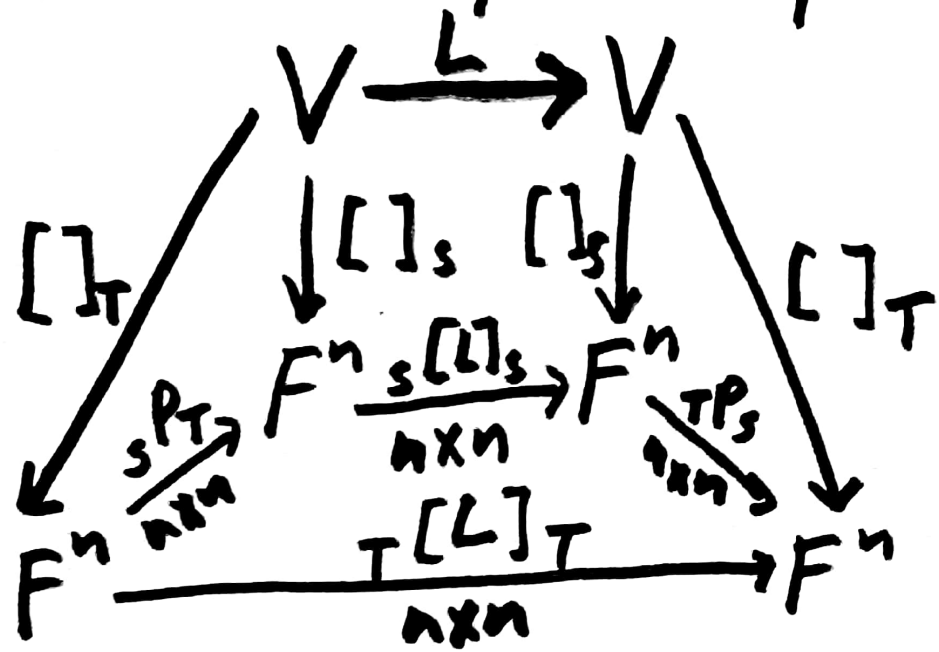


Let's study the special case of $L \in \text{End}(V)$ 163



Let $S = \{v_1, \dots, v_n\}$ and $T = \{w_1, \dots, w_n\}$ be bases of V ,

${}_T P_S, {}_S P_T \in F^n$ the transition matrices s.t. ${}_T P_S = {}_S P_T^{-1}$. Then we have

from the general Theorem, but the relation between $B = {}_T[L]_T$ and $A = {}_S[L]_S$

$${}_T[L]_T = {}_T P_S {}_S[L]_S {}_S P_T$$

$$B = P^{-1} A P$$

is that. Det. Say $A, B \in F^n$ are similar when $B = P^{-1} A P$ for some invertible $P \in F^n$. Write: $A \sim B$

Th. \sim is an equivalence relation on F_n [164
(reflexive, symmetric, transitive).

Questions: ① If you could choose any basis T of
 V , what choice would give "nice" ${}_T[L]_T$?

② If you could pick any invertible $P \in F_n$,
what choice would give "nice" $B = P^{-1}AP$?

Answers: First try to find T or $P = {}_s P_T$
making ${}_T[L]_T$ and B diagonal, that is,

$$D = \begin{bmatrix} \lambda_1 & & 0 \\ & \lambda_2 & \\ 0 & & \ddots \\ & & & \lambda_n \end{bmatrix} \text{ for some scalars } \lambda_1, \dots, \lambda_n \in F$$

(not necessarily distinct).

That is the "diagonalization" topic in elem. lin.
alg. courses. We will ~~cover~~ it.

If L or A cannot be diagonalized, 165
what are the next best options?

Two options: ① Jordan Canonical Form,
② Rational Canonical Form.

We ~~would~~ study both of these in Math 404

Th. Let $L: V \rightarrow V$, $S = \{v_1, \dots, v_n\}$ a basis of V ,
 $A = {}_S[L]_S \in F^n$. For basis $T = \{w_1, \dots, w_n\}$ of V ,
 ${}_T[L]_T = D = \text{diag}(\lambda_1, \dots, \lambda_n)$ iff $L(w_j) = \lambda_j \cdot w_j$
for $1 \leq j \leq n$.

Pf. $\text{Col}_j({}_T[L]_T) = [L(w_j)]_T = \text{Col}_j(D)$ iff
 $L(w_j) = 0w_1 + \dots + \lambda_j w_j + 0w_{j+1} + \dots + 0w_n$ by def. of
coordinates. \square

Def. For $L: V \rightarrow V$, say $0 \neq v \in V$ is an eigenvector for L with eigenvalue $\lambda \in F$ when $L(v) = \lambda v$. For $A \in F^n$, say $0 \neq x \in F^n$ is an e. vect. for A with e. val. $\lambda \in F$ when $Ax = \lambda x$.

Def. Say $L: V \rightarrow V$ is diagonalizable when there is some basis T of V for which ${}_T[L]_T = D$ is diagonal. Say $A \in F^n$ is diag. able when $\exists P \in F^n$ invertible s.t. $P^{-1}AP = D$ is diagonal.

Th. $L: V \rightarrow V$ is diag. able iff $\exists w_1, \dots, w_n \in V$ s.t. $T = \{w_1, \dots, w_n\}$ is a basis of V with each $w_j \in T$ an e. vector for L with some e. value λ_j .

Goals ① Find an algorithm to find such an 167
e.basis T , if possible, or prove one does not
exist for the given $L: V \rightarrow V$.

② Find an algorithm to get an invertible
 $P \in F_n^n$ s.t. $P^{-1}AP = D$ is diag. if possible for
the given $A \in F_n^n$.

Def. For $L: V \rightarrow V$, $\lambda \in F$, let $L_\lambda = \{v \in V \mid L(v) = \lambda v\}$

If $L_\lambda \neq \{\emptyset\}$, call it the λ e-space for L .

For $A \in F_n^n$, $\lambda \in F$, let $A_\lambda = \{X \in F_n^n \mid AX = \lambda X\}$.

If $A_\lambda \neq \{0^n\}$ call it the λ e-space for A .

Note: $\forall \lambda \in F$, $L_\lambda \subseteq V$ and $A_\lambda \subseteq F_n^n$ since

$0_v \in L_\lambda$, $0^n \in A_\lambda$, and both are closed under $+$ and \cdot .

Pf. L_λ is closed under + since: If $v_1, v_2 \in L_\lambda$ then [167.1]

$$L(v_1) = \lambda v_1 \text{ and } L(v_2) = \lambda v_2 \text{ so } L(v_1 + v_2) = L(v_1) + L(v_2) \\ = \lambda v_1 + \lambda v_2 = \lambda(v_1 + v_2) \text{ so } v_1 + v_2 \in L_\lambda.$$

If $v \in L_\lambda$ and $c \in F$ then $L(v) = \lambda v$ so $L(cv) = cL(v)$
 $= c(\lambda v) = (c\lambda)v = (\lambda c)v = \lambda(cv)$ means $cv \in L_\lambda$.

A_λ is closed under + since: If $x_1, x_2 \in A_\lambda$ then
 $Ax_1 = \lambda x_1$ and $Ax_2 = \lambda x_2$ so $A(x_1 + x_2) = Ax_1 + Ax_2 =$

$$\lambda x_1 + \lambda x_2 = \lambda(x_1 + x_2) \text{ so } x_1 + x_2 \in A_\lambda.$$

If $x \in A_\lambda$ and $c \in F$ then $Ax = \lambda x$ so $A(cx) =$
 $c(Ax) = c(\lambda x) = (c\lambda)x = (\lambda c)x = \lambda(cx)$ so $cx \in A_\lambda$.

These results show that $L_\lambda \leq V$ and $A_\lambda \leq F^n$. \square

For most $\lambda \in F$, $L_\lambda = \{0_V\}$ and $A_\lambda = \{0^n\}$ 1168
means there are no e.vectors for L and A with
e.val. λ . How can we find the few special
 $\lambda \in F$ whose non-trivial e-spaces L_λ and A_λ
contain e.vectors that can contribute to T
or to P (as columns)?

Viewpoints on L_λ and A_λ which help:

$v \in L_\lambda$ iff $L(v) = \lambda v = \lambda I_V(v)$ iff
 $(L - \lambda I_V)(v) = 0_V$ where we understand

$L - \lambda I_V \in \text{End}(V) = \text{Lin}(V, V)$. So

$L_\lambda = \text{Ker}(L - \lambda I_V) = \text{Ker}(\lambda I_V - L)$

is an easy way to see that $L_\lambda \subseteq V$.

$$X \in A_\lambda \text{ iff } AX = \lambda X = \lambda I_n X \text{ iff } \underline{1169}$$
$$(A - \lambda I_n)(X) = 0^n \text{ so } A_\lambda = \text{Nul}(A - \lambda I_n)$$
$$= \text{Nul}(\lambda I_n - A) \subseteq F^n.$$

We know how to use row reduction to find $\text{Ker}(L - \lambda I_V)$ and $\text{Nul}(A - \lambda I_n)$ but the issue is to find those $\lambda \in F$ s.t. these subspaces are nontrivial, that is, s.t. there is at least one free variable in the solution set. We do that for $\text{Ker}(L - \lambda I_V)$ by using the matrix $A = {}_S[L]_S$ so that

$${}_S[L - \lambda I_V]_S = {}_S[L]_S - \lambda {}_S[I_V]_S = A - \lambda I_n.$$

Th: Let $L: V \rightarrow V$ and $A = {}_S[L]$, for some [170]
basis S of V . Then $\lambda \in F$ is an e. value for L
and for A iff $\text{rank}(A - \lambda I_n) < n$.

For small n it is not hard to check this
condition directly, as shown in the examples below.
But the usual method uses \det , and gives us
the important concept of characteristic poly.

Example: Let $L: \mathbb{R}^n \rightarrow \mathbb{R}^n$ be

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$$L \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix} = \begin{bmatrix} \lambda_1 x_1 \\ \lambda_2 x_2 \\ \vdots \\ \lambda_n x_n \end{bmatrix} = \begin{bmatrix} \lambda_1 & & 0 \\ & \lambda_2 & \\ 0 & & \ddots \\ & & & \lambda_n \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix}$$

then for $S = \{e_1, e_2, \dots, e_n\}$ std basis of \mathbb{R}^n ,
 ${}_S[L]_S = \begin{bmatrix} \lambda_1 & & 0 \\ & \ddots & \\ 0 & & \lambda_n \end{bmatrix} = D$ is diagonal so L is
diag-able. $L(e_j) = \lambda_j e_j$ for $1 \leq j \leq n$, works
for any choices of $\lambda_1, \dots, \lambda_n \in \mathbb{R}$.

Special case: $\lambda_1 = \dots = \lambda_n = c$, $D = cI_n$
is a scalar matrix, $L = cI_{\mathbb{R}^n}$ is a scalar
operator on $V = \mathbb{R}^n$.

Example: $L: \mathbb{R}^2 \rightarrow \mathbb{R}^2$ by $L \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} y \\ x \end{bmatrix}$ 1172

$= \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix}$ so ${}_S[L]_S = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$ (for S std basis)

is not diagonal. $T = \left\{ \begin{bmatrix} 1 \\ 1 \end{bmatrix} = w_1, \begin{bmatrix} 1 \\ -1 \end{bmatrix} = w_2 \right\}$

$L \begin{bmatrix} 1 \\ 1 \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \end{bmatrix} = 1 \cdot \begin{bmatrix} 1 \\ 1 \end{bmatrix}$ means w_1 is an e-vector

for this L with e-value $\lambda_1 = 1$.

$L \begin{bmatrix} 1 \\ -1 \end{bmatrix} = \begin{bmatrix} -1 \\ 1 \end{bmatrix} = -1 \begin{bmatrix} 1 \\ -1 \end{bmatrix}$ means w_2 is an e-vector

for this L with e-value $\lambda_2 = -1$.

${}_T[L]_T = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} = \begin{bmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{bmatrix} = D$ is diagonal

so this L is diag-able.

Problem: Let $A = \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix} \in \mathbb{R}^2$, and let 173

$$L = L_A: \mathbb{R}^2 \rightarrow \mathbb{R}^2 \text{ so } L \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} x+y \\ x+y \end{bmatrix}$$

Is L diag-able? What would it mean for

$w = \begin{bmatrix} x \\ y \end{bmatrix}$ to be an e-vector for L with e-value $\lambda \in \mathbb{R}$? It would mean that

$$\begin{bmatrix} x+y \\ x+y \end{bmatrix} = \lambda \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} \lambda x \\ \lambda y \end{bmatrix} \text{ iff } \begin{matrix} x+y = \lambda x \\ x+y = \lambda y \end{matrix} \text{ and$$

$$\text{so } \begin{matrix} (1-\lambda)x + y = 0 \\ x + (1-\lambda)y = 0 \end{matrix} \text{ and } \left[\begin{array}{cc|c} (1-\lambda) & 1 & 0 \\ 1 & (1-\lambda) & 0 \end{array} \right]$$

would have to have a non-trivial solution.

Have two approaches:

row reduction:

$$\begin{pmatrix} \rightarrow \left[\begin{array}{cc|c} (1-\lambda) & 1 & 0 \\ 1 & (1-\lambda) & 0 \end{array} \right] \rightarrow \left[\begin{array}{cc|c} -\lambda & \lambda & 0 \\ 1 & 1-\lambda & 0 \end{array} \right] \rightarrow \left[\begin{array}{cc|c} 1 & 1 & 0 \\ 0 & 0 & 0 \end{array} \right] \end{pmatrix} \begin{array}{l} \underline{174} \\ \text{if } \lambda=0 \end{array}$$

but if $\lambda \neq 0$, get $\left[\begin{array}{cc|c} 1 & -1 & 0 \\ 1 & 1-\lambda & 0 \end{array} \right] \rightarrow \left[\begin{array}{cc|c} 1 & -1 & 0 \\ 0 & 2-\lambda & 0 \end{array} \right]$

for $\lambda \neq 2$ this has only trivial solution.

For $\lambda = 2$ get $\left[\begin{array}{cc|c} 1 & -1 & 0 \\ 0 & 0 & 0 \end{array} \right] \begin{array}{l} x_1 = x_2 = r \\ x_2 = r \in \mathbb{R} \text{ free} \end{array}$

For $\lambda = 0$ get $\left[\begin{array}{cc|c} 1 & 1 & 0 \\ 0 & 0 & 0 \end{array} \right] \begin{array}{l} x_1 = -x_2 = -r \\ x_2 = r \in \mathbb{R} \text{ free} \end{array}$

Let $w_1 = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$ so $L \begin{bmatrix} 1 \\ 1 \end{bmatrix} = \begin{bmatrix} 2 \\ 2 \end{bmatrix} = 2 \begin{bmatrix} 1 \\ 1 \end{bmatrix}$

$w_2 = \begin{bmatrix} 1 \\ -1 \end{bmatrix}$ so $L \begin{bmatrix} 1 \\ -1 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} = 0 \begin{bmatrix} 1 \\ -1 \end{bmatrix}$

For $T = \left\{ \begin{bmatrix} 1 \\ 1 \end{bmatrix} = w_1, \begin{bmatrix} 1 \\ -1 \end{bmatrix} = w_2 \right\}$ basis of \mathbb{R}^2 175

$$L(w_1) = 2w_1 \quad \text{and} \quad L(w_2) = \theta = 0 \cdot w_2$$

$${}_T[L]_T = \begin{bmatrix} 2 & 0 \\ 0 & 0 \end{bmatrix} = D = \begin{bmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{bmatrix} \text{ is } \underline{\text{diagonal}}.$$

Second approach: $\begin{bmatrix} (1-\lambda) & 1 \\ 1 & (1-\lambda) \end{bmatrix}$ not invertible when

its determinant is zero:

$$(1-\lambda)^2 - 1 = 0 \quad \text{iff} \quad (1-2\lambda+\lambda^2) - 1 = \lambda^2 - 2\lambda =$$

$\lambda(\lambda-2) = 0$. Only roots of this polynomial

are $\lambda_1 = 0$ and $\lambda_2 = 2$. Solve two lin. systems:

For $\lambda_1 = 0$: $\left[\begin{array}{cc|c} 1 & 1 & 0 \\ 1 & 1 & 0 \end{array} \right]$ and for $\lambda_2 = 2$: $\left[\begin{array}{cc|c} -1 & 1 & 0 \\ 1 & -1 & 0 \end{array} \right]$.