Th. Let $L: V \to W$ be linear and $\dim(V) = n$ finite. Then

$$\dim(V) = \dim(\mathrm{Ker}(L)) + \dim(\mathrm{Range}(L)).$$

Pf. Let $B = \{v_1, \dots, v_k\}$ be a basis of $K = \mathrm{Ker}(L)$, so $k = \dim(K)$. Extend B to a basis of V, $S = \{v_1, \dots, v_k, v_{k+1}, \dots, v_n\}$. Then

$$\mathrm{Range}(L) = \{L(v) \in W \mid v \in V\} = \left\{L\left(\sum_{j=1}^{n} a_j v_j\right) \;\middle|\; a_j \in F\right\}$$

$$= \left\{\sum_{j=1}^{n} a_j L(v_j) \in W \;\middle|\; a_j \in F\right\} = \langle L(v_1), \dots, L(v_n)\rangle = \langle L(S)\rangle$$

But $L(v_1) = L(v_2) = \dots = L(v_k) = \theta_W$ since $B \subseteq K$, so

$$\mathrm{Range}(L) = \langle L(v_{k+1}), \dots, L(v_n)\rangle.$$

Claim: $T = \{L(v_{k+1}), \ldots, L(v_n)\}$ is indep. 138.96

Suppose $\theta_W = \sum_{j=k+1}^{n} a_j L(v_j) = L\left(\sum_{j=k+1}^{n} a_j v_j\right)$. Then $\sum_{j=k+1}^{n} a_j v_j \in K = \mathrm{Ker}(L)$ which has basis B, so

$$\sum_{j=k+1}^{n} a_j v_j = \sum_{j=1}^{k} b_j v_j$$

for some $b_j \in F$. Thus,

$$\theta_V = \sum_{j=1}^{k} b_j v_j - \sum_{j=k+1}^{n} a_j v_j$$

is a lin. comb. from S (basis of V) which is indep. so all coefficients are 0, $a_j = 0$ for $k+1 \le j \le n$. T indep and $\mathrm{Range}(L) = \langle T \rangle$ so T is a basis of $\mathrm{Range}(L)$, $\dim(\mathrm{Range}(L)) = n-k$.

$\square$

Note: The last Theorem generalizes the rank-nullity theorem. Recall that for $A \in F_n^m$, $L_A : F^n \to F^m$, $r = \text{rank}(A) = \dim(\text{Range}(L_A))$ and $n - r = \dim(\text{Ker}(L_A))$

so $\dim(V) = n = (n-r) + r$

$= \dim(\text{Ker}(L_A)) + \dim(\text{Range}(L_A))$

Applications:

Th: Let $L : V \to W$ with $\dim(V) = \dim(W)$ finite.

(a) If L is injective then L is surjective.

(b) If L is surjective then L is injective.

Pf. (a) If L is inj then $\dim(\text{Ker}(L)) = 0$ so $\dim(V) = 0 + \dim(\text{Range}(L)) = \dim(W)$ so

Range $(L) = W$ so L is surj.

(b) If L is surj then Range$(L) = W$ so
$\dim(\text{Range}(L)) = \dim(W)$ so
$\dim(W) = \dim(V) = \dim(\text{Ker}(L)) + \dim(W)$ says
$\dim(\text{Ker}(L)) = 0$ so $\text{Ker}(L) = \{\theta_V\}$ so L is injo

---

<u>Th</u>: Let $L : V \to W$ with $\dim(V) = n$ finite.
(a) If $n > \dim(W) = m$ then $\dim(\text{Ker}(L)) \geq n - m$.
(b) $\dim(\text{Range}(L)) \leq n$.

<u>Pf</u>(a) Know $n = \dim(\text{Ker}(L)) + \dim(\text{Range}(L))$ so
$\dim(\text{Ker}(L)) = n - \dim(\text{Range}(L))$. Also Range$(L) \subseteq W$
so $\dim(\text{Range}(L)) \leq \dim(W) = m$ so
$-\dim(\text{Range}(L)) \geq -m$. Add n to both sides,
get the result.

(b) $Ker(L) \leq V$, $0 \leq \dim(Ker(L)) \leq n$ and $\dim(V) = n = \dim(Ker(L)) + \dim(Range(L))$ means $0 \leq \dim(Range(L)) \leq n$. □ 138.99

<u>Note</u>: $\dim(Range(L)) \leq m = \dim(W)$ also true. This gives $\dim(Range(L)) \leq Min(m, n)$ and connects with the case when $L = L_A$ and $r = rank(A) = \dim(Range(L_A)) \leq Min(m, n)$ for $A \in F^m_n$.

<u>Cor.</u> If $L : V \to W$ is invertible for finite $\dim(V) = n$ and $\dim(W) = m$ then $m = n$.

<u>Pf.</u> L invertible means L is bijective so $\dim(Ker(L)) = 0$, $Range(L) = W$ so $\dim(V) = \dim(W)$. □

Th: Let $L : V \longrightarrow W$ be linear, $S = \{v_1, \dots, v_n\}$ a basis of V, $T = \{w_1, \dots, w_m\}$ a basis of W. Then we can find $A = {}_T[L]_S \in F^m_n$ such that

$$\forall v \in V, \quad \underbrace{{}_T[L]_S}_{m\times n}\ \underbrace{[v]_S}_{n\times 1} = \underbrace{[L(v)]_T}_{m\times 1}.$$

Pf. The appropriate diagram is:

$$\begin{tikzcd} V \arrow[r, "L"] \arrow[d, "{[\,]_S}"] & W \arrow[d, "{[\,]_T}"] \\ F^n \arrow[r, "{{}_T[L]_S}", "m\times n"'] & F^m \end{tikzcd}$$

What would ${}_T[L]_S$ have to be if the equation is true for all $v_j \in S$?

$${}_T[L]_S\,[v_j]_S = {}_T[L]_S\, e_j = Col_j\left({}_T[L]_S\right) = [L(v_j)]_T$$

so the columns of ${}_T[L]_S$ are the coordinates w.r.t. T of the n images $L(v_j) \in W$. This gives the following algorithm to find it:

$[T \mid L(S)] \xrightarrow{\text{r.r.}} [I_m \mid {}_T[L]_S]$ and this can

as columns always be done since T is a basis of W so $T \underset{\text{row}}{\sim} I_m$ (T vectors written as col's).

Then $\forall v \in V$, write $v = \sum_{j=1}^{n} a_j v_j$ and check

$${}_T[L]_S [v]_S = {}_T[L]_S \left[\sum_{j=1}^{n} a_j v_j\right]_S = \sum_{j=1}^{n} a_j \, {}_T[L]_S [v_j]_S =$$

$$\sum_{j=1}^{n} a_j [L(v_j)]_T = \left[\sum_{j=1}^{n} a_j L(v_j)\right]_T = \left[L\left(\sum_{j=1}^{n} a_j v_j\right)\right]_T$$

$$= [L(v)]_T. \quad \square$$

Def. The matrix ${}_T[L]_S \in F_n^m$ is called the matrix representing L from S to T (w.r.t. S and T).

(Any $L: V \to W$ has been related to an $L_A: F^n \to F^m$.)

Example: Let $L : \mathbb{R}^2_2 \to \mathbb{R}^3$ be the linear map

$$L\begin{bmatrix} a & b \\ c & d \end{bmatrix} = \begin{bmatrix} -a+2b+c-d \\ a+b+2c+d \\ a-b+c-d \end{bmatrix}.$$

Let $S = \{v_1, v_2, v_3, v_4\}$ be the std. basis of $\mathbb{R}^2_2$, $T = \{w_1, w_2, w_3\}$ be the std. basis of $\mathbb{R}^3$,

$S' = \left\{ v_1' = \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}, v_2' = \begin{bmatrix} 1 & 1 \\ 1 & 0 \end{bmatrix}, v_3' = \begin{bmatrix} 1 & 1 \\ 0 & 0 \end{bmatrix}, v_4' = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} \right\}$ another basis of $\mathbb{R}^2_2$ and

$T' = \left\{ w_1' = \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}, w_2' = \begin{bmatrix} 0 \\ 1 \\ 2 \end{bmatrix}, w_3' = \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix} \right\}$ another basis of $\mathbb{R}^3$.

① Find ${}_T[L]_S$ ② Find ${}_{T'}[L]_{S'}$

③ Find ${}_{S'}P_S, {}_SP_{S'} \in \mathbb{R}^4_4$ transition matrices.

④ Find ${}_{T'}Q_T, {}_TQ_{T'} \in \mathbb{R}^3_3$ transition matrices.

⑤ What are the relations among these matrices?

One Diagram to Rule them All:

$$
\begin{tikzcd}
 & \mathbb{R}^2_2 \arrow[r, "L"] \arrow[d, "{[\ ]_S}"] \arrow[dl, "{[\ ]_{S'}}"'] & \mathbb{R}^3 \arrow[d, "{[\ ]_T}"'] \arrow[dr, "{[\ ]_{T'}}"] & \\
 & \mathbb{R}^4 \arrow[r, "{{}_T[L]_S}", "3\times 4"'] \arrow[dl, shift left, "{{}_{S'}P_S}"] & \mathbb{R}^3 \arrow[dr, shift left, "{{}_{T'}Q_T}", "3\times 3"'] & \\
\mathbb{R}^4 \arrow[rrr, "{{}_{T'}[L]_{S'}}", "3\times 4"'] \arrow[ur, shift left, "{{}_S P_{S'}}", "4\times 4"'] & & & \mathbb{R}^3 \arrow[ul, shift left, "{{}_T Q_{T'}}"]
\end{tikzcd}
$$

$${}_T[L]_S\,[v]_S = [L(v)]_T \qquad {}_S P_{S'}\,[v]_{S'} = [v]_S$$

$${}_{T'}[L]_{S'}\,[v]_{S'} = [L(v)]_{T'} \qquad {}_{T'}Q_T\,[w]_T = [w]_{T'}$$

① To find ${}_T[L]_S$ first compute $L(S)$:

$$L(v_1)=\begin{bmatrix}-1\\1\\1\end{bmatrix},\ L(v_2)=\begin{bmatrix}2\\1\\-1\end{bmatrix},\ L(v_3)=\begin{bmatrix}1\\2\\1\end{bmatrix},\ L(v_4)=\begin{bmatrix}-1\\1\\-1\end{bmatrix}$$

Solve

$$[T\,|\,L(S)]=\left[\begin{array}{ccc|cccc}1&0&0&-1&2&1&-1\\0&1&0&1&1&2&1\\0&0&1&1&-1&1&-1\end{array}\right]$$

already RREF so

$${}_T[L]_S=\begin{bmatrix}-1&2&1&-1\\1&1&2&1\\1&-1&1&-1\end{bmatrix}$$

<u>Check</u>:

$${}_T[L]_S\,[v]_S=\begin{bmatrix}-1&2&1&-1\\1&1&2&1\\1&-1&1&-1\end{bmatrix}\begin{bmatrix}a\\b\\c\\d\end{bmatrix}=\begin{bmatrix}-a+2b+c-d\\a+b+2c+d\\a-b+c-d\end{bmatrix}$$

$= [L(v)]_T$ since for

$$w=\begin{bmatrix}x\\y\\z\end{bmatrix}\in\mathbb{R}^3,\ w=x\underbrace{\begin{bmatrix}1\\0\\0\end{bmatrix}}_{w_1}+y\underbrace{\begin{bmatrix}0\\1\\0\end{bmatrix}}_{w_2}+z\underbrace{\begin{bmatrix}0\\0\\1\end{bmatrix}}_{w_3}\ \text{so}\ [w]_T=\begin{bmatrix}x\\y\\z\end{bmatrix}=w$$

<u>Note</u>: ${}_T[L]_S$ could be "read off" the formula for $L(v)$ since S and T were std. bases.

② To find ${}_{T'}[L]_{S'}$ "directly" from the algorithm (without using transition matrices), first find $L(S')$:

$$L(v_1') = \begin{bmatrix} 1 \\ 5 \\ 0 \end{bmatrix},\ L(v_2') = \begin{bmatrix} 2 \\ 4 \\ 1 \end{bmatrix},\ L(v_3') = \begin{bmatrix} 1 \\ 2 \\ 0 \end{bmatrix},\ L(v_4') = \begin{bmatrix} -1 \\ 1 \\ 1 \end{bmatrix}$$

Solve

$$[T' \mid L(S')] = \left[\begin{array}{ccc|cccc} 1 & 0 & 1 & 1 & 2 & 1 & -1 \\ 2 & 1 & 1 & 5 & 4 & 2 & 1 \\ 3 & 2 & 0 & 0 & 1 & 0 & 1 \end{array}\right] \to \left[\begin{array}{ccc|cccc} 1 & 0 & 1 & 1 & 2 & 1 & -1 \\ 0 & 1 & -1 & 3 & 0 & 0 & 3 \\ 0 & 2 & -3 & -3 & -5 & -3 & 4 \end{array}\right]$$

$-2 \quad 0 \quad -2 \quad -2 \quad -4 \quad -2 \quad 2$ $\qquad$ $0 \quad -2 \quad 2 \quad -6 \quad 0 \quad 0 \quad -6$

$-3 \quad 0 \quad -3 \quad -3 \quad -6 \quad -3 \quad 3$

$$\to \left[\begin{array}{ccc|cccc} 1 & 0 & 1 & 1 & 2 & 1 & -1 \\ 0 & 1 & -1 & 3 & 0 & 0 & 3 \\ 0 & 0 & -1 & -9 & -5 & -3 & -2 \end{array}\right] \to \left[\begin{array}{ccc|cccc} 1 & 0 & 0 & -8 & -3 & -2 & -3 \\ 0 & 1 & 0 & 12 & 5 & 3 & 5 \\ 0 & 0 & 1 & 9 & 5 & 3 & 2 \end{array}\right]$$

gives ${}_{T'}[L]_{S'} \in \mathbb{R}^3_4$.

To check $_{T'}[L]_{S'}[v]_{S'} = [L(v)]_{T'}$ we need to find $[v]_{S'}$: Solve $\sum_{j=1}^{4} x_j \cdot v_j' = v$ by row reducing

$$\underset{\text{"as columns"}}{\left[\begin{array}{cccc|c} 1&1&1&1&a\\ 1&1&1&0&b\\ 1&1&0&0&c\\ 1&0&0&0&d \end{array}\right]} \to \left[\begin{array}{cccc|c} 1&0&0&0&d\\ 0&1&0&0&c-d\\ 0&0&1&0&b-c\\ 0&0&0&1&a-b \end{array}\right]$$

S' "as columns" v $\qquad I_4 \quad [v]_{S'}$

was done on page ~~[illegible]~~ (but we called that basis T on p. ~~[illegible]~~).

$$\underbrace{\begin{bmatrix} -8 & -3 & -2 & -3 \\ 12 & 5 & 3 & 5 \\ 9 & 5 & 3 & 2 \end{bmatrix}}_{_{T'}[L]_{S'}} \underbrace{\begin{bmatrix} d \\ c-d \\ b-c \\ a-b \end{bmatrix}}_{[v]_{S'}} = \begin{bmatrix} -3a+b-c-5d \\ 5a-2b+2c+7d \\ 2a+b+2c+4d \end{bmatrix} \stackrel{?}{=} [L(v)]_{T'},$$

The easiest check is: $L(v) \stackrel{?}{=}$

$$(-3a+b-c-5d)\underbrace{\begin{bmatrix}1\\2\\3\end{bmatrix}}_{w_1'} + (5a-2b+2c+7d)\underbrace{\begin{bmatrix}0\\1\\2\end{bmatrix}}_{w_2'} + (2a+b+2c+4d)\underbrace{\begin{bmatrix}1\\1\\0\end{bmatrix}}_{w_3'} =$$

$$\begin{bmatrix} -a+2b+c-d \\ a+b+2c+d \\ a-b+c-d \end{bmatrix} = L(v)$$ so that was $[L(v)]_{T'}$.

③ We already found on p. ~~###~~ 138 that

$${}_S P_{S'} = \begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 0 \\ 1 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{bmatrix} \text{ and } {}_{S'}P_S = \begin{bmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & -1 \\ 0 & 1 & -1 & 0 \\ 1 & -1 & 0 & 0 \end{bmatrix} = {}_S P_{S'}^{\,-1}.$$

④ $$\left[\begin{array}{ccc|ccc} 1 & 0 & 0 & 1 & 0 & 1 \\ 0 & 1 & 0 & 2 & 1 & 1 \\ 0 & 0 & 1 & 3 & 2 & 0 \end{array}\right]$$ is in RREF so ${}_T Q_{T'} = \begin{bmatrix} 1 & 0 & 1 \\ 2 & 1 & 1 \\ 3 & 2 & 0 \end{bmatrix}$

$T \qquad T'$

$$\left[\begin{array}{ccc|ccc} 1 & 0 & 1 & 1 & 0 & 0 \\ 2 & 1 & 1 & 0 & 1 & 0 \\ 3 & 2 & 0 & 0 & 0 & 1 \end{array}\right] \xrightarrow{\text{r.r.}} \left[\begin{array}{ccc|ccc} 1 & 0 & 1 & 1 & 0 & 0 \\ 0 & 1 & -1 & -2 & 1 & 0 \\ 0 & 2 & -3 & -3 & 0 & 1 \end{array}\right] \to \left[\begin{array}{ccc|ccc} 1 & 0 & 1 & 1 & 0 & 0 \\ 0 & 1 & -1 & -2 & 1 & 0 \\ 0 & 0 & -1 & 1 & -2 & 1 \end{array}\right] \to$$

$T' \qquad T$

$0 \; -2 \; 2 \; 4 \; -2 \; 0$

$-2 \; 0 \; -2 \; -2 \; 0 \; 0$

$-3 \; 0 \; -3 \; -3 \; 0 \; 0$

$$\left[\begin{array}{ccc|ccc} 1 & 0 & 0 & 2 & -2 & 1 \\ 0 & 1 & 0 & -3 & 3 & -1 \\ 0 & 0 & 1 & -1 & 2 & -1 \end{array}\right] \quad {}_{T'}Q_T$$

⑤ The relations among transition mat's $\underline{|147}$ are known: ${}_{S}P_{S'}^{-1} = {}_{S'}P_{S}$ and ${}_{T}Q_{T'}^{-1} = {}_{T'}Q_{T}$.

The bottom rectangle of the diagram says:

$$\underset{3\times 4}{{}_{T'}[L]_{S'}} = \underset{3\times 3}{{}_{T'}Q_{T}} \; \underset{3\times 4}{{}_{T}[L]_{S}} \; \underset{4\times 4}{{}_{S}P_{S'}}$$

so let's check:

$$\begin{bmatrix} 2 & -2 & 1 \\ -3 & 3 & -1 \\ -1 & 2 & -1 \end{bmatrix} \begin{bmatrix} -1 & 2 & 1 & -1 \\ 1 & 1 & 2 & 1 \\ 1 & -1 & 1 & -1 \end{bmatrix} \begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 0 \\ 1 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{bmatrix}$$

$$= \begin{bmatrix} -3 & 1 & -1 & -5 \\ 5 & -2 & 2 & 7 \\ 2 & 1 & 2 & 4 \end{bmatrix} \begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 0 \\ 1 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{bmatrix} = \begin{bmatrix} -8 & -3 & -2 & -3 \\ 12 & 5 & 3 & 5 \\ 9 & 5 & 3 & 2 \end{bmatrix} = {}_{T'}[L]_{S'}$$

<u>checks!</u>

This example illustrates a general Theorem.