Def: For $L: V \to V$, $\lambda \in \mathbb{R}$, let $L_\lambda = \{v \in V \mid L(v) = \lambda v\}$. Note: $L(\theta) = \theta = \lambda \cdot \theta$ so $\theta \in L_\lambda$.

Th: $L_\lambda \leq V$, called the λ eigenspace of L if $L_\lambda \neq \{\theta\}$ non-trivial subspace whose nonzero vectors are all e-vectors for L with e-value λ.

Pf: $\theta \in L_\lambda$ done. If $u, v \in L_\lambda$ then $L(u) = \lambda u$ and $L(v) = \lambda v$ so $L(u+v) = L(u) + L(v) = \lambda u + \lambda v = \lambda(u+v)$ so $u+v \in L_\lambda$.

If $u \in L_\lambda$ and $c \in \mathbb{R}$, $L(u) = \lambda u$ and $L(cu) = cL(u) = c(\lambda u) = (c\lambda)u = (\lambda c)u = \lambda(cu)$ so $cu \in L_\lambda$. $\square$

<u>General Procedure</u>: Given $L: V \to V$

① Find all $\lambda \in \mathbb{R}$ s.t. $L_\lambda \neq \{\theta\}$

② List them $\lambda_1, \lambda_2, \ldots, \lambda_r$ (distinct)

③ For each λ_i find a <u>basis</u> of L_{λ_i},
$T_i = \{w_{i1}, w_{i2}, \ldots, w_{ig_i}\}$, $g_i = \dim(L_{\lambda_i})$
called <u>geometric multiplicity</u> of λ_i for L.

④ Is $T = T_1 \cup T_2 \cup \cdots \cup T_r$ a basis for V?

5) If T is a basis for V, it is an e-basis, L is diag-able, and

$$ {}_T[L]_T = \begin{bmatrix} \boxed{\lambda_1 I_{g_1}} & & & 0 \\ & \boxed{\lambda_2 I_{g_2}} & & \\ & & \ddots & \\ 0 & & & \boxed{\lambda_r I_{g_r}} \end{bmatrix} = D $$

is a diagonal matrix representing L w.r.t. T.

If $A = {}_S[L]_S$, $\boxed{P = {}_SP_T}$ = transition matrix from T to S

then $D = P^{-1}AP$ has "diagonalized" A.

We can just start with $A \in \mathbb{R}^n_n$ and try to "diagonalize" it, try to find invertible $P \in \mathbb{R}^n_n$ s.t. $D = P^{-1}AP$ is diagonal.

<u>Def</u> For $\lambda \in \mathbb{R}$, let $A_\lambda = \{X \in \mathbb{R}^n \mid AX = \lambda X\}$

<u>Th</u>: $A_\lambda \leq \mathbb{R}^n$. Called λ e-space of A if $A_\lambda \neq \{0^n_1\}$ nontrivial.

<u>Special Cases:</u> For $\lambda = 0$, $A_0 = Nul(A)$

$L_0 = Ker(L)$.

<u>Generally:</u> $L_\lambda = Ker(L - \lambda I_V)$, $A_\lambda = Nul(A - \lambda I_n)$

$L(v)=\lambda v$ iff $L(v)-\lambda v=\theta$

$I_V:V\longrightarrow V$ is lin. map s.t. $I_V(v)=v$ so

$\lambda v=\lambda I_V(v)$ and $L(v)-\lambda v=L(v)-\lambda I_V(v)$

$\overset{\text{if}}{=}(L-\lambda I_V)(v)$ then $L_\lambda=\mathrm{Ker}(L-\lambda I_V)$

Why is $(L-\lambda I_V):V\longrightarrow V$ a linear map?

Can we make $\{L:V\to V \mid L \text{ is lin.}\}$ a vector space?

<u>Def</u>: Let $Lin(V,W)=\{L:V\to W \mid L \text{ is linear}\}$ with $+$ and $\cdot$ defined by

$(L_1+L_2)(v)=L_1(v)\underset{\text{in } W}{+}L_2(v)$ and $(c\cdot L_1)(v)=c\underset{\text{in } W}{\cdot}(L_1(v))$

Th: $Lin(V,W)$ with the above + and $\cdot$ is a vector space and its "zero vector" is the "zero lin. map" $O_W^V : V \longrightarrow W$ defined by $O_W^V(v) = \theta_W, \forall v \in V$.

Proof: Long an tedious but straight forward.

EX: Let $L_1, L_2 : \mathbb{R}^2 \longrightarrow \mathbb{R}^2$ be in $Lin(\mathbb{R}^2, \mathbb{R}^2)$ with $L_1\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 2x+y \\ -x+3y \end{bmatrix}$ and $L_2\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} -5x+2y \\ 3x-4y \end{bmatrix}$

so $(L_1+L_2)\begin{bmatrix} x \\ y \end{bmatrix} = L_1\begin{bmatrix} x \\ y \end{bmatrix} + L_2\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} -3x+3y \\ 2x-y \end{bmatrix}$ is linear.

$_S[L_1]_S + {}_S[L_2]_S = \begin{bmatrix} 2 & 1 \\ -1 & 3 \end{bmatrix} + \begin{bmatrix} -5 & 2 \\ 3 & -4 \end{bmatrix} = \begin{bmatrix} -3 & 3 \\ 2 & -1 \end{bmatrix} = {}_S[L_1+L_2]_S$

Also $(2L_1)\begin{bmatrix} x \\ y \end{bmatrix} = 2\begin{bmatrix} 2x+y \\ -x+3y \end{bmatrix} = \begin{bmatrix} 4x+2y \\ -2x+6y \end{bmatrix}$

$${}_S[2L_1]_S = \begin{bmatrix} 4 & 2 \\ -2 & 6 \end{bmatrix} = 2\begin{bmatrix} 2 & 1 \\ -1 & 3 \end{bmatrix} = 2\,{}_S[L_1]_S$$

<u>Th</u>: Let V have basis S, W have basis T, so $\forall L \in Lin(V,W)$ have ${}_T[L]_S \in \mathbb{R}^m_n$ when $\dim(V)=n$ and $\dim(W)=m$. This defines a map ${}_T\mathcal{M}_S : Lin(V,W) \longrightarrow \mathbb{R}^m_n$ by ${}_T\mathcal{M}_S(L) = {}_T[L]_S$. Then ${}_T\mathcal{M}_S$ is <u>linear</u>, and bijective, so is an isomorphism.

Cor: $\dim(\mathrm{Lin}(V,W)) = (\dim V)(\dim W) = n \cdot m$

Ex: $\mathrm{Lin}(\mathbb{R}^2_2, \mathbb{R}^3) = \{L : \mathbb{R}^2_2 \to \mathbb{R}^3 \mid L \text{ is lin.}\}$

bases S T

${}_T M_S(L) \in \mathbb{R}^3_4$

$\dim(\mathrm{Lin}(\mathbb{R}^2_2, \mathbb{R}^3)) =$
$\dim(\mathbb{R}^2_2)\dim(\mathbb{R}^3) = (4)(3) = 12$

$$\begin{array}{ccc} \mathbb{R}^2_2 & \xrightarrow{L} & \mathbb{R}^3 \\ \downarrow [\ ]_S & & \downarrow [\ ]_T \\ \mathbb{R}^4 & \xrightarrow{{}_T[L]_S} & \mathbb{R}^3 \end{array}$$

3×4

$m \times n$

<u>Example</u>: Try to diagonalize $A = \begin{bmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{bmatrix} \in \mathbb{R}^3_3$.

Step ①: Find all possible e-values $\lambda \in \mathbb{R}$ s.t. $AX = \lambda X$ for $0^3_1 \neq X \in \mathbb{R}^3$. $X = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$

Suppose $AX = \begin{bmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \lambda \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$ then it means

$$\begin{bmatrix} x_1 + x_2 + x_3 \\ x_1 + x_2 + x_3 \\ x_1 + x_2 + x_3 \end{bmatrix} = \begin{bmatrix} \lambda x_1 \\ \lambda x_2 \\ \lambda x_3 \end{bmatrix} = \begin{bmatrix} \lambda & 0 & 0 \\ 0 & \lambda & 0 \\ 0 & 0 & \lambda \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \lambda I_3 X$$

same as (*) $(A - \lambda I_3) X = 0^3_1$ so looking for $\lambda \in \mathbb{R}$ s.t. $Nul(A - \lambda I_3)$ is <u>non-trivial</u>

(*) has non-zero solutions.

$$[A-\lambda I_3 \mid 0^3_1] = \left[\begin{array}{ccc|c} 1-\lambda & 1 & 1 & 0 \\ 1 & 1-\lambda & 1 & 0 \\ 1 & 1 & 1-\lambda & 0 \end{array}\right]$$

$$-1 \quad -1 \quad \lambda-1$$

try row op's
$-R_3+R_1 \to R_1$
$-R_3+R_2 \to R_2$

get

$$\left[\begin{array}{ccc|c} -\lambda & 0 & \lambda & 0 \\ 0 & -\lambda & \lambda & 0 \\ 1 & 1 & 1-\lambda & 0 \end{array}\right] \to \left[\begin{array}{ccc|c} 1 & 1 & 1-\lambda & 0 \\ \lambda & 0 & -\lambda & 0 \\ 0 & \lambda & -\lambda & 0 \end{array}\right]$$

Two cases:
$\lambda = 0$ or
$\lambda \neq 0$

Case 1 $\lambda = 0$: $\boxed{\lambda_1 = 0}$

$$\left[\begin{array}{ccc|c} 1 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right] \quad \begin{array}{l} x_1 = -r-s \\ \left.\begin{array}{l} x_2 = r \\ x_3 = s \end{array}\right\} \text{free} \end{array}$$

$$\mathrm{Nul}(A-\lambda I_3) = \mathrm{Nul}(A) = A_0 = \left\{ \begin{bmatrix} -r-s \\ r \\ s \end{bmatrix} \in \mathbb{R}^3 \,\middle|\, r,s \in \mathbb{R} \right\}$$

has basis $T_1 = \left\{ \underbrace{\begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix}}_{w_{11}}, \underbrace{\begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix}}_{w_{12}} \right\}$ $\quad g_{\lambda_1} = \dim(A_{\lambda_1}) = 2$

Case ②: $\lambda \neq 0$: $\left[\begin{array}{ccc|c} 1 & 1 & (1-\lambda) & 0 \\ \lambda & 0 & -\lambda & 0 \\ 0 & \lambda & -\lambda & 0 \end{array}\right] \rightarrow \left[\begin{array}{ccc|c} 1 & 1 & (1-\lambda) & 0 \\ 1 & 0 & -1 & 0 \\ 0 & 1 & -1 & 0 \end{array}\right] \rightarrow$

$\begin{array}{ccc} -1 & 0 & 1 \\ 0 & -1 & 1 \end{array}$

$\left[\begin{array}{ccc|c} 1 & 0 & -1 & 0 \\ 0 & 1 & -1 & 0 \\ 0 & 0 & 3-\lambda & 0 \end{array}\right]$ has free varidbles iff $\lambda = 3$. So only other e-value is $\boxed{\lambda_2 = 3}$ To get e-space $A_3 = Nul(A - 3I_3)$

solve $\left[\begin{array}{ccc|c} 1 & 0 & -1 & 0 \\ 0 & 1 & -1 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right]$ $x_1 = r$, $x_2 = r$, $x_3 = r$ free. $A_3 = \left\{ \begin{bmatrix} r \\ r \\ r \end{bmatrix} \in \mathbb{R}^3 \middle| r \in \mathbb{R} \right\}$

has basis $T_2 = \left\{ \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} = w_{21} \right\}$, $g_{\lambda_2} = \dim(A_3) = 1$

Is $T = T_1 \cup T_2 = \left\{ \begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} \right\}$ an e-basis of $\mathbb{R}^3$ for A? If so, find $P = {}_S P_T$, $D = P^{-1} A P$.

Check: $\underbrace{\begin{bmatrix}1&1&1\\1&1&1\\1&1&1\end{bmatrix}}_{A}\underbrace{\begin{bmatrix}-1\\1\\0\end{bmatrix}}_{w_{11}}=\begin{bmatrix}0\\0\\0\end{bmatrix}=\underbrace{0\begin{bmatrix}-1\\1\\0\end{bmatrix}}_{=0\,w_{11}}$

$\underbrace{\begin{bmatrix}1&1&1\\1&1&1\\1&1&1\end{bmatrix}}_{A}\underbrace{\begin{bmatrix}-1\\0\\1\end{bmatrix}}_{w_{12}}=\begin{bmatrix}0\\0\\0\end{bmatrix}=\underbrace{0\begin{bmatrix}-1\\0\\1\end{bmatrix}}_{=0\,w_{12}}$ and $\underbrace{\begin{bmatrix}1&1&1\\1&1&1\\1&1&1\end{bmatrix}}_{A}\underbrace{\begin{bmatrix}1\\1\\1\end{bmatrix}}_{w_{21}}=\begin{bmatrix}3\\3\\3\end{bmatrix}=\underbrace{3\begin{bmatrix}1\\1\\1\end{bmatrix}}_{=3\,w_{21}}$

$\left[\begin{array}{ccc|ccc}1&0&0&-1&-1&1\\0&1&0&1&0&1\\0&0&1&0&1&1\end{array}\right]$ is in RREF so ${}_S P_T=\begin{bmatrix}-1&-1&1\\1&0&1\\0&1&1\end{bmatrix}=P$

$S \quad T$ (as columns)

$\left[\begin{array}{ccc|ccc}-1&-1&1&1&0&0\\1&0&1&0&1&0\\0&1&1&0&0&1\end{array}\right]\to\left[\begin{array}{ccc|ccc}1&0&1&0&1&0\\0&1&1&0&0&1\\0&0&3&1&1&1\end{array}\right]\to\left[\begin{array}{ccc|ccc}1&0&0&-1/3&2/3&-1/3\\0&1&0&-1/3&-1/3&2/3\\0&0&1&1/3&1/3&1/3\end{array}\right]$

$0 \quad 0 \quad -1 \quad -1/3 \quad -1/3 \quad -1/3$

$T \quad S$ (as columns)

${}_T P_S=\frac{1}{3}\begin{bmatrix}-1&2&-1\\-1&-1&2\\1&1&1\end{bmatrix}=P^{-1}$

$$P^{-1}AP = \frac{1}{3}\begin{bmatrix}-1 & 2 & -1\\ -1 & -1 & 2\\ 1 & 1 & 1\end{bmatrix}\begin{bmatrix}1 & 1 & 1\\ 1 & 1 & 1\\ 1 & 1 & 1\end{bmatrix}\begin{bmatrix}-1 & -1 & 1\\ 1 & 0 & 1\\ 0 & 1 & 1\end{bmatrix}$$

$$= \frac{1}{3}\begin{bmatrix}-1 & 2 & -1\\ -1 & -1 & 2\\ 1 & 1 & 1\end{bmatrix}\begin{bmatrix}0 & 0 & 3\\ 0 & 0 & 3\\ 0 & 0 & 3\end{bmatrix} = \frac{1}{3}\begin{bmatrix}0 & 0 & 0\\ 0 & 0 & 0\\ 0 & 0 & 9\end{bmatrix} = \begin{bmatrix}0 & 0 & 0\\ 0 & 0 & 0\\ 0 & 0 & 3\end{bmatrix}$$

$= D = \begin{bmatrix}\lambda_1 & 0 & 0\\ 0 & \lambda_1 & 0\\ 0 & 0 & \lambda_2\end{bmatrix}$ is diagonal, $\boxed{\begin{matrix}\lambda_1 = 0\\ \lambda_2 = 3\end{matrix}}$

Note: $\lambda_1 = 0$ was repeated corresponding to $g_{\lambda_1} = 2$.

<u>Exercise</u>: Show that $A = \begin{bmatrix}0 & 1\\ -1 & 0\end{bmatrix}$ can <u>not</u> be diagonalized using only real numbers.

$\det(A - \lambda I_2) = \det\begin{bmatrix}-\lambda & 1\\ -1 & -\lambda\end{bmatrix} = (-\lambda)^2 - (1)(-1) = \lambda^2 + 1$ has no real roots.