Ex. $V = F^2$, $S = \left\{ \underset{v_1}{\begin{bmatrix} 1 \\ 0 \end{bmatrix}}, \underset{v_2}{\begin{bmatrix} 0 \\ 1 \end{bmatrix}} \right\}$, $T = \left\{ \underset{w_1}{\begin{bmatrix} 1 \\ 1 \end{bmatrix}}, \underset{w_2}{\begin{bmatrix} 1 \\ 2 \end{bmatrix}} \right\}$

For $v = \begin{bmatrix} a \\ b \end{bmatrix} \in F^2$, $v = a\begin{bmatrix} 1 \\ 0 \end{bmatrix} + b\begin{bmatrix} 0 \\ 1 \end{bmatrix}$ so $[v]_S = \begin{bmatrix} a \\ b \end{bmatrix}$

To get $[v]_T = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$ solve $v = x_1 w_1 + x_2 w_2$

$$\begin{bmatrix} a \\ b \end{bmatrix} = x_1 \begin{bmatrix} 1 \\ 1 \end{bmatrix} + x_2 \begin{bmatrix} 1 \\ 2 \end{bmatrix}$$

solve $\underset{\displaystyle T \quad\quad v}{\left[\begin{array}{cc|c} 1 & 1 & a \\ 1 & 2 & b \end{array}\right]} \xrightarrow{r.r.} \left[\begin{array}{cc|c} 1 & 1 & a \\ 0 & 1 & b-a \end{array}\right] \to \left[\begin{array}{cc|c} 1 & 0 & 2a-b \\ 0 & 1 & b-a \end{array}\right]$ so

$+$ $\quad -1 \quad -1 \quad -a$ $\quad\quad\quad$ $0 \quad -1 \quad a-b$

$x_1 = 2a-b$
$x_2 = b-a$ $\quad$ so $[v]_T = \begin{bmatrix} 2a-b \\ b-a \end{bmatrix}$

Check:
$(2a-b)\begin{bmatrix} 1 \\ 1 \end{bmatrix} + (b-a)\begin{bmatrix} 1 \\ 2 \end{bmatrix} = \begin{bmatrix} a \\ b \end{bmatrix}$

Thus, $[v]_T = \begin{bmatrix} 2a-b \\ b-a \end{bmatrix} = \overset{{}_T P_S}{\begin{bmatrix} 2 & -1 \\ -1 & 1 \end{bmatrix}} \overset{[v]_S}{\begin{bmatrix} a \\ b \end{bmatrix}}$

By algorithm:

$$\underset{\quad T \qquad\quad S}{\left[\begin{array}{cc|cc} 1 & 1 & 1 & 0 \\ 1 & 2 & 0 & 1 \end{array}\right]} \xrightarrow{r.r.} \left[\begin{array}{cc|cc} 1 & 1 & 1 & 0 \\ 0 & 1 & -1 & 1 \end{array}\right] \rightarrow \underset{I_2 \qquad {}_T P_S}{\left[\begin{array}{cc|cc} 1 & 0 & 2 & -1 \\ 0 & 1 & -1 & 1 \end{array}\right]} \text{ matches}$$

$$-1 \ -1 \ -1 \ 0 \ + \qquad 0 \ -1 \ 1 \ -1 \ +$$

$\underset{\quad S \qquad\quad T}{\left[\begin{array}{cc|cc} 1 & 0 & 1 & 1 \\ 0 & 1 & 1 & 2 \end{array}\right]}$ is already RREF $= [I_2 \mid {}_S Q_T]$ so ${}_S Q_T = \begin{bmatrix} 1 & 1 \\ 1 & 2 \end{bmatrix}$

$\begin{bmatrix} 1 & 1 \\ 1 & 2 \end{bmatrix} \begin{bmatrix} 2a-b \\ b-a \end{bmatrix} = \begin{bmatrix} a \\ b \end{bmatrix} = \underset{{}_S Q_T}{\begin{bmatrix} 1 & 1 \\ 1 & 2 \end{bmatrix}} \underset{{}_T P_S}{\begin{bmatrix} 2 & -1 \\ -1 & 1 \end{bmatrix}} \begin{bmatrix} a \\ b \end{bmatrix}$ shows ${}_S Q_T = ({}_T P_S)^{-1}$

${}_S Q_T \, [v]_T = [v]_S$ checks

Cor: If ${}_T P_S [v]_S = [v]_T$ and ${}_S Q_T [v]_T = [v]_S$ then ${}_T P_S = {}_S Q_T^{-1}$.

Pf. By substitution: ${}_T P_S ({}_S Q_T [v]_T) = [v]_T$ and ${}_S Q_T ({}_T P_S [v]_S) = [v]_S$. By assoc. of matrix mult. and using $v = w_j \in T$ in the first eq. and $v = v_j \in S$ in the second eq., we get

$$({}_T P_S \, {}_S Q_T) e_j = e_j \text{ and } ({}_S Q_T \, {}_T P_S) e_j = e_j, \quad 1 \le j \le n.$$

So ${}_T P_S \, {}_S Q_T = I_n = {}_S Q_T \, {}_T P_S$. $\square$

Usually I will write ${}_T P_S [v]_S = [v]_T$ and ${}_S P_T [v]_T = [v]_S$.

Another way to see why these two transition matrices are inverses of each other is to look at the algorithms for finding them.

Since $[T \mid S] \xrightarrow{①} [I_n \mid {}_T P_S]$ and $[S \mid T] \xrightarrow{②} [I_n \mid {}_S P_T]$

we could start with switching the two sides:

$$[{}_T P_S \mid I_n] \xrightarrow[\text{steps of ①}]{\text{reverse}} [S \mid T] \xrightarrow{②} [I_n \mid {}_S P_T]$$ says

$${}_S P_T = ({}_T P_S)^{-1}.$$

<u>Example</u>: Let $V = \mathbb{R}^2_2$, $S = \left\{ \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix} \right\}$

the <u>std</u>. basis, $T = \left\{ \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}, \begin{bmatrix} 1 & 1 \\ 1 & 0 \end{bmatrix}, \begin{bmatrix} 1 & 1 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} \right\}$ (see page ~~[illegible]~~)

For $v = \begin{bmatrix} a & b \\ c & d \end{bmatrix} \in \mathbb{R}^2_2$ find $[v]_S$ and $[v]_T$ and the transition mat's: ${}_T P_S$ and ${}_S P_T$.

To find $[v]_S \in \mathbb{R}^4$, solve $\sum_{j=1}^{4} x_j v_j = v$ with $S = \{v_1, v_2, v_3, v_4\}$, get lin. sys. $[S \mid v] =$ as col's

$$\begin{array}{c} \left[\begin{array}{cccc|c} 1 & 0 & 0 & 0 & a \\ 0 & 1 & 0 & 0 & b \\ 0 & 0 & 1 & 0 & c \\ 0 & 0 & 0 & 1 & d \end{array}\right] \\ \quad S \qquad\qquad\quad v \end{array} \quad \begin{array}{l} x_1 = a \\ x_2 = b \\ x_3 = c \\ x_4 = d \end{array} \quad \text{so } [v]_S = \begin{bmatrix} a \\ b \\ c \\ d \end{bmatrix} \in \mathbb{R}^4.$$

To find $[v]_T \in \mathbb{R}^4$, solve $\sum_{j=1}^{4} x_j w_j = v$ with $T = \{w_1, w_2, w_3, w_4\}$, get lin. sys. $[T \mid v] =$ as col's

$$\begin{array}{c} \left[\begin{array}{cccc|c} 1 & 1 & 1 & 1 & a \\ 1 & 1 & 1 & 0 & b \\ 1 & 1 & 0 & 0 & c \\ 1 & 0 & 0 & 0 & d \end{array}\right] \\ \quad T \qquad\qquad\quad v \end{array} \xrightarrow{r.r.} \left[\begin{array}{cccc|c} 1 & 0 & 0 & 0 & d \\ 0 & 1 & 0 & 0 & c-d \\ 0 & 0 & 1 & 0 & b-c \\ 0 & 0 & 0 & 1 & a-b \end{array}\right] \quad \begin{array}{l} x_1 = d \\ x_2 = c-d \\ x_3 = b-c \\ x_4 = a-b \end{array} \quad \text{so } [v]_T = \begin{bmatrix} d \\ c-d \\ b-c \\ a-b \end{bmatrix}$$

and we can check: $d\begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix} + (c-d)\begin{bmatrix} 1 & 1 \\ 1 & 0 \end{bmatrix} + (b-c)\begin{bmatrix} 1 & 1 \\ 0 & 0 \end{bmatrix} + (a-b)\begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$

But notice: $[v]_T = \begin{bmatrix} d \\ c-d \\ b-c \\ a-b \end{bmatrix} = \underbrace{\begin{bmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & -1 \\ 0 & 1 & -1 & 0 \\ 1 & -1 & 0 & 0 \end{bmatrix}}_{{}_T P_S} \underbrace{\begin{bmatrix} a \\ b \\ c \\ d \end{bmatrix}}_{[v]_S}$

and by the algorithm:

$$\underset{\text{as col's}}{[T \mid S]} = \left[\begin{array}{cccc|cccc} 1 & 1 & 1 & 1 & 1 & 0 & 0 & 0 \\ 1 & 1 & 1 & 0 & 0 & 1 & 0 & 0 \\ 1 & 1 & 0 & 0 & 0 & 0 & 1 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{array}\right] \xrightarrow{\text{r.r.}} \left[\begin{array}{cccc|cccc} 1 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 & 0 & 0 & 1 & -1 \\ 0 & 0 & 1 & 0 & 0 & 1 & -1 & 0 \\ 0 & 0 & 0 & 1 & 1 & -1 & 0 & 0 \end{array}\right]$$

$${}_T P_S$$

while

$$\underset{\text{as col's}}{[S \mid T]} = \left[\begin{array}{cccc|cccc} 1 & 0 & 0 & 0 & 1 & 1 & 1 & 1 \\ 0 & 1 & 0 & 0 & 1 & 1 & 1 & 0 \\ 0 & 0 & 1 & 0 & 1 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 1 & 0 & 0 & 0 \end{array}\right]$$

is already in RREF on leftside so $\;= {}_S P_T$.

Diagram:

$$\begin{tikzcd} & \mathbb{R}^2_2 \arrow[dl, "{[\;]_S}"'] \arrow[dr, "{[\;]_T}"] & \\ \mathbb{R}^4 \arrow[rr, shift left, "{}_T P_S"] & & \mathbb{R}^4 \arrow[ll, shift left, "{}_S P_T"] \end{tikzcd}$$

Recall that list of vectors in V,

$S = \{v_1, \ldots, v_m\}$ is **dep.** when $\exists a_1, \ldots, a_m \in F$ s.t.

$\sum_{i=1}^{m} a_i v_i = \theta$ with not all a_i are zero.

Say $1 \le k \le m$ is the largest index s.t. $a_k \neq 0$

so $\sum_{i=1}^{k} a_i v_i = \theta$. Then $a_1 v_1 + \cdots + a_{k-1} v_{k-1} + a_k v_k = \theta$

allows us to "solve for v_k", that is,

$$v_k = \frac{1}{a_k}\left(a_1 v_1 + \cdots + a_{k-1} v_{k-1}\right) \in \langle v_1, \ldots, v_{k-1} \rangle$$

shows v_k is redundant in S so

$\langle S \rangle = \langle S - \{v_k\} \rangle$. Continue this process.

Keep removing redundant vectors until you get an <u>indep.</u> set having the same span as S, so got a <u>basis</u> for $\langle S \rangle$.

<u>Th</u>: A spanning set for $W \leq V$ can be cut down to a basis for W.

<u>Pf.</u> Use dep. relations to remove redundant vectors from the spanning set until you get an indep. spanning set. ∎

<u>Application</u>: Use basis vectors of $\mathrm{Ker}(L_A)$ to remove redundant columns of A to get a basis for $\mathrm{Range}(L_A)$.

<u>Ex</u>: $S = \left\{ \begin{bmatrix} 1 & 1 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 \\ 1 & 1 \end{bmatrix}, \begin{bmatrix} 1 & 0 \\ 1 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 1 \\ 0 & 1 \end{bmatrix} \right\}$

$\qquad\quad w_1 \qquad\quad w_2 \qquad\quad w_3 \qquad\quad w_4$

$W = \langle S \rangle \leq F_2^2$ but $w_1 + w_2 - w_3 - w_4 = 0_2^2$

so $w_4 = w_1 + w_2 - w_3$ is redundant

$w_4 \in \langle w_1, w_2, w_3 \rangle$

$S - \{w_4\} = \{w_1, w_2, w_3\}$ also spans W.

Check ✓ <u>indep</u>:

$$\left[\begin{array}{ccc|c} 1 & 0 & 1 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 1 & 0 & 0 \end{array}\right] \to \left[\begin{array}{ccc|c} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right] \begin{array}{l} x_1 = 0 \\ x_2 = 0 \\ x_3 = 0 \end{array}$$

So $\{w_1, w_2, w_3\}$ is a <u>basis</u> of $W \lneqq F_2^2$