Where did A come from? Let

$$S = \left\{ v_1 = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}, v_2 = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}, v_3 = \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix}, v_4 = \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix} \right\} \subseteq F_2^2$$

so $v = \begin{bmatrix} a & b \\ c & d \end{bmatrix} = av_1 + bv_2 + cv_3 + dv_4$ and

$$L(v) = \begin{bmatrix} a+b \\ b+c \\ c+d \end{bmatrix} = L(av_1 + bv_2 + cv_3 + dv_4)$$
$$= aL(v_1) + bL(v_2) + cL(v_3) + dL(v_4)$$
$$= a\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} + b\begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix} + c\begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix} + d\begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$$

shows $\mathrm{Col}_j(A) = L(v_j)$ for $1 \le j \le 4$.

Also note that $v = \begin{bmatrix} -1 & 1 \\ -1 & 1 \end{bmatrix} \in \ker(L)$ means that

$$L(v) = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} = -1\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} + 1\begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix} - 1\begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix} + 1\begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$$ so we get the

<u>relation</u> $\theta_{F^3} = -w_1 + w_2 - w_3 + w_4$.

That "dependence relation" has several interpretations, one of which is that

$w_4 = w_1 - w_2 + w_3$

so any linear combination of all four vectors is a lin. comb. of just w_1, w_2, w_3 as follows:

$$\sum_{j=1}^{4} a_j w_j = \sum_{j=1}^{3} a_j w_j + a_4(w_1 - w_2 + w_3)$$

$$= (a_1+a_4)w_1 + (a_2-a_4)w_2 + (a_3+a_4)w_3$$

We say that w_4 is "redundant" in this case, so $\text{Range}(L) = \left\{ \sum_{j=1}^{3} a_j w_j \,\middle|\, a_j \in F \right\}$.

We will now make important definitions that capture these ideas in any vector space.

Def. For V a vector space over field F and any subset $S \subseteq V$, a linear combination from S is an element $\sum_{i=1}^{m} x_i s_i \in V$ for some $x_i \in F$, $s_i \in S$, m finite. The set of all such elements is called the span of S, denoted by $\langle S \rangle$. If $S = \emptyset$ is the empty set then we define $\langle S \rangle = \{\theta_V\}$.

Th: For any subset $S \subseteq V$ we have $\langle S \rangle \leq V$.

Pf. If $S = \emptyset$ then $\langle \emptyset \rangle = \{\theta_V\}$ is a subspace of V since it satisfies the conditions of the subspace Theorem: $\theta_V + \theta_V = \theta_V \in \langle S \rangle$, $\alpha \cdot \theta_V = \theta_V \in \langle S \rangle$, $\forall \alpha \in F$, and $\theta_V \in \langle S \rangle$. Let $S \neq \emptyset$.

For $w_1, w_2 \in \langle S \rangle$, write
$w_1 = \sum_{i=1}^{m} x_i s_i$ and $w_2 = \sum_{j=1}^{n} y_j s_j'$ for some
$x_i, y_j \in \mathbb{F}$, $s_i, s_j' \in S$. Then

$$w_1 + w_2 = \sum_{i=1}^{m} x_i s_i + \sum_{j=1}^{n} y_j s_j' \in \langle S \rangle$$ is a
(finite) lin. comb. from S, so $\langle S \rangle$ is closed under $+$.

$\forall \alpha \in \mathbb{F}$, $w_1 \in \langle S \rangle$ as above, we have

$$\alpha \cdot w_1 = \alpha \cdot \sum_{i=1}^{m} x_i \cdot s_i = \sum_{i=1}^{m} \alpha \cdot (x_i \cdot s_i) = \sum_{i=1}^{m} (\alpha \cdot x_i) \cdot s_i$$
$\in \langle S \rangle$ so $\langle S \rangle$ is closed under $\cdot$.

$S \subseteq \langle S \rangle$ so for any $s_1 \in S$, $\theta = 0 \cdot s_1 \in \langle S \rangle$.

By last Theorem, $\langle S \rangle \leq V$. □

Basic Concepts for Vector spaces:

Def. Let $S \subseteq V$ be any subset of v.s. V over field F. Say S is **independent** when $\sum_{i=1}^{m} x_i v_i = \theta$ for $v_1, \dots, v_m \in S$, $x_1, \dots, x_m \in F$ implies $x_i = 0$ for $1 \le i \le m$. Otherwise, say S is **dependent**, and call the above equation (with at least one non-zero coeff.) a **dependence relation on S**.

The empty set is considered to be **indep.**

If S is an infinite set, indep. means any choice of finitely many $v_i \in S$ satisfies the condition above. For S finite, write $S = \{v_1, \dots, v_m\}$.

<u>Ex.</u> $V = \mathbb{R}^2$, $S = \left\{ \begin{bmatrix} 1 \\ 0 \end{bmatrix} \right\}$ is <u>indep.</u> since
$x_1 \cdot \begin{bmatrix} 1 \\ 0 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \Rightarrow x_1 = 0$. But
$T = \left\{ \begin{bmatrix} 0 \\ 0 \end{bmatrix} \right\}$ is <u>dep.</u> since $1 \cdot \begin{bmatrix} 0 \\ 0 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$.

<u>Th.</u> In any v.s. V, $S = \{v\}$ where $v \neq \theta$, is <u>indep</u> but $T = \{\theta_V\}$ is <u>dep.</u> So $\{v\}$ is indep. iff $v \neq \theta_V$.

<u>Pf.</u> Recall that $\alpha \cdot v = \theta_V$ implies $\alpha = 0 \in F$ or $v = \theta_V \in V$. So given $\alpha \cdot v = \theta_V$ we get: $v \neq \theta_V \Rightarrow \alpha = 0$ and

$$\alpha \neq 0 \Rightarrow v = \theta_V. \qquad \square$$

Ex: $V=\mathbb{R}^2$, $S=\left\{\begin{bmatrix}1\\0\end{bmatrix},\begin{bmatrix}0\\1\end{bmatrix}\right\}$ is indep. since

$$x_1\begin{bmatrix}1\\0\end{bmatrix}+x_2\begin{bmatrix}0\\1\end{bmatrix}=\begin{bmatrix}0\\0\end{bmatrix} \Rightarrow \begin{bmatrix}x_1\\x_2\end{bmatrix}=\begin{bmatrix}0\\0\end{bmatrix} \Rightarrow x_1=x_2=0.$$

But $T=\left\{\begin{bmatrix}1\\0\end{bmatrix},\begin{bmatrix}0\\1\end{bmatrix},\begin{bmatrix}2\\3\end{bmatrix}\right\}$ is **dep** since

$$-2\begin{bmatrix}1\\0\end{bmatrix}-3\begin{bmatrix}0\\1\end{bmatrix}+1\begin{bmatrix}2\\3\end{bmatrix}=\begin{bmatrix}0\\0\end{bmatrix},$$ that is,

$$x_1\begin{bmatrix}1\\0\end{bmatrix}+x_2\begin{bmatrix}0\\1\end{bmatrix}+x_3\begin{bmatrix}2\\3\end{bmatrix}=\begin{bmatrix}0\\0\end{bmatrix}$$ has non-trivial solution(s).

Corresp. matrix calculation: Does lin sys.

$$\underbrace{\left[\begin{array}{ccc|c}1&0&2&0\\0&1&3&0\end{array}\right]}_{A}$$ have any non-triv. solutions?

Yes, since $\text{rank}(A)=2<n=3$

so get $n-r=3-2=1$ free var.

<u>Th.</u> In $V = F^m$ let $S = \{v_1, \dots, v_n\} \subseteq V$ with $n > m$. Then S is <u>dep.</u>

<u>pf.</u> Look at the lin. sys. corresponding to the vector equation: $\sum_{j=1}^{n} x_j \cdot v_j = 0_1^m$.

Write $v_j = \begin{bmatrix} a_{1j} \\ \vdots \\ a_{mj} \end{bmatrix} \in F^m$, so the lin. sys. is $\underset{(m\times n)(n\times 1)}{AX} = 0_1^m$, $A = [a_{ij}] \in F_n^m$ with $X \in F^n$. Does $[A \mid 0_1^m]$ have any free variables in its solution set?

For $n > m$, $r = \text{rank}(A) \le \text{Min}(m,n) = m < n$ so must have $\underline{n - r > 0 \text{ free variables}}$, so S is dep. $\square$

$\boxed{\text{Note: Each free variable gives a dep. relation on } S. \text{ Examples later.}}$

Cor. In F^m, the maximum size of an indep. set is m.

EX: The set $S = \{e_1, \dots, e_m\}$ of "standard basis" vectors in F^m is indep. since
$$\sum_{i=1}^{m} x_i e_i = \begin{bmatrix} x_1 \\ \vdots \\ x_m \end{bmatrix} = \begin{bmatrix} 0 \\ \vdots \\ 0 \end{bmatrix}$$ iff each $x_i = 0$.

EX: In F_n^m, the set $S = \{E_{ij} \mid 1 \le i \le m, 1 \le j \le n\}$ is indep since $\sum_{i=1}^{m} \sum_{j=1}^{n} a_{ij} E_{ij} = A = [a_{ij}] = O_n^m$ iff each $a_{ij} = 0$.
This set is called the "standard basis" of F_n^m.

Th: Let $T \subseteq S \subseteq V$ be subsets of v.s. V.

(a) If S is indep. then T is indep.

(b) If T is dep. then S is dep.

Pf. (a) Suppose S is indep. Any lin. comb. of vectors in T is also a lin. comb. of vectors in S since $T \subseteq S$. So if the combo. is θ_V, all its coeffs are 0 by indep. of S, so T is indep.

(b) Is the contrapositive of (a). □

(a) Subset of indep set is indep.

(b) If a set is dep, so is any containing "superset".

Ex: In $V = \mathbb{R}^2_2$, is $S = \left\{ \underset{v_1}{\begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}}, \underset{v_2}{\begin{bmatrix} 1 & 1 \\ 1 & 0 \end{bmatrix}}, \underset{v_3}{\begin{bmatrix} 1 & 1 \\ 0 & 0 \end{bmatrix}}, \underset{v_4}{\begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}} \right\}$ indep or dep? Solve

$$\sum_{j=1}^{4} x_j v_j = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix} = x_1 \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix} + x_2 \begin{bmatrix} 1 & 1 \\ 1 & 0 \end{bmatrix} + x_3 \begin{bmatrix} 1 & 1 \\ 0 & 0 \end{bmatrix} + x_4 \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}$$

4 eq's in 4 var's:

$$\begin{aligned} x_1 + x_2 + x_3 + x_4 &= 0 \\ x_1 + x_2 + x_3 &= 0 \\ x_1 + x_2 &= 0 \\ x_1 &= 0 \end{aligned}$$

$$\left[\begin{array}{cccc|c} 1 & 1 & 1 & 1 & 0 \\ 1 & 1 & 1 & 0 & 0 \\ 1 & 1 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 \end{array}\right] \xrightarrow{r.r.} \left[\begin{array}{cccc|c} 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \end{array}\right]$$

has <u>only</u> the trivial solution, $x_1 = x_2 = x_3 = x_4 = 0$.

So S is <u>indep</u>. So any subset of S is indep.

Ex: Is $T = \left\{ \underset{w_1}{\begin{bmatrix} 1 & 1 \\ 0 & 0 \end{bmatrix}}, \underset{w_2}{\begin{bmatrix} 0 & 0 \\ 1 & 1 \end{bmatrix}}, \underset{w_3}{\begin{bmatrix} 1 & 0 \\ 1 & 0 \end{bmatrix}}, \underset{w_4}{\begin{bmatrix} 0 & 1 \\ 0 & 1 \end{bmatrix}} \right\}$ indep or dep?

$\sum_{j=1}^{4} x_j w_j = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix} = x_1 \begin{bmatrix} 1 & 1 \\ 0 & 0 \end{bmatrix} + x_2 \begin{bmatrix} 0 & 0 \\ 1 & 1 \end{bmatrix} + x_3 \begin{bmatrix} 1 & 0 \\ 1 & 0 \end{bmatrix} + x_4 \begin{bmatrix} 0 & 1 \\ 0 & 1 \end{bmatrix}$ 11/18

is lin. sys.

$$\begin{aligned} x_1 + x_3 &= 0 \\ x_1 + x_4 &= 0 \\ x_2 + x_3 &= 0 \\ x_2 + x_4 &= 0 \end{aligned}$$

So we need to solve by row red.

$$\left[\begin{array}{cccc|c} 1 & 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 1 & 0 \\ 0 & 1 & 1 & 0 & 0 \\ 0 & 1 & 0 & 1 & 0 \end{array}\right] \rightarrow \left[\begin{array}{cccc|c} 1 & 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 & 0 \\ 0 & 0 & 1 & -1 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right] \quad \begin{aligned} x_1 &= -r \\ x_2 &= -r \\ x_3 &= r \\ x_4 &= r \in \mathbb{R} \text{ free} \end{aligned}$$

Get nontrivial solutions so T is <u>dep.</u>

$r = 1$ gives the dep. relation

$-w_1 - w_2 + w_3 + w_4 = 0_2^2$ so

$w_4 = w_1 + w_2 - w_3$