Th: Let $L: V \to V$ have distinct e-values $\lambda_1, \dots, \lambda_r \in \mathbb{F}$ with corresponding e-vectors $w_1, \dots, w_r \in V$, so $L(w_i) = \lambda_i w_i$ for $1 \le i \le r$. Then $S = \{w_1, \dots, w_r\} \subseteq V$ is independent.

Proof. Suppose $\theta = \sum_{i=1}^{m} c_i w_i$ is a shortest dep. relation on S, so all $c_i \neq 0$, $1 \le i \le m$. We may have relabeled vectors for convenience.

Apply L to get $\theta = \sum_{i=1}^{m} c_i L(w_i) = \sum_{i=1}^{m} c_i \lambda_i w_i$

Could also just multiply the dep rel. by λ_1, get $\theta = \sum_{i=1}^{m} c_i \lambda_1 w_i$. Subtract 2^{nd} eq. from 1^{st}:

Get $\theta = \sum_{i=1}^{m} c_i(\lambda_i - \lambda_1) w_i = \sum_{i=2}^{m} c_i(\lambda_i - \lambda_1) w_i$

Note: $m \geq 2$ since $\theta = c_1 w_1$ can't happen
($c_1 \neq 0$, $w_1 \neq \theta$).
The $i=1$ term is $c_1(\lambda_1 - \lambda_1) w_1 = c_1(0) w_1 = \theta$.
For $2 \leq i \leq m$, $\lambda_i - \lambda_1 \neq 0$ (distinct e-values),
so $\theta = \sum_{i=2}^{m} c_i (\lambda_i - \lambda_1) w_i$ is a <u>shorter</u> dep. rel on S, contradicting "shortest" chosen. □

<u>Th</u>: Let $L : V \to V$ have distinct e-values $\lambda_1, \dots, \lambda_r \in \mathbb{F}$ and for $1 \leq i \leq r$ let $T_i = \{w_{i1}, \dots, w_{ig_i}\}$ be a basis of e-space L_{λ_i}.
Then $T = T_1 \cup T_2 \cup \dots \cup T_r$ is <u>independent</u>.

Note: $T = \{w_{11}, \dots, w_{1g_1}, w_{21}, \dots, w_{2g_2}, \dots, w_{r1}, \dots, w_{rg_r}\}$ is a list of $g_1 + g_2 + \dots + g_r$ e-vectors in V.

root. Since $w_{ij} \in T_i \subseteq L_{\lambda_i}$ for $1 \le i \le r$ and $1 \le j \le g_i$ we have $L(w_{ij}) = \lambda_i w_{ij}$.
Suppose T were dependent, so there is a "shortest" dep. rel. on T. If necessary, by relabeling sets T_i and vectors in T_i, we could write that dep. rel. as

$$\theta = \sum_{i=1}^{m} \sum_{j=1}^{h_i} c_{ij} w_{ij} \quad \text{where each } c_{ij} \neq 0.$$

$$= \sum_{i=1}^{m} w_i \quad \text{where } w_i = \sum_{j=1}^{h_i} c_{ij} w_{ij} \in L_{\lambda_i}$$

and each $w_i \neq \theta$ since T_i is indep and $c_{ij} \neq 0$.
But that contradicts the last Theorem, so T indep. $\square$

This means T is a basis of V iff $g_1+g_2+\cdots+g_r = n = \dim(V)$, so L is diag-able iff we get a basis of e-vectors for V, enough from each e-space L_{λ_i} to make an e-basis for all of $V = \langle T \rangle$.

Focus now on finding all distinct e-values $\lambda_1, \ldots, \lambda_r$ for $L : V \to V$ or for $A \in \mathbb{F}^n_n$.
Find all $\lambda \in \mathbb{F}$ s.t. $(L - \lambda I_V)(v) = \theta$ has solutions $v \neq \theta$, that is, s.t. $\ker(L - \lambda I_V) \neq \{\theta_V\}$. Let S be any basis of V, and $A = {}_S[L]_S$ so $A - \lambda I_n = {}_S[L - \lambda I_V]_S$.

$\vec{\mathcal{N}}ul(A-\lambda I_n) \neq \{\theta_1^n\}$ iff $\text{rank}(A-\lambda I_n) < n$
iff $A-\lambda I_n$ is <u>not invertible</u>.
Use determinants to study this, and for other uses.

<u>Def.</u> For $1 \leq n \in \mathbb{Z}$ (integer) let
$S = \{1, 2, \dots, n\}$ and let
$S_n = \text{Perm}(S) = \{f : S \to S \mid f \text{ is bijective}\}$

Notation: Write $f = \begin{pmatrix} 1 & 2 & \cdots & i & \cdots & n \\ f(1) & f(2) & \cdots & f(i) & \cdots & f(n) \end{pmatrix}$
like a table of values. <u>Examples:</u>

$S_2 = \left\{ \begin{pmatrix} 1 & 2 \\ 1 & 2 \end{pmatrix}, \begin{pmatrix} 1 & 2 \\ 2 & 1 \end{pmatrix} \right\}$ has only two elements.

$S_3 = \left\{ \begin{pmatrix} 1 & 2 & 3 \\ 1 & 2 & 3 \end{pmatrix}, \begin{pmatrix} 1 & 2 & 3 \\ 1 & 3 & 2 \end{pmatrix}, \begin{pmatrix} 1 & 2 & 3 \\ 2 & 1 & 3 \end{pmatrix}, \begin{pmatrix} 1 & 2 & 3 \\ 2 & 3 & 1 \end{pmatrix}, \begin{pmatrix} 1 & 2 & 3 \\ 3 & 1 & 2 \end{pmatrix}, \begin{pmatrix} 1 & 2 & 3 \\ 3 & 2 & 1 \end{pmatrix} \right\}$ has six elements.

For $f = \begin{pmatrix} 1 & 2 & \cdots & n \\ f(1) & f(2) & \cdots & f(n) \end{pmatrix}$ there are

# of choices: $n\,(n-1)\cdots(2)(1) = n!$ ("n factorial")

So S_n contains $n!$ distinct elements, each one a bijection from S to S.

Composition of any two elements of S_n is another one, so have a <u>binary operation</u> on S_n, $\circ$, composition.

<u>Example</u>: For $f = \begin{pmatrix} 1 & 2 & 3 \\ 2 & 3 & 1 \end{pmatrix}$ and $g = \begin{pmatrix} 1 & 2 & 3 \\ 2 & 1 & 3 \end{pmatrix}$

find $f \circ g$ and $g \circ f$.

$$g = \begin{pmatrix} 1 & 2 & 3 \\ 2 & 1 & 3 \end{pmatrix}$$
$$f = \begin{pmatrix} 1 & 2 & 3 \\ 2 & 3 & 1 \end{pmatrix}$$
$$f \circ g = \begin{pmatrix} 1 & 2 & 3 \\ 3 & 2 & 1 \end{pmatrix}$$

$$f = \begin{pmatrix} 1 & 2 & 3 \\ 2 & 3 & 1 \end{pmatrix}$$
$$g = \begin{pmatrix} 1 & 2 & 3 \\ 2 & 1 & 3 \end{pmatrix}$$
$$g \circ f = \begin{pmatrix} 1 & 2 & 3 \\ 1 & 3 & 2 \end{pmatrix}$$

Note: $f \circ g \neq g \circ f$ can happen.

Note: $I = I_S = \begin{pmatrix} 1 & 2 & \cdots & i & \cdots & n \\ 1 & 2 & \cdots & i & \cdots & n \end{pmatrix}$ the identity map on S is bijective so $I \in S_n$ and $\forall f \in S_n$, $f \circ I = f = I \circ f$, so have an identity element for $\circ$ in S_n.

Also: $\forall f \in S_n$, $f^{-1} \in S_n$ since bijections are invertible and their inverses are bijective.

Ex: $f = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 \\ 2 & 5 & 4 & 3 & 1 \end{pmatrix}$ has $f^{-1} = \begin{pmatrix} 2 & 5 & 4 & 3 & 1 \\ 1 & 2 & 3 & 4 & 5 \end{pmatrix}$

(just permute columns to get top row in order) $= \begin{pmatrix} 1 & 2 & 3 & 4 & 5 \\ 5 & 1 & 4 & 3 & 2 \end{pmatrix}$

Th: $(S_n, \circ, I)$ is a group under composition with id. elt. I.

Def: For $f = \begin{pmatrix} 1 & \cdots & i & \cdots & j & \cdots & n \\ f(1) & & f(i) & & f(j) & & f(n) \end{pmatrix} \in S_n$

say f has an <u>inversion</u> for the pair (i, j), $1 \le i < j \le n$, when $f(i) > f(j)$.

EX: Inversions of $f = \begin{pmatrix} 1 & 2 & 3 \\ 2 & 1 & 3 \end{pmatrix} \in S_3$ (only one)

are marked below:

$f = \begin{pmatrix} 1 & 2 & 3 \\ 3 & 2 & 1 \end{pmatrix}$ has 3 inversions

Def: Let $\mathrm{Inv}(f)$ = Total number of inversions in $f \in S_n$.

Def: For $f \in S_n$ let $\mathrm{sgn}(f) = (-1)^{\mathrm{Inv}(f)} \in \{1, -1\}$ (say f is even or odd)

Def: For $A=[a_{ij}]\in \mathbb{F}_n^n$ define [197]

$$|A| = \det(A) = \sum_{\sigma\in S_n} \operatorname{sgn}(\sigma)\, a_{1\sigma(1)}\, a_{2\sigma(2)} \cdots a_{n\sigma(n)}$$

a sum ($n!$ terms) one term for each $\sigma\in S_n$
each term a product of n entries from A, one from each row, column number depends on σ.

Ex: $n=2$: $A=\begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22}\end{bmatrix}$ $\quad S_2=\left\{\begin{pmatrix}1&2\\1&2\end{pmatrix}, \begin{pmatrix}1&2\\2&1\end{pmatrix}\right\}$

$\operatorname{sgn}(\sigma)$: $1 \quad -1$

$\det(A) = (+1)a_{11}a_{22} + (-1)a_{12}a_{21}$

$= a_{11}a_{22} - a_{12}a_{21}$ is usual 2×2 $\det(A)$ formula

"Crosshatching"

Ex: $n = 3$: $A = [a_{ij}] \in \mathbb{F}^3_3$

$$S_3 = \left\{ \begin{pmatrix} 1&2&3 \\ 1&2&3 \end{pmatrix}, \begin{pmatrix} 1&2&3 \\ 1&3&2 \end{pmatrix}, \begin{pmatrix} 1&2&3 \\ 2&1&3 \end{pmatrix}, \begin{pmatrix} 1&2&3 \\ 2&3&1 \end{pmatrix}, \begin{pmatrix} 1&2&3 \\ 3&1&2 \end{pmatrix}, \begin{pmatrix} 1&2&3 \\ 3&2&1 \end{pmatrix} \right\}$$

$\operatorname{sgn}(\sigma)$: 1, -1, -1, $+1$, $+1$, -1

$$\det(A) = 1\,a_{11}a_{22}a_{33} + 1\,a_{12}a_{23}a_{31} + 1\,a_{13}a_{21}a_{32}$$
$$-1\,a_{11}a_{23}a_{32} - 1\,a_{12}a_{21}a_{33} - 1\,a_{13}a_{22}a_{31}$$

Crosshatching Method:

$$\begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix} \begin{matrix} a_{11} & a_{12} \\ a_{21} & a_{22} \\ a_{31} & a_{32} \end{matrix}$$

Warning: Crosshatching ONLY works for $n = 2, 3$

Products of three "+1" terms $\diagdown\diagdown\diagdown$

Products of three "−1" terms $\diagup\diagup\diagup$

Ex: $\det \left[\begin{array}{ccc} 1 & -1 & 2 \\ 3 & 1 & -1 \\ 0 & 4 & 5 \end{array}\right] \begin{array}{cc} 1 & -1 \\ 3 & 1 \\ 0 & 4 \end{array}$

$$= (1)(1)(5) + (-1)(-1)(0) + (2)(3)(4)$$
$$- (2)(1)(0) - (1)(-1)(4) - (-1)(3)(5)$$
$$= 5 + 24 + 4 + 15 = 48$$

$$A = \begin{bmatrix} 1 & -1 & 2 \\ 3 & 1 & -1 \\ 0 & 4 & 5 \end{bmatrix} \xrightarrow[\text{adder}]{\text{[redacted]}} \begin{bmatrix} 1 & -1 & 2 \\ 0 & 4 & -7 \\ 0 & 4 & 5 \end{bmatrix} \xrightarrow{\text{adder}} \begin{bmatrix} 1 & -1 & 2 \\ 0 & 4 & -7 \\ 0 & 0 & 12 \end{bmatrix}$$

$-3 \quad 3 \quad -6$ $\qquad$ $0 \quad -4 \quad 7$ $\qquad$ B

$$\det(B) = \left[\begin{array}{ccc} 1 & -1 & 2 \\ 0 & 4 & -7 \\ 0 & 0 & 12 \end{array}\right] \begin{array}{cc} 1 & -1 \\ 0 & 4 \\ 0 & 0 \end{array} = (1)(4)(12) = 48$$

How do row operations affect $\det(A)$?

Facts about det(A).

Th: Let $A=[a_{ij}]\in \mathbb{F}^n_n$. Then we have

(a) If A has a row of zeros then $\det(A)=0$

(b) $\det(A^T)=\det(A)$

(c) If A has two identical rows, then $\det A=0$

(d) If $\operatorname{rank}(A)<n$ then $\det(A)=0$

(e) $\det(A)=0$ implies $\operatorname{rank}(A)<n$

(f) $\det(A)=0$ iff A is <u>not</u> invertible

Goal: Understand how elementary row operations affect $\det(A)$.