A basis for subspace W gives us a parameterization of W and coordinates for each vector $w \in W$. In the last example,

$$W = \{ r w_1 + s w_2 + t w_3 \in F_2^2 \mid r, s, t \in F \}$$

$$= \left\{ r\begin{bmatrix} 1 & 1 \\ 0 & 0 \end{bmatrix} + s\begin{bmatrix} 0 & 0 \\ 1 & 1 \end{bmatrix} + t\begin{bmatrix} 1 & 0 \\ 1 & 0 \end{bmatrix} = \begin{bmatrix} (r+t) & r \\ (s+t) & s \end{bmatrix} \in F_2^2 \;\middle|\; r, s, t \in F \right\}$$

but another description of W is by a consistency condition required for $\begin{bmatrix} a & b \\ c & d \end{bmatrix} \in W$.

Solve
$$\left[\begin{array}{ccc|c} 1 & 0 & 1 & a \\ 1 & 0 & 0 & b \\ 0 & 1 & 1 & c \\ 0 & 1 & 0 & d \end{array}\right] \xrightarrow{r.r.} \left[\begin{array}{ccc|c} 1 & 0 & 0 & b \\ 0 & 1 & 0 & d \\ 0 & 0 & 1 & c-d \\ 1 & 0 & 1 & a \end{array}\right] \rightarrow \left[\begin{array}{ccc|c} 1 & 0 & 0 & b \\ 0 & 1 & 0 & d \\ 0 & 0 & 1 & c-d \\ 0 & 0 & 0 & a-b-c+d \end{array}\right]$$

$$\begin{array}{cccc} -1 & 0 & 0 & -b \\ 0 & 0 & -1 & -c+d \end{array}$$

consistent when $0 = a - b - c + d$

So
$$W = \left\{ \begin{bmatrix} a & b \\ c & d \end{bmatrix} \in F_2^2 \;\middle|\; 0 = a - b - c + d \right\}.$$

For basis $B = \left\{ w_1 = \begin{bmatrix} 1 & 1 \\ 0 & 0 \end{bmatrix}, w_2 = \begin{bmatrix} 0 & 0 \\ 1 & 1 \end{bmatrix}, w_3 = \begin{bmatrix} 1 & 0 \\ 1 & 0 \end{bmatrix} \right\}$ $\boxed{138.5}$

of W we have coordinate map

$[\cdot]_B : W \to F^3$ defined by $[w]_B = \begin{bmatrix} r \\ s \\ t \end{bmatrix}$ when

$w = \begin{bmatrix} (r+t) & r \\ (s+t) & s \end{bmatrix} \in W$.

<u>Question:</u> Can we extend B to get a basis of F_2^2? What would be required of $w_4 = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$ so that $B \cup \{w_4\} = \{w_1, w_2, w_3, w_4\}$ is a basis of F_2^2? Need it to be indep and to span F_2^2.

Both true iff rank $\begin{bmatrix} 1 & 0 & 1 & a \\ 1 & 0 & 0 & b \\ 0 & 1 & 1 & c \\ 0 & 1 & 0 & d \end{bmatrix} = 4$ but from the row reduction on the last page that happens iff $a-b-c+d \neq 0$ iff $w_4 \notin W$.

There are infinitely many choices of $w_4 \notin W$, but some stand out as being very simple, for example, any of the standard basis vectors of F_2^2.

General theory about extending an indep set to a basis:

<u>Th.</u> Let $S \subset V$ be an indep. subset of vector space V. Then there is a subset T with $S \subseteq T \subset V$ and T is a basis of V.

<u>Lemma</u>: If $S \subset V$ is indep and $v \in V$ then $T = S \cup \{v\}$ is indep iff $v \notin \langle S \rangle$ iff $\langle S \rangle \subsetneq \langle T \rangle$.

<u>Pf.</u> Suppose T is indep. but $v \in \langle S \rangle$.

Write $v = \sum_{i=1}^{m} c_i s_i$ for some $s_i \in S$, $c_i \in F$. 138.7

Then $\theta = \sum_{i=1}^{m} c_i s_i - v$ is a lin. comb. from T with not all coefficients 0, contradicting T indep, so T indep $\Rightarrow v \notin \langle S \rangle$. Conversely, suppose $v \notin \langle S \rangle$ for S indep but T is dep. Let a dep. rel. on T be written $\theta = \sum_{i=1}^{m} c_i s_i + cv$ for some $s_i \in S$, $c, c_i \in F$. If $c = 0$ this is a dep rel. on S, so $c \neq 0$, and then we can write

$$v = -c^{-1} \sum_{i=1}^{m} c_i s_i = \sum_{i=1}^{m} \frac{-c_i}{c} s_i \in \langle S \rangle$$ a contradiction.

Now we prove that $v \notin \langle S \rangle$ iff $\langle S \rangle \subsetneq \langle T \rangle$.

Since $S \subseteq T$ we know $\langle S \rangle \leq \langle T \rangle$. To get **138.8**
$\langle S \rangle \neq \langle T \rangle$ we just need to show $\langle T \rangle$ contains
some vector not in $\langle S \rangle$. Of course, $v \in T$ is given
so $v \in \langle T \rangle$. If $v \notin \langle S \rangle$ then $\langle S \rangle \lneq \langle T \rangle$ is
true. Now show that $v \in \langle S \rangle$ implies $\langle S \rangle = \langle T \rangle$.
Write $v = \sum_{j=1}^{n} d_j s_j \in \langle S \rangle$. Then $\forall w \in \langle T \rangle$ we can
write $w = \sum_{i=1}^{m} c_i s_i' + cv$ for some $s_i' \in S$, $c, c_i \in F$.
By substitution of the formula for $v \in \langle S \rangle$ get

$$w = \sum_{i=1}^{m} c_i s_i' + c \sum_{j=1}^{n} d_j s_j \in \langle S \rangle.$$

This means
$\langle T \rangle \leq \langle S \rangle$ so they must be equal. ∎

Pf. of extension Thm. Let $S \subset V$ be indep.

If $\langle S \rangle = V$ then $T = S$ is a basis of V.

If $\langle S \rangle \neq V$ then $\exists v_1 \in V$ with $v_1 \notin \langle S \rangle$ so

$T_1 = S \cup \{v_1\}$ is indep and $\langle S \rangle \subsetneq \langle T_1 \rangle$.

If $\langle T_1 \rangle = V$ then T_1 is the desired basis of V.

If $\langle T_1 \rangle \neq V$ then $\exists v_2 \in V$ with $v_2 \notin \langle T_1 \rangle$ so

$T_2 = T_1 \cup \{v_2\}$ is indep and $\langle T_1 \rangle \subsetneq \langle T_2 \rangle$.

If $\langle T_2 \rangle = V$ then T_2 is the desired basis of V.

If $\langle T_2 \rangle \neq V$ then $\exists v_3 \in V$ with $v_3 \notin \langle T_2 \rangle$ so

$T_3 = T_2 \cup \{v_3\}$ is indep and $\langle T_2 \rangle \subsetneq \langle T_3 \rangle$.

Continue this way getting bigger indep. sets until get a basis for V. $\square$

Th: Let $S=\{v_1,\dots,v_n\}$ and $T=\{w_1,\dots,w_m\}$ both be bases of vector space V. Then $m=n$. 138.91

Pf. Since $\langle S\rangle = V$ and $w_1\in V$, get that list $\{w_1\}\cup S=\{w_1,v_1,\dots,v_n\}$ is dep. so $\theta=c_1w_1+\sum_{j=1}^{n}d_jv_j$ for coefficients not all zero. Claim that for some $1\le j\le n$, $d_j\ne 0$. Otherwise $\theta=c_1w_1$ and $c_1\ne 0$ contradicts T indep. By renumbering the vectors in S we may say $d_n\ne 0$ so $v_n\in\langle w_1,v_1,\dots,v_{n-1}\rangle$ is redundant and $S_1=\{w_1,v_1,\dots,v_{n-1}\}$ spans V. Since $w_2\in V$ get list $\{w_2\}\cup S_1=\{w_1,w_2,v_1,\dots,v_{n-1}\}$ is dep so $\theta=c_1w_1+c_2w_2+\sum_{j=1}^{n-1}d_jv_j$

with not all coeff's zero. Claim: for some $1 \le j \le n-1$, $d_j \neq 0$, since otherwise $\theta = c_1 w_1 + c_2 w_2$ with not both coeffs. 0, so $\{w_1, w_2\}$ dep contradicts T indep. For convenience we may renumber $v_1, \ldots, v_{n-1}$ so that $d_{n-1} \neq 0$.

As before, get $v_{n-1} \in \langle w_1, w_2, v_1, \ldots, v_{n-2} \rangle$ is redundant so $S_2 = \{w_1, w_2, v_1, \ldots, v_{n-2}\}$ spans V. Can continue in this way replacing a v_j by a w_i. Suppose $m > n$, so after n steps of replacement, get $S_n = \{w_1, \ldots, w_n\}$ spans V. But $n < m$ means $\exists\, w_{n+1} \in T$ and $w_{n+1} \in \langle S_n \rangle$ so $\{w_1, \ldots, w_n, w_{n+1}\}$ is dep.

contradicts T indep. Thus $m \le n$.
The same argument applied with S and T switched says $n \le m$ is also true, so $m = n$. $\square$

This means we can define $\dim(V)$ to be the number of vectors in any basis of V.

Ex: For $W = \langle B \rangle$ on p. 138.3, $\dim(W) = 3$ but $\dim(F_2^2) = 4$.

Th. Assume V has $\dim(V) = n$ finite. Let $W \le V$. Then $W = V$ iff $\dim(W) = \dim(V)$. Generally, $\dim(W) \le \dim(V)$.

Pf. If $W = V$ then any basis of W is a basis of V so $\dim(W) = \dim(V)$. 138.94

If $W \subsetneq V$ then can extend a basis $S = \{w_1, \dots, w_m\}$ for W to a basis for V by adding at least one vector $v \in V$ but $v \notin W = \langle S \rangle$, so $\dim(W) = m < m+1 \leq \dim(V)$. $\square$