Review Questions for Exam 1:

Let $T = \{v_1, \dots, v_m\}$ be a list of vectors in V. The span of T is

$$\langle T \rangle = \left\{ \sum_{i=1}^{m} x_i v_i \in V \;\middle|\; x_i \in F \right\} \leq V$$

<u>Ex:</u> $T = \left\{ \underbrace{\begin{bmatrix}1\\1\\1\end{bmatrix}}_{v_1}, \underbrace{\begin{bmatrix}1\\1\\0\end{bmatrix}}_{v_2}, \underbrace{\begin{bmatrix}1\\0\\0\end{bmatrix}}_{v_3} \right\} \subseteq F^3$

What is $\langle T \rangle$? Is $v = \begin{bmatrix}x\\y\\z\end{bmatrix} \in \langle T \rangle$? When Is

$x_1 v_1 + x_2 v_2 + x_3 v_3 = \begin{bmatrix}x\\y\\z\end{bmatrix}$ consistent? Solve lin. sys.

$$\left[\begin{array}{ccc|c} 1 & 1 & 1 & x \\ 1 & 1 & 0 & y \\ 1 & 0 & 0 & z \end{array}\right] \xrightarrow{r.r.} \left[\begin{array}{ccc|c} 1 & 0 & 0 & z \\ 0 & 1 & 0 & y-z \\ 0 & 0 & 1 & x-y \end{array}\right]$$

$T \quad v \qquad$ RREF

$x_1 = z$

$x_2 = y - z$

$x_3 = x - y$

is consistent $\forall v = \begin{bmatrix} x \\ y \\ z \end{bmatrix} \in F^3$

$-1 -1\ 0\ -y$

$-1\ 0\ 0\ -z$

$\mathrm{Rank}(A) = 3$ So $\langle T \rangle = F^3$

so T spans F^3.

Is T indep.? Does $\sum_{i=1}^{3} x_i v_i = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$ have only the triv. solution?

Use $\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$ above, get unique sol'n

$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$ is only sol'n. so T is <u>indep.</u>

So T is a <u>basis</u> of F^3.

Let $S=\left\{e_1=\begin{bmatrix}1\\0\\0\end{bmatrix}, e_2=\begin{bmatrix}0\\1\\0\end{bmatrix}, e_3=\begin{bmatrix}0\\0\\1\end{bmatrix}\right\}$ be the std. basis of F^3. Find $[v]_S=\begin{bmatrix}x_1\\x_2\\x_3\end{bmatrix}$ s.t. $v=\begin{bmatrix}x\\y\\z\end{bmatrix}=\sum_{i=1}^{3} x_i \cdot e_i$

Solve $x_1\begin{bmatrix}1\\0\\0\end{bmatrix}+x_2\begin{bmatrix}0\\1\\0\end{bmatrix}+x_3\begin{bmatrix}0\\0\\1\end{bmatrix}=\begin{bmatrix}x\\y\\z\end{bmatrix}$

$$\underbrace{\left[\begin{array}{ccc|c}1&0&0&x\\0&1&0&y\\0&0&1&z\end{array}\right]}_{S \quad v} \quad \begin{array}{l}x_1=x\\x_2=y\\x_3=z\end{array}$$

$[v]_S=\begin{bmatrix}x\\y\\z\end{bmatrix}=v$

Find $[v]_T=\begin{bmatrix}z\\y-z\\x-y\end{bmatrix}$ from before, and can check that

$$\underset{v_1}{z\begin{bmatrix}1\\1\\1\end{bmatrix}}+(y-z)\underset{v_2}{\begin{bmatrix}1\\1\\0\end{bmatrix}}+(x-y)\underset{v_3}{\begin{bmatrix}1\\0\\0\end{bmatrix}}=\underset{v}{\begin{bmatrix}x\\y\\z\end{bmatrix}}$$

What is the relationship between $[v]_S$ and $[v]_T$?

$$[v]_T = \begin{bmatrix} z \\ y-z \\ x-y \end{bmatrix} = \underbrace{\begin{bmatrix} 0 & 0 & 1 \\ 0 & 1 & -1 \\ 1 & -1 & 0 \end{bmatrix}}_{3\times 3} \underbrace{\begin{bmatrix} x \\ y \\ z \end{bmatrix}}_{3\times 1} = {}_T P_S \, [v]_S$$

3×1

${}_T P_S$: Transition matrix from S to T

<u>Ex</u>: Let $T = \left\{ \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 2 \\ 2 \\ 1 \end{bmatrix} \right\}$ Find $\langle T \rangle$.

$\quad v_1 \quad\quad v_2 \quad\quad v_3$

When is $\begin{bmatrix} x \\ y \\ z \end{bmatrix} \in \langle T \rangle$? When is $\left[\begin{array}{ccc|c} 1 & 1 & 2 & x \\ 1 & 1 & 2 & y \\ 1 & 0 & 1 & z \end{array}\right]$ consistent?

$$\xrightarrow{r.r.} \left[\begin{array}{ccc|c} 1 & 0 & 1 & z \\ 0 & 1 & 1 & x-z \\ 0 & 0 & 0 & y-x \end{array}\right]$$

$\text{rank}(A) = 2$

only consist. when $x = y$

So $\langle T\rangle = \left\{\begin{bmatrix} x \\ y \\ z \end{bmatrix} \in F^3 \,\middle|\, x=y\right\} = \left\{\begin{bmatrix} y \\ y \\ z \end{bmatrix} \in F^3 \,\middle|\, y, z \in F\right\}$

$= \left\{ y\begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix} + z\begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} \in F^3 \,\middle|\, y, z \in F\right\} = \left\langle \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} \right\rangle$

Is T indep or dep? Since $\left[\begin{array}{ccc|c} 1 & 1 & 2 & 0 \\ 1 & 1 & 2 & 0 \\ 1 & 0 & 1 & 0 \end{array}\right] \xrightarrow{\text{r.r.}}$

$\left[\begin{array}{ccc|c} 1 & 0 & 1 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right]$ $\begin{aligned} x_1 &= -x_3 = -r \\ x_2 &= -x_3 = -r \\ x_3 &= r \in F \end{aligned}$ has non-triv. sol'ns (a free var.)

So T is dep and $-rv_1 - rv_2 + rv_3 = 0^3_1$

for $r=1$ says $-v_1 - v_2 + v_3 = 0^3_1$, so $v_3 = v_1 + v_2$ is redundant in T

$\langle T\rangle = \langle v_1, v_2\rangle$

A very important use for a basis of a vector space V is to give "coordinates" for each $v \in V$.

<u>Def</u>. Let $S = \{v_1, \ldots, v_n\}$ be a basis for V, so $\forall v \in V$, $\exists a_1, \ldots, a_n \in F$ s.t. $v = \sum_{j=1}^{n} a_j v_j$. It is understood that S is actually an ordered list, so the corresponding list of scalar coefficients, $a_1, \ldots, a_n \in F$ is uniquely determined by v. So we define the "coordinate vector of v w.r.t. S" to be

$$[v]_S = \begin{bmatrix} a_1 \\ \vdots \\ a_n \end{bmatrix} \in F^n .$$

<u>Th</u>: The map $[\cdot]_S : V \to F^n$ is linear and bijective, so it is invertible and an isomorphis

Pf. Suppose $v, w \in V$ with $v = \sum_{j=1}^{n} a_j v_j$ and $w = \sum_{j=1}^{n} b_j v_j$, so $[v]_S = \begin{bmatrix} a_1 \\ \vdots \\ a_n \end{bmatrix}$ and $[w]_S = \begin{bmatrix} b_1 \\ \vdots \\ b_n \end{bmatrix}$. Then $v + w = \sum_{j=1}^{n} (a_j + b_j) v_j$ so

$$[v+w]_S = \begin{bmatrix} a_1+b_1 \\ \vdots \\ a_n+b_n \end{bmatrix} = \begin{bmatrix} a_1 \\ \vdots \\ a_n \end{bmatrix} + \begin{bmatrix} b_1 \\ \vdots \\ b_n \end{bmatrix} = [v]_S + [w]_S .$$

For any $\alpha \in F$, $\alpha v = \sum_{j=1}^{n} (\alpha a_j) v_j$ so $[\alpha v]_S = \begin{bmatrix} \alpha a_1 \\ \vdots \\ \alpha a_n \end{bmatrix} = \alpha \begin{bmatrix} a_1 \\ \vdots \\ a_n \end{bmatrix} = \alpha [v]_S$. This shows $[\cdot]_S$ is <u>linear</u>.

$\mathrm{Ker}([\cdot]_S) = \{ v \in V / [v]_S = 0_1^n \}$ so $v \in \mathrm{Ker}([\cdot]_S)$ iff $v = \sum_{j=1}^{n} 0 v_j = \theta_V$ so the map is <u>injective</u>.

$\forall \begin{bmatrix} a_1 \\ \vdots \\ a_n \end{bmatrix} \in F^n$, the vector $v = \sum_{j=1}^{n} a_j v_j \in V$ has $[v]_S = \begin{bmatrix} a_1 \\ \vdots \\ a_n \end{bmatrix}$, so $[\cdot]_S$ is <u>surjective</u>. This gives the rest of the properties, invert, isom. ∎

<u>Note</u>: For $1 \le j \le n$, $\forall v_j \in S$, $[v_j]_S = e_j$ is the j^{th} standard basis vector of F^n, so

$\{[v_1]_S, [v_2]_S, \ldots, [v_n]_S\}$ is the <u>std.</u> basis of F^n.

Fundamental Question about coordinates:
If V has two bases, $S = \{v_1, \ldots, v_n\}$ and $T = \{w_1, \ldots, w_n\}$, how are $[v]_S$ and $[v]_T$ related?

Diagram:

$$\begin{tikzcd} & V \arrow[dl, "{[\ ]_S}"'] \arrow[dr, "{[\ ]_T}"] & \\ F^n \arrow[rr, shift left, "P"] & & F^n \arrow[ll, shift left, "Q"] \end{tikzcd}$$

Can we find $P \in F^n_n$ s.t. $\forall v \in V$, $P[v]_S = [v]_T$ and $Q \in F^n_n$ s.t. $Q[v]_T = [v]_S$

The maps on the bottom lines are actually L_P and L_Q, but we just label the arrows with P and Q.

What would these matrices have to be if they existed? <u>Answer</u>: Certainly we would need $P[v_j]_S = [v_j]_T$ and $Q[w_j]_T = [w_j]_S$ for $1 \le j \le n$. But $[v_j]_S = e_j = [w_j]_T$ so $Col_j(P) = [v_j]_T$ and $Col_j(Q) = [w_j]_S$.

Thus, to find P, find its columns, which are the coordinates w.r.t T of the basis S vectors. Similarly, the columns of Q are the coordinates w.r.t. S of the basis T vectors.

How do we find the coordinates of a vector $v \in V$ w.r.t. a basis, S or T, of V?

Solve a lin. sys., of cource!

$\sum_{j=1}^{n} x_j \cdot v_j = v$ is solved by row reducing

$$\underset{\text{as columns}}{[S \mid v]} \xrightarrow{\text{r.r.}} [I_n \mid [v]_S]$$

$\sum_{j=1}^{n} x_j \cdot w_j = v$ is solved by $\underset{\text{as columns}}{[T \mid v]} \xrightarrow{\text{r.r.}} [I_n \mid [v]_T]$

So to get P s.t. $P[v_j]_S = [v_j]_T$ we need to solve n linear systems, $1 \le j \le n$, but all have the same coeff. matrix, so solve all at once:

① $[T|S] \xrightarrow{\text{r.r.}} [I_n|{}_TP_S]$ gives the "transition matrix" ${}_TP_S$ from
as columns
S to T s.t. ${}_TP_S[v]_S = [v]_T$

② $[S|T] \xrightarrow{\text{r.r.}} [I_n|{}_SQ_T]$ gives the "transition matrix" ${}_SQ_T$ from
as columns
T to S s.t. ${}_SQ_T[v]_T = [v]_S$

<u>Th</u>: If $S = \{v_1, \dots, v_n\}$ and $T = \{w_1, \dots, w_n\}$ are bases of V, then $\exists\ {}_TP_S, {}_SQ_T \in F_n^n$ s.t. $\forall v \in V$, ${}_TP_S[v]_S = [v]_T$ and ${}_SQ_T[v]_T = [v]_S$.

<u>Pf.</u> We have an algorithm by row reduction guaranteed to find the matrices P and Q s.t. the formulas are true for $v = v_j \in S$ in the equation $P[v_j]_S = [v_j]_T$ and for $v = w_j \in T$ for $Q[w_j]_T = [w_j]_S$. But $\forall v \in V$, can write $v = \sum_{j=1}^{n} a_j v_j$ so $P[v]_S = P\left[\sum_{j=1}^{n} a_j v_j\right]_S = \sum_{j=1}^{n} a_j P[v_j]_S$ (by lin. of $[\cdot]_S$) $= \sum_{j=1}^{n} a_j [v_j]_T = \left[\sum_{j=1}^{n} a_j v_j\right]_T$ (by lin. of $[\cdot]_T$) $= [v]_T$ so $P[v]_S = [v]_T$, $\forall v \in V$.

The argument for any $w = \sum_{j=1}^{n} b_j w_j \in V$ is similar. $\square$