Answer: Solve $AX = B = \begin{bmatrix} b_1 \\ b_2 \end{bmatrix}$ by legal row ops. of $\angle 17$

$$\begin{bmatrix} 1 & 2 & 3 & | & b_1 \\ 2 & 3 & 4 & | & b_2 \end{bmatrix} \xrightarrow{-2R_1 + R_2 \to R_2} \begin{bmatrix} 1 & 2 & 3 & | & b_1 \\ 0 & -1 & -2 & | & b_2 - 2b_1 \end{bmatrix}$$

$$+ \quad -2 \; -4 \; -6 \; -2b_1$$

$$+ \quad 0 \; -2 \; -4 \; (2b_2 - 4b_1)$$

$$\begin{bmatrix} 1 & 0 & -1 & | & -3b_1 + 2b_2 \\ 0 & -1 & -2 & | & b_2 - 2b_1 \end{bmatrix} \to \begin{bmatrix} 1 & 0 & -1 & | & -3b_1 + 2b_2 \\ 0 & 1 & 2 & | & 2b_1 - b_2 \end{bmatrix}$$

Interp: $x_1 = x_3 - 3b_1 + 2b_2 = r - 3b_1 + 2b_2$

$x_2 = -2x_3 + 2b_1 - b_2 = -2r + 2b_1 - b_2$

$x_3 = r \in F$ free var.

So $W = \{X \in F^3 \mid AX = B\} = \left\{ X = \begin{bmatrix} r - 3b_1 + 2b_2 \\ -2r + 2b_1 - b_2 \\ r \end{bmatrix} \in \mathbb{F}^3 \;\middle|\; r \in \mathbb{F} \right\}$

$= \left\{ X = r\begin{bmatrix} 1 \\ -2 \\ 1 \end{bmatrix} + \begin{bmatrix} -3b_1 + 2b_2 \\ +2b_1 - b_2 \\ 0 \end{bmatrix} \in \mathbb{F}^3 \;\middle|\; r \in \mathbb{F} \right\}$ is always consistent for any $B \in \mathbb{F}^2$.

Ex. Let $A = \begin{bmatrix} 1 & 2 \\ 2 & 3 \\ 3 & 4 \end{bmatrix} \in F^3_2$. Solve $AX = 0^3_1$.

$$\left[\begin{array}{cc|c} 1 & 2 & 0 \\ 2 & 3 & 0 \\ 3 & 4 & 0 \end{array}\right] \longrightarrow \left[\begin{array}{cc|c} 1 & 2 & 0 \\ 0 & -1 & 0 \\ 0 & -2 & 0 \end{array}\right] \rightarrow \left[\begin{array}{cc|c} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{array}\right]$$

$$\begin{matrix} -2 & -4 & 0 \\ -3 & -6 & 0 \end{matrix} \qquad \begin{matrix} 0 & 2 & 0 \end{matrix}$$

Interp

$x_1 = 0$

$x_2 = 0$

$W = \{X \in F^2 \mid AX = 0^3_1\} = \left\{\begin{bmatrix} 0 \\ 0 \end{bmatrix}\right\}$

has only the trivial solution (no free variables).

For which $B \in F^3$ does $AX = B$ have a solution?

Answer:

$$\left[\begin{array}{cc|c} 1 & 2 & b_1 \\ 2 & 3 & b_2 \\ 3 & 4 & b_3 \end{array}\right] \longrightarrow \left[\begin{array}{cc|c} 1 & 2 & b_1 \\ 0 & -1 & b_2 - 2b_1 \\ 0 & -2 & b_3 - 3b_1 \end{array}\right] \rightarrow \left[\begin{array}{cc|c} 1 & 2 & b_1 \\ 0 & 1 & 2b_1 - b_2 \\ 0 & 0 & b_1 - 2b_2 + b_3 \end{array}\right]$$

$$\begin{matrix} -2 & -4 & -2b_1 \\ -3 & -6 & -3b_1 \end{matrix} \qquad \begin{matrix} 0 & 2 & (-2b_2 + 4b_1) \end{matrix}$$

when is this consistent?

It is consistent when $0 = b_1 - 2b_2 + b_3$

This "consistency condition" on B tells exactly for which $B \in F^3$ is $AX = B$ consistent.

This happened because of the last row having all zeros on the left side.

<u>Conclusion</u>: $\left\{ B = \begin{bmatrix} b_1 \\ b_2 \\ b_3 \end{bmatrix} \in F^3 \middle| AX = B \text{ is consistent} \right\}$

$= \left\{ B = \begin{bmatrix} b_1 \\ b_2 \\ b_3 \end{bmatrix} \in F^3 \middle| b_1 - 2b_2 + b_3 = 0 \right\}$

$= \left\{ B = \begin{bmatrix} 2b_2 - b_3 \\ b_2 \\ b_3 \end{bmatrix} = b_2 \begin{bmatrix} 2 \\ 1 \\ 0 \end{bmatrix} + b_3 \begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix} \in F^3 \middle| b_2, b_3 \in F \right\}.$

Ex. Let $A = \begin{bmatrix} 1 & 2 \\ 2 & 4 \\ 3 & 6 \end{bmatrix}$. Solve $AX = 0_1^3$.

$$\begin{bmatrix} 1 & 2 & | & 0 \\ 2 & 4 & | & 0 \\ 3 & 6 & | & 0 \end{bmatrix} \to \begin{bmatrix} 1 & 2 & | & 0 \\ 0 & 0 & | & 0 \\ 0 & 0 & | & 0 \end{bmatrix}$$

$$\begin{matrix} -2 & -4 & 0 \\ -3 & -6 & 0 \end{matrix}$$

Interp

$x_1 = -2x_2 = -2r$

$x_2 = r \in F$ free var.

so

$$W = \left\{ X = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} \in F^2 \,\middle|\, AX = 0_1^3 \right\} = \left\{ r \begin{bmatrix} -2 \\ 1 \end{bmatrix} \in F^2 \,\middle|\, r \in F \right\}$$

has non-trivial solutions (1 free var.)

For which $B \in F^3$ does $AX = B$ have solutions?

Answer:

$$\begin{bmatrix} 1 & 2 & | & b_1 \\ 2 & 4 & | & b_2 \\ 3 & 6 & | & b_3 \end{bmatrix} \to \begin{bmatrix} 1 & 2 & | & b_1 \\ 0 & 0 & | & b_2 - 2b_1 \\ 0 & 0 & | & b_3 - 3b_1 \end{bmatrix}$$

$$\begin{matrix} + & -2 & -4 & -2b_1 \\ + & -3 & -6 & -3b_1 \end{matrix}$$

Interp.

This is consistent when two conditions hold:

$$\boxed{\begin{aligned} 0 &= b_2 - 2b_1 \\ 0 &= b_3 - 3b_1 \end{aligned}} \iff \boxed{\begin{aligned} b_2 &= 2b_1 \\ b_3 &= 3b_1 \end{aligned}}$$

so

For $A = \begin{bmatrix} 1 & 2 \\ 2 & 4 \\ 3 & 6 \end{bmatrix}$, $\{B \in \mathbb{F}^3 \mid AX = B \text{ is consistent}\}$

$$= \left\{ B = \begin{bmatrix} b_1 \\ 2b_1 \\ 3b_1 \end{bmatrix} = b_1 \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix} \in \mathbb{F}^3 \;\middle|\; b_1 \in \mathbb{F} \right\}.$$

Look for patterns in these examples and unifying principles that explain them.

<u>Principle 1</u>: Each lin. sys. $AX = B$ can be encoded as an augmented matrix $[A|B]$ where each <u>row</u> corresponds to an <u>equation</u>, and each <u>column</u> of A corresponds to a <u>variable</u>.

Principle 2: There are 3 basic equation manipulations which don't change the set of solutions, $W = \{X \in \mathbb{F}^n \mid AX = B\}$. They are:

① **Switch** the order of two of the equations.

② **Multiply** one of the equations by $0 \neq c \in \mathbb{F}$.

③ **Add** a multiple of one equation to another equation.

Principle 3: Each of the above equation manipulations corresponds to a **row operation** on $[A|B]$ changing it to some $[C|D]$ whose lin. sys. $CX = D$ has the same solution set as $AX = B$.

Principle 4: Using these "elementary row operations" it is always possible to change $[A|B]$ in a finite number of steps into a special one called **Reduced Row Echelon Form** (RREF) which has a nice simple interpretation that immediately gives all the solutions.

Examples: Let $A = [\delta_{ij}] \in F_n^n$ where $\delta_{ij} = \begin{cases} 1 & \text{if } i=j \\ 0 & \text{if } i \neq j \end{cases}$

so for $n=2$, $A = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$ for $n=3$, $A = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$, etc.

Then $AX = 0_1^n$ has only the trivial solution $X = 0_1^n$ since the equations are:

$$x_1 = 0$$
$$x_2 = 0$$
$$\vdots$$
$$x_n = 0$$

$[A|0_1^n]$ is in RREF.

$\boxed{\text{Notation: } I_n = [\delta_{ij}] \text{ } n \times n \text{ Iden. matrix}}$

<u>Ex</u>: Let $A = \begin{bmatrix} 1 & 0 & -1 & 2 \\ 0 & 1 & 3 & -4 \end{bmatrix}$. A is in RREF.

$[A \mid 0^2_1] = \left[\begin{array}{cccc|c} 1 & 0 & -1 & 2 & 0 \\ 0 & 1 & 3 & -4 & 0 \end{array}\right]$ has solutions

$x_1 = 1r - 2s$
$x_2 = -3r + 4s$
$\left.\begin{array}{l} x_3 = r \in F \\ x_4 = s \in F \end{array}\right\}$ free vars

$W = \{X \in F^4 \mid AX = 0^2_1\}$

$= \left\{ X = \begin{bmatrix} r - 2s \\ -3r + 4s \\ r \\ s \end{bmatrix} \in F^4 \;\middle|\; r, s \in F \right\}$

$= \left\{ X = r\begin{bmatrix} 1 \\ -3 \\ 1 \\ 0 \end{bmatrix} + s\begin{bmatrix} -2 \\ 4 \\ 0 \\ 1 \end{bmatrix} \in F^4 \;\middle|\; r, s \in F \right\}$ is expressed as the set of all <u>linear combinations</u> of the two specific vectors in

$S = \left\{ v_1 = \begin{bmatrix} 1 \\ -3 \\ 1 \\ 0 \end{bmatrix}, v_2 = \begin{bmatrix} -2 \\ 4 \\ 0 \\ 1 \end{bmatrix} \right\}$

Ex: $A = \begin{bmatrix} 1 & 0 & -1 & 2 \\ 0 & 1 & 3 & -4 \\ 0 & 0 & 0 & 0 \end{bmatrix}$ is also in RREF, and $AX = 0_1^3$ has the same solution set as the last example. The third equation of this lin. sys. says $0 = 0$.

Ex $A = \begin{bmatrix} 1 & 2 & 0 & 3 \\ 0 & 0 & 1 & -1 \end{bmatrix}$ is in RREF and the solutions of $AX = 0_1^2$ are given by

$$\left[\begin{array}{cccc|c} 1 & 2 & 0 & 3 & 0 \\ 0 & 0 & 1 & -1 & 0 \end{array}\right]$$

Interp

$x_1 = -2r - 3s$

$x_2 = r \in F$ (free)

$x_3 = s$

$x_4 = s \in F$ (free)

so $W = \left\{ X = \begin{bmatrix} -2r-3s \\ r \\ s \\ s \end{bmatrix} \in F^4 \;\middle|\; r, s \in F \right\}$

$= \left\{ r \begin{bmatrix} -2 \\ 1 \\ 0 \\ 0 \end{bmatrix} + s \begin{bmatrix} -3 \\ 0 \\ 1 \\ 1 \end{bmatrix} \in F^4 \;\middle|\; r, s \in F \right\}$

Def. We say $A = [a_{ij}] \in F_n^m$ is in RREF if

① Any "zero rows" are at the bottom.

② Each "non-zero row" has a leftmost ("leading") non-zero entry which is a 1.

③ If row i of A has a leading 1 in column j and row k of A has a leading 1 in column ℓ and $i < k$, then $j < \ell$.

④ If $a_{ij} = 1$ is a leading 1 in row i, col. j, then all other entries in col. j are <u>zero</u>.

<u>Notation</u>: If A is in RREF, then the columns of A containing leading 1's are called <u>pivot</u> columns.