -2 \\ 1 & 2 \end{bmatrix}}_{D}\begin{bmatrix} x \\ y \end{bmatrix} = L_D(X) = (L \circ K)(X)$$

$L \circ K = L_B \circ L_A = L_{BA} = L_D$ since

$$BA = \begin{bmatrix} 3 & 1 \\ -1 & 1 \end{bmatrix}\begin{bmatrix} 1 & -1 \\ 2 & 1 \end{bmatrix} = \begin{bmatrix} 5 & -2 \\ 1 & 2 \end{bmatrix} = D$$

Def. For V and W vector spaces over F, say that a function $L: V \to W$ is *linear* when

① $L(v_1 + v_2) = L(v_1) + L(v_2)$, $\forall v_1, v_2 \in V$, and

② $L(\alpha \cdot v) = \alpha \cdot L(v)$, $\forall v \in V$, $\forall \alpha \in F$.

Ex. Let $L: F_2^2 \to F^3$ be the map defined by formula $L\begin{bmatrix} a & b \\ c & d \end{bmatrix} = \begin{bmatrix} a+b \\ b+c \\ c+d \end{bmatrix}$. Check that L is linear.

Pf. Let $v_1 = \begin{bmatrix} a_1 & b_1 \\ c_1 & d_1 \end{bmatrix}$, $v_2 = \begin{bmatrix} a_2 & b_2 \\ c_2 & d_2 \end{bmatrix} \in V = F_2^2$ so

$$v_1 + v_2 = \begin{bmatrix} (a_1+a_2) & (b_1+b_2) \\ (c_1+c_2) & (d_1+d_2) \end{bmatrix} \text{ and } L(v_1+v_2) = \begin{bmatrix} (a_1+a_2)+(b_1+b_2) \\ (b_1+b_2)+(c_1+c_2) \\ (c_1+c_2)+(d_1+d_2) \end{bmatrix}$$

$$= \begin{bmatrix} (a_1+b_1)+(a_2+b_2) \\ (b_1+c_1)+(b_2+c_2) \\ (c_1+d_1)+(c_2+d_2) \end{bmatrix} = \begin{bmatrix} a_1+b_1 \\ b_1+c_1 \\ c_1+d_1 \end{bmatrix} + \begin{bmatrix} a_2+b_2 \\ b_2+c_2 \\ c_2+d_2 \end{bmatrix} = L(v_1) + L(v_2).$$

Next: Let $v = \begin{bmatrix} a & b \\ c & d \end{bmatrix} \in F_2^2$, $\alpha \in F$, so

$$\alpha \cdot v = \begin{bmatrix} (\alpha a) & (\alpha b) \\ (\alpha c) & (\alpha d) \end{bmatrix} \text{ and } L(\alpha \cdot v) = \begin{bmatrix} \alpha a + \alpha b \\ \alpha b + \alpha c \\ \alpha c + \alpha d \end{bmatrix} = \begin{bmatrix} \alpha(a+b) \\ \alpha(b+c) \\ \alpha(c+d) \end{bmatrix}$$

$$= \alpha \cdot \begin{bmatrix} a+b \\ b+c \\ c+d \end{bmatrix} = \alpha \cdot L(v). \quad \square$$

Ex. Let $L : \mathcal{P}_2[t] \to F^3$ be defined by $L(a_0 + a_1 t + a_2 t^2) = \begin{bmatrix} a_0 \\ a_1 \\ a_2 \end{bmatrix}$. Exercise: Show this map is linear.

Question: Would the map $L(a_0 + a_1 t + a_2 t^2) = \begin{bmatrix} 1 \\ a_1 \\ a_2 \end{bmatrix}$ be linear? What about $L(a_0 + a_1 t + a_2 t^2) = \begin{bmatrix} 0 \\ a_1 \\ a_2 \end{bmatrix}$?

Do these as exercises yourself.

<u>Th</u>. If $L: V \to W$ is linear then we have:

① $L(\theta_V) = \theta_W$, ② $\forall v \in V$, $L(-v) = -L(v)$,

③ $\forall v_1, v_2 \in V$, $\forall a_1, a_2 \in F$, $L(a_1 v_1 + a_2 v_2) = a_1 L(v_1) + a_2 L(v_2)$,

④ $\forall v_1, \dots, v_m \in V$, $\forall a_1, \dots, a_m \in F$, $L\left(\sum_{i=1}^{m} a_i v_i\right) = \sum_{i=1}^{m} a_i L(v_i)$.

<u>Pf</u>. ① Let $L(\theta_V) = w \in W$. Then $L(\theta_V) = L(\theta_V + \theta_V) = L(\theta_V) + L(\theta_V)$ says $w = w + w$. As we have seen in the proof on p. 95, this implies $w = \theta_W$.

③ We have $L(a_1 v_1 + a_2 v_2) = L(a_1 v_1) + L(a_2 v_2) = a_1 L(v_1) + a_2 L(v_2)$.

② $L(-v) = L((-1) \cdot v) = (-1) \cdot L(v) = -L(v)$ by Lemma (c), p. 95.

④ A rigorous proof by induction on m works using ③ for case of $m = 2$ as a base case. If you are not so familiar with proof by induction then use "$\cdots$":

$L(a_1 v_1 + \cdots + a_m v_m) = L(a_1 v_1) + \cdots + L(a_m v_m) = a_1 L(v_1) + \cdots + a_m L(v_m)$.

$\square$

Def. For any lin. map $L: V \to W$ define

$\mathrm{Ker}(L) = \{v \in V \mid L(v) = \theta_W\}$ and

$\mathrm{Range}(L) = \{L(v) \in W \mid v \in V\} = \{w \in W \mid \exists v \in V, L(v) = w\}$.

Th: For any lin. map $L: V \to W$ we have

$\mathrm{Ker}(L) \leq V$ and $\mathrm{Range}(L) \leq W$.

Pf. This is analogous to the proof on p. 99 for L_A.

Let $K = \mathrm{Ker}(L)$. If $v_1, v_2 \in K$ then $L(v_1) = \theta_W = L(v_2)$ so $L(v_1 + v_2) = L(v_1) + L(v_2) = \theta_W + \theta_W = \theta_W$ so $v_1 + v_2 \in K$.

If $v \in K$ and $\alpha \in F$, $L(v) = \theta_W$ so $L(\alpha \cdot v) = \alpha \cdot L(v) = \alpha \cdot \theta_W = \theta_W$ (by Lemma (a), p. 95 applied to W) so $\alpha \cdot v \in K$. $L(\theta_V) = \theta_W$ means $\theta_V \in K$. So $K \leq V$.

Let $R = \mathrm{Range}(L)$. If $w_1, w_2 \in R$ then $\exists v_1, v_2 \in V$, $w_1 = L(v_1)$ and $w_2 = L(v_2)$ so $w_1 + w_2 = L(v_1) + L(v_2) = L(v_1 + v_2)$ means $w_1 + w_2 \in R$.

If $w \in R$ and $\alpha \in F$ then $\exists v \in V$, $L(v)=w$

so $L(\alpha \cdot v) = \alpha \cdot L(v) = \alpha \cdot w$ means $\alpha \cdot w \in R$.

$L(\theta_V) = \theta_W$ means $\theta_W \in R$, so $R \leq W$. □

<u>Note</u>: This Theorem generalizes the previous one on p. 99.

<u>EX</u>: For $L : F_2^2 \to F^3$ given on p. 101, find

$\ker(L) = \left\{ \begin{bmatrix} a & b \\ c & d \end{bmatrix} \in F_2^2 \,\middle|\, \begin{bmatrix} a+b \\ b+c \\ c+d \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \right\}$. Just solve the lin. sys.

$$\left[\begin{array}{cccc|c} 1 & 1 & 0 & 0 & 0 \\ 0 & 1 & 1 & 0 & 0 \\ 0 & 0 & 1 & 1 & 0 \end{array}\right] \xrightarrow{\text{r.r.}} \left[\begin{array}{cccc|c} 1 & 0 & -1 & 0 & 0 \\ 0 & 1 & 0 & -1 & 0 \\ 0 & 0 & 1 & 1 & 0 \end{array}\right] \to \left[\begin{array}{cccc|c} 1 & 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & -1 & 0 \\ 0 & 0 & 1 & 1 & 0 \end{array}\right]$$

$0 \ -1 \ -1 \ 0 \ 0$

$0 \ 0 \ -1 \ -1 \ 0$

$a = -d$
$b = d$
$c = -d$
$d \in F$ is free.

$$\ker(L) = \left\{ \begin{bmatrix} -d & d \\ -d & d \end{bmatrix} \in F_2^2 \,\middle|\, d \in F \right\} \leq F_2^2$$

$$= \left\{ d \begin{bmatrix} -1 & 1 \\ -1 & 1 \end{bmatrix} \in F_2^2 \,\middle|\, d \in F \right\}$$

Find $\text{Range}(L) = \left\{ \begin{bmatrix} a+b \\ b+c \\ c+d \end{bmatrix} \in F^3 \;\middle|\; a,b,c,d \in F \right\}$

$= \left\{ a\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} + b\begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix} + c\begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix} + d\begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} \in F^3 \;\middle|\; a,b,c,d \in F \right\}$

How much of F^3 is Range (L)? All or less?

Find which $\begin{bmatrix} x \\ y \\ z \end{bmatrix} \in F^3$ are in Range(L). When is

$\underset{A}{\left[\begin{array}{cccc|c} 1 & 1 & 0 & 0 & x \\ 0 & 1 & 1 & 0 & y \\ 0 & 0 & 1 & 1 & z \end{array}\right]}$ consistent? Rank$(A) = 3$ shows it is consistent for all $\begin{bmatrix} x \\ y \\ z \end{bmatrix} \in F^3$

so Range $(L) = F^3$.

Notice that the description above of Range(L) is the set of all linear combinations of the four vectors in $\left\{ \underset{w_1}{\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}}, \underset{w_2}{\begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}}, \underset{w_3}{\begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix}}, \underset{w_4}{\begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}} \right\}$, the columns of matrix A.