Th. Let $L: V \to W$, $S = \{v_1, \dots, v_n\}$ and $S' = \{v_1', \dots, v_n'\}$ bases of V, $T = \{w_1, \dots, w_m\}$ and $T' = \{w_1', \dots, w_m'\}$ bases of W, ${}_T[L]_S \in F_n^m$ the matrix rep'ing L from S to T, ${}_{T'}[L]_{S'} \in F_n^m$ the matrix rep'ing L from S' to T', and let

${}_S P_{S'} \in F_n^n$ be the transition matrix from S' to S,
${}_{T'} Q_T \in F_m^m$ " " " " T to T'.

$$\text{Then } {}_{T'}[L]_{S'} = \underset{m\times m}{{}_{T'}Q_T}\ \underset{m\times n}{{}_T[L]_S}\ \underset{n\times n}{{}_S P_{S'}}.$$

Pf. We have the diagram and equations defining these matrices as follows:

$$
\begin{tikzcd}
 & V \arrow[r, "L"] \arrow[d, "{[\,]_S}"] \arrow[dl, "{[\,]_{S'}}"'] & W \arrow[d, "{[\,]_T}"'] \arrow[dr, "{[\,]_{T'}}"] & \\
 & F^n \arrow[r, "{_T[L]_S}"] & F^m \arrow[dr, "{_{T'}Q_T}"] & \\
F^n \arrow[ur, "{_SP_{S'}}"] \arrow[rrr, "{_{T'}[L]_{S'}}"] & & & F^m
\end{tikzcd}
$$

Equations

① $_T[L]_S[v]_S = [L(v)]_T$

② $_{T'}[L]_{S'}[v]_{S'} = [L(v)]_{T'}$

③ $_SP_{S'}[v]_{S'} = [v]_S$

④ $_{T'}Q_T[w]_T = [w]_{T'}$

So $\forall v \in V$,

$$_{T'}[L]_{S'}[v]_{S'} \overset{②}{=} [L(v)]_{T'} \overset{④}{=} {_{T'}Q_T}[L(v)]_T \overset{①}{=} {_{T'}Q_T}\,{_T[L]_S}[v]_S$$

$$\overset{③}{=} {_{T'}Q_T}\,{_T[L]_S}\,{_SP_{S'}}[v]_{S'}.$$

Using $v = v_j' \in S'$, have $[v_j']_{S'} = e_j$ std. basis vector in F^n, get for $1 \le j \le n$, $\mathrm{Col}_j({_{T'}[L]_{S'}}) = \mathrm{Col}_j({_{T'}Q_T}\,{_T[L]_S}\,{_SP_{S'}})$, so the whole matrices are equal. □

<u>Question</u>: If you had any choice of bases S' in V and T' in W, what would be the "nicest" matrix ${}_{T'}[L]_{S'}$ rep'ing L?

<u>Alternative</u>: If you had any choice of invertible matrices $Q \in F^m_m$ and $P \in F^n_n$ what is the "nicest" matrix $B = QAP$ you could get from a given $A \in F^m_n$?

<u>Answer</u>: We can interpret Q as doing row op's to A, and P as doing col. op's to QA. The "best" QA is in RREF. Allowing col. op's on QA can give Block Identity Form (BIF)

$$B = QAP = \left[\begin{array}{c|c} I_r & 0 \\ \hline 0 & 0 \end{array}\right] \text{ where } r = \operatorname{rank}(A).$$

From the point of view of linear maps and bases and matrices rep'ing maps, we can see it as follows:

For $L : V \to W$ let $K = \operatorname{Ker}(L)$ have basis $\{k_1, \dots, k_{n-r}\}$ for $r = \operatorname{rank}(A) = \dim(\operatorname{Range}(L))$

Extend this to a basis of V but add the new vectors at the beginning of the list; get $S' = \{v_1', \dots, v_r', v_{r+1}' = k_1, \dots, v_n' = k_{n-r}\}$. Then

$$L(S') = \{w_1' = L(v_1'), \dots, w_r' = L(v_r'), \underbrace{\theta_W = L(k_1), \dots, \theta_W = L(k_{n-r})}_{\text{redundant}}\}$$

get a basis for Range(L) to be $\{w_1', \dots, w_r'\}$ and can extend it to a basis T' for W, $T' = \{w_1', \dots, w_r', \dots w_m'\}$ any way you like.

To find ${}_{T'}[L]_{S'} = B$, find $[L(v_j')]_{T'} = Col_j(B)$.

for $1 \le j \le r$, $L(v_j') = w_j'$ so $[L(v_j')]_{T'} = e_j \in F^m$

but for $r < j \le n$, $L(v_j') = \theta_W$ since $v_j' = k_{j-r} \in K$

so $[L(v_j')]_{T'} = O_1^m \in F^m$.

This gives ${}_{T'}[L]_{S'} = B = \left[\begin{array}{c|c} I_r & 0 \\ \hline 0 & 0 \end{array}\right] \in F_n^m$ in BIF.

Th: Let $L : V \to W$ for $\dim(V) = n < \infty$. Then $\dim(V) = \dim(Ker(L)) + \dim(Range(L))$.

Pf. Let $B = \{v_1, \ldots, v_k\}$ be a basis for $Ker(L)$, $k \ge 0$.

Extend B to a basis $S = \{v_1, \ldots, v_k, \ldots, v_n\}$ for all of V.

Then $Range(L) = \left\{ \sum_{j=1}^{n} a_j L(v_j) \in W \mid a_j \in F \right\} = \langle L(S) \rangle$, but

$L(B) = \{L(v_1), \ldots, L(v_k)\} = \{\theta_W, \ldots, \theta_W\}$ so the vectors $L(v_j) = \theta_W$ for $1 \le j \le k$ are redundant in $L(S)$ and $\text{Range}(L) = \langle L(v_{k+1}), \ldots, L(v_n) \rangle$ is spanned by just $n-k$ vectors. <u>Claim</u>: These $n-k$ vectors are indep.

<u>Pf</u>. If $\sum_{j=k+1}^{n} a_j L(v_j) = \theta_W$ then $L\left(\sum_{j=k+1}^{n} a_j v_j\right) = \theta_W$

so $\sum_{j=k+1}^{n} a_j v_j \in \text{Ker}(L) = \langle B \rangle$ so

$= \sum_{j=1}^{k} b_j v_j$ giving $\theta_V = \sum_{j=1}^{k} -b_j v_j + \sum_{j=k+1}^{n} a_j v_j$ a

dep. relation on basis S, so all its coeff's are 0. $\dim(\text{Range}(L)) = n-k$ and $\dim(\text{Ker}(L)) = k$ so they add up to $n = \dim(V)$. □

Applications of the dim formula (rank/nullity)

Def. $Nul(A) = Ker(L_A)$ for $A \in F^m_n$

$Nullity(A) = \dim(Nul(A)) = n - r$ if $rank(A) = r$

$= \dim(Range(L_A))$.

Th: Suppose $L : V \to W$, $\dim(V) = n$, $\dim(W) = m$.

(a) If $n > m$ then $\dim(Ker(L)) \geq n - m > 0$.

(b) If $n < m$ then $\dim(Range(L)) \leq n < m$.

(c) If $n = m$ then L inj $\Leftrightarrow$ L surj

Pf. (a) $n = \dim(Ker(L)) + \dim(Range(L))$ but

$0 \leq \dim(Range(L)) \leq m$ so

$\dim(Ker(L)) = n - \dim(Range(L)) \geq n - m > 0$ if $n > m$.

(b) $\dim(Ker(L)) \geq 0$ so $n \geq \dim(Range(L))$ is always true.

(c) Suppose $n = m$. ($\Rightarrow$) If L is inj then $\dim(\operatorname{Ker}(L)) = 0$ so $m = n = \dim(\operatorname{Range}(L))$ gives $\dim(W) = \dim(\operatorname{Range}(L))$. For W fin.dim'l, get $W = \operatorname{Range}(L)$ so L is <u>surj</u>.

($\Leftarrow$) Suppose L is <u>surj</u>. So $\operatorname{Range}(L) = W$ so $\dim(\operatorname{Range}(L)) = \dim(W) = m = n$ so $n = \dim(\operatorname{Ker}(L)) + n$ gives $\dim(\operatorname{Ker}(L)) = 0$ so L is <u>inj</u>. $\square$

<u>Ex</u>: For $L : F_5^2 \to F^7$ what are all possible values of $\dim(\operatorname{Ker}(L))$?

$10 = \dim(F_5^2) = \dim(\operatorname{Ker}(L)) + \dim(\operatorname{Range}(L))$

Since $0 \le \dim(\operatorname{Range}(L)) \le 7$ we must have

$3 \le \dim(\operatorname{Ker}(L)) \le 10$.

(3) (Webwork #6)

$$a(-16-15x-9x^2)+b(7+7x+4x^2)+c(30+28x+17x^2)=0$$

$$\underbrace{(-16a+7b+30c)}_{=0}+\underbrace{(-15a+7b+28c)}_{=0}x+\underbrace{(-9a+4b+17c)}_{=0}x^2=0$$

$$\left[\begin{array}{ccc|c} +16 & -7 & -30 & 0 \\ -15 & 7 & 28 & 0 \\ -9 & 4 & 17 & 0 \end{array}\right] \to \left[\begin{array}{ccc|c} 1 & 0 & -2 & 0 \\ -15 & 7 & 28 & 0 \\ -9 & 4 & 17 & 0 \end{array}\right] \to \left[\begin{array}{ccc|c} 1 & 0 & -2 & 0 \\ 0 & 7 & -2 & 0 \\ 0 & 4 & -1 & 0 \end{array}\right] \to \left[\begin{array}{ccc|c} 1 & 0 & -2 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 4 & -1 & 0 \end{array}\right]$$

$9 \quad 0 \quad -18 \quad 0$

$15 \quad 0 \quad -30 \quad 0$

-2

$0 \quad -8 \quad 2 \quad 0$

$0 \quad -4 \quad 0 \quad 0$

$$\downarrow$$

$$\left[\begin{array}{ccc|c} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{array}\right] \begin{array}{l} a=0 \\ b=0 \\ c=0 \end{array}$$

---

(4)

$$\left[\begin{array}{ccc|c} 6 & 10 & 1 & 0 \\ 15 & 10 & 10 & 0 \\ 9 & -2 & 10 & 0 \end{array}\right] \to \left[\begin{array}{ccc|c} 6 & 10 & 1 & 0 \\ 3 & 2 & 2 & 0 \\ 9 & -2 & 10 & 0 \end{array}\right] \to \left[\begin{array}{ccc|c} 3 & 2 & 2 & 0 \\ 0 & 6 & -3 & 0 \\ 0 & -8 & 4 & \end{array}\right] \to \left[\begin{array}{ccc|c} 3 & 2 & 2 & 0 \\ 0 & 2 & -1 & 0 \\ 0 & 2 & -1 & 0 \end{array}\right] \to \left[\begin{array}{ccc|c} 3 & 0 & 3 & 0 \\ 0 & 2 & -1 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

$-9 \quad -6 \quad -6 \quad 0$

$-6 \quad -4 \quad -4 \quad 0$

$$\begin{bmatrix} a \\ b \\ c \end{bmatrix} = \begin{bmatrix} -2 \\ 1 \\ 2 \end{bmatrix}$$

$a = -c$
$b = \frac{1}{2}c$
$c \in \mathbb{R}$ free

$$\left[\begin{array}{ccc|c} 1 & 0 & 1 & 0 \\ 0 & 1 & -\frac{1}{2} & 0 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

(5) (1) Indep

(2) Dep

(3) Indep

(4) Dep.

$$\longrightarrow \left[\begin{array}{ccc|c} 1 & -2 & -2 & 0 \\ -2 & 1 & 4 & 0 \\ 4 & 1 & 1 & 0 \\ 3 & 4 & 6 & 0 \end{array}\right] \to \left[\begin{array}{ccc|c} 1 & -2 & -2 & 0 \\ 0 & -3 & 0 & 0 \\ 0 & 9 & 9 & 0 \\ 0 & 10 & 6 & 0 \end{array}\right] \to \left[\begin{array}{ccc|c} 1 & -2 & -2 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 5 & 3 & 0 \end{array}\right] \to \left[\begin{array}{ccc|c} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

$2 \ -4 \ -4 \ 0$

$-4 \ 8 \ 8 \ 0$

$-3 \ 6 \ 6 \ 0$

So $x_1 = 0$, $x_2 = 0$, $x_3 = 0$

Set is Indep.

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(6) $S = \{r, u, d\}$ indep. $\quad x = 4r + 3u + 4d \in \langle S \rangle$

Is $T = \{r, u, x\}$ indep?

$\{r, u\}$ is indep so is $x \in \langle r, u \rangle$? No, needs d

$a, b, c \in F$

If $ar + bu + cx = \theta = ar + bu + c(4r + 3u + 4d)$

$= (a + 4c)r + (b + 3c)u + 4c\,d$ Then S indep $\Rightarrow$

$a + 4c = 0$

$b + 3c = 0$

$4c = 0 \Rightarrow c = 0 \Rightarrow b = 0 \Rightarrow a = 0$

(7) $Sym = \{A \in \mathbb{R}^3_3 \mid A^T = A\} = \left\{ \begin{bmatrix} a & b & c \\ b & d & e \\ c & e & f \end{bmatrix} \in \mathbb{R}^3_3 \;\middle|\; a,b,c,d,e,f \in \mathbb{R} \right\}$

If $A^T = A$ and $B^T = B$ then $(A+B)^T = A^T + B^T = A + B$

$(rA)^T = r(A^T) = rA \quad (0^3_3)^T = 0^3_3$

$$= \left\{ a\underbrace{\begin{bmatrix} 1&0&0\\0&0&0\\0&0&0\end{bmatrix}}_{v_1} + b\underbrace{\begin{bmatrix} 0&1&0\\1&0&0\\0&0&0\end{bmatrix}}_{v_2} + c\underbrace{\begin{bmatrix} 0&0&1\\0&0&0\\1&0&0\end{bmatrix}}_{v_3} + d\underbrace{\begin{bmatrix} 0&0&0\\0&1&0\\0&0&0\end{bmatrix}}_{v_4} + e\underbrace{\begin{bmatrix} 0&0&0\\0&0&1\\0&1&0\end{bmatrix}}_{v_5} + f\underbrace{\begin{bmatrix} 0&0&0\\0&0&0\\0&0&1\end{bmatrix}}_{v_6} \in \mathbb{R}^3 \;\middle|\; a,b,c,d,e,f \in \mathbb{R} \right\}$$

$= \langle v_1, \dots, v_6 \rangle$ and $\{v_1, \dots, v_6\}$ is indep. so $\dim(Sym) = 6$.

$$W = \{A \in \mathbb{R}^7_7 \mid Tr(A) = 0\} = \left\{ A = [a_{ij}] \in \mathbb{R}^7_7 \;\middle|\; \overbrace{\sum_{i=1}^{7} a_{ii}}^{Tr(A)} = 0 \right\} \quad \dim(W) = 48$$

One linear condition on entries.

$\dim(\mathbb{R}^7_7) = 49$.

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(8) $P_2 = \{a_0 + a_1 x + a_2 x^2 \mid a_0, a_1, a_2 \in \mathbb{R}\}$ $\quad H = \langle \underbrace{x^2 - x}_{p_1(x)}, \underbrace{-(2x+1)}_{p_2(x)}, \underbrace{x^2 + 7x + 4}_{p_3(x)} \rangle$

Note $p_3(x) - p_1(x) = 8x + 4 = 4(2x+1) = -4p_2(x)$

so $p_1(x) - 4p_2(x) - p_3(x) = 0$

$\dim(H) = 2$, $\{p_1, p_2, p_3\}$ not a basis for H

$\{x^2 - x, 2x+1\}$ is a basis for H