<u>Def</u>. For $A=[a_{ij}]\in F^m_n$ define the row-space of A to be the span of the rows of A,

$$\text{row-sp}(A)=Row(A)=\langle Row_1(A),\dots,Row_m(A)\rangle,$$

which is a subspace of F_n since each $Row_i(A)\in F_n$.

Also define the column-space of A to be

$$\text{Col-sp}(A)=Col(A)=\langle Col_1(A),\dots,Col_n(A)\rangle\le F^m.$$

What do we already know about these?

Know: $L_A:F^n\to F^m$ has $Range(L_A)=$

$$\{AX\in F^m \mid X\in F^n\}=\left\{\sum_{j=1}^n x_j Col_j(A)\in F^m \,\middle|\, x_j\in F\right\}$$

$$=\langle Col_1(A),\dots,Col_n(A)\rangle = Col(A) \text{ so}$$

$$\dim(Range(L_A))=\dim(Col(A))=rank(A).$$

<u>Th.</u> If $A \underset{\text{row}}{\sim} B$ then $\mathrm{Row}(A) = \mathrm{Row}(B)$.

<u>Pf.</u> It is enough to check this when B is obtained from A by just one elem. row op. Let $S = \{R_1, \dots, R_m\}$ be the set of row vectors of A. If B came from A by switching rows i and j, then $\mathrm{Row}(B) = \langle R_1, \dots, \underset{(i^{th}}{R_j}, \dots, \underset{j^{th}\text{ place})}{R_i}, \dots, R_m \rangle = \mathrm{Row}(A)$ since the span of the two lists is the same subspace.

If B came from A by multiplying row$_k$ by $0 \neq c \in F$, then $\mathrm{Row}(B) = \langle R_1, \dots, cR_k, \dots, R_m \rangle = \mathrm{Row}(A)$ since $\left\{ \sum_{i=1}^{m} x_i R_i \,\middle|\, x_i \in F \right\} = \left\{ x_1 R_1 + \dots + x_k R_k + \dots + x_m R_m \,\middle|\, x_i \in F \right\}$

$= \left\{ y_1 R_1 + \dots + y_k (cR_k) + \dots + y_m R_m \,\middle|\, y_i \in F \right\}$.

If B came from A by an elem. adder row op. we can use switchers to make it $cR_1+R_2 \to R_2$ for convenience. Then $\mathrm{Row}(B) = \langle R_1, cR_1+R_2, R_3, \ldots, R_m \rangle = \langle R_1, R_2, \ldots, R_m \rangle = \mathrm{Row}(A)$

since $cR_1+R_2 \in \langle R_1, R_2 \rangle \leq \mathrm{Row}(A)$ so $\mathrm{Row}(B) \leq \mathrm{Row}(A)$

and $R_2 \in \langle R_1, cR_1+R_2 \rangle \leq \mathrm{Row}(B)$ since $R_2 = -cR_1 + cR_1 + R_2$

so $\mathrm{Row}(A) \leq \mathrm{Row}(B)$.

In each case $\mathrm{Row}(A) = \mathrm{Row}(B)$. If B came from A by a sequence of elem. row ops., say

$$A \underset{\text{row}}{\sim} B_1 \underset{\text{row}}{\sim} B_2 \underset{\text{row}}{\sim} \cdots \underset{\text{row}}{\sim} B_t = B$$

then

$$\mathrm{Row}(A) = \mathrm{Row}(B_1) = \mathrm{Row}(B_2) = \cdots = \mathrm{Row}(B_t) = \mathrm{Row}(B). \square$$

Cor. If $A \underset{\text{row}}{\sim} B$ and B is in RREF with $r = \text{rank}(A)$ non-zero rows, $Row_1(B), \ldots, Row_r(B)$, then $T = \{Row_1(B), \ldots, Row_r(B)\}$ is a basis of $Row(A)$.

Pf. $Row(A) = Row(B)$ from the last theorem. $Row(B)$ is spanned by the non-zero rows of B since any zero rows are redundant. Each $Row_i(B)$, $1 \le i \le r$, has a leading 1 in some column k_i and all other entries in those pivot columns are 0 in B. To see that T is indep. look at

$$0'_n = \sum_{i=1}^{r} c_i \, Row_i(B) = \begin{array}{c} \left[\ldots, c_1, \ldots, c_2, \ldots, c_r, \ldots\right] \\ \text{col: } k_1 \quad k_2 \quad k_r \end{array}$$

and see that the coefficients $c_1, \ldots, c_r$ occur separately in columns $k_1, \ldots, k_r$, so each $c_i = 0$. $\square$

Def. For $A \in F_n^m$ define
row-rank$(A) = \dim(\mathrm{Row}(A))$ and
col-rank$(A) = \dim(\mathrm{Col}(A))$.

Th: For $A \in F_n^m$ we have row-rank$(A) =$ col-rank$(A) =$ rank$(A) = r$ is the number of leading 1's in the RREF $B \underset{\text{row}}{\sim} A$, that is, the number of pivot columns, and it equals $\dim(\mathrm{Range}(L_A))$.

Application: If $A \in F_5^2$ what are the possible values of col-rank(A)?

Solution: col-rank$(A) =$ rank$(A) \le \mathrm{Min}(2,5) = 2$ so $0 \le$ col-rank$(A) \le 2$. If $A \neq 0_5^2$, $1 \le$ rank$(A) \le 2$.

Connection with the dimension formula $\dim(V) = \dim \operatorname{Ker}(L) + \dim \operatorname{Range}(L)$: For map $L_A : F^5 \to F^2$ have $5 = \dim(F^5) = \dim \operatorname{Ker}(L_A) + \dim \operatorname{Range}(L_A)$
$\operatorname{rank}(A) = \dim(\operatorname{Col}(A)) = \dim(\operatorname{Range}(L_A)) \leq 2$ so
$3 \leq \dim \operatorname{Ker}(L_A) \leq 5$ and $\dim \operatorname{Ker}(L_A) = 5 - \operatorname{rank}(A)$.
$\operatorname{rank}(A) = 0$ iff $A = O_5^2$ iff $\operatorname{Ker}(L_A) = F^5$.

<u>Def</u>. For vector spaces V and W we say V is <u>isomorphic</u> to W when $\exists L : V \to W$ such that L is a bijective lin. map (isomorphism).
Write this relation $V \cong W$ or $V \simeq W$.

<u>Th</u>: The relation $\simeq$ is symmetric, reflexive and transitive (an equivalence relation).

Th: Let V and W be finite dimensional vector spaces with $\dim(V)=n$ and $\dim(W)=m$. Then $V \simeq W$ iff $n=m$.

Pf. Let $S=\{v_1,\dots,v_n\}$ be a basis of V and $T=\{w_1,\dots,w_m\}$ be a basis of W. Suppose $V \simeq W$ so $\exists L: V \to W$ bijective. So $n=\dim(V)=\dim \mathrm{Ker}(L) + \dim \mathrm{Range}(L) = 0+\dim(W)=m$ since L inj. gives $\dim \mathrm{Ker}(L)=0$ and L surj. gives $\mathrm{Range}(L)=W$.

Conversely, suppose $n=m$, so using coordinate maps we have bijections $[\cdot]_S : V \to F^n$, $[\cdot]_T : W \to F^n$ and composing $[\cdot]_T^{-1} \circ [\cdot]_S : V \to W$ gives a bijection from V to W so $V \simeq W$. $\square$

$$\begin{tikzcd} & V \arrow[dl, leftrightarrow, "{[\,]_S}"'] & \\ F^n & & W \arrow[ll, leftrightarrow, "{[\,]_T}"'] \end{tikzcd}$$