Let $W \leq \mathbb{R}^3$ be the subspace

$$W = \left\{ \begin{bmatrix} x \\ y \\ z \end{bmatrix} \in \mathbb{R}^3 \;\middle|\; 1x + 2y + 3z = 0 \right\} \qquad [1\ 2\ 3 \mid 0] \quad \begin{matrix} \cancel{x} = -2y - 3z \\ \left.\begin{matrix} y \in \mathbb{R} \\ z \in \mathbb{R} \end{matrix}\right\} \text{free} \end{matrix}$$

$$= \left\{ \begin{bmatrix} -2y-3z \\ y \\ z \end{bmatrix} = y \begin{bmatrix} -2 \\ 1 \\ 0 \end{bmatrix} + z \begin{bmatrix} -3 \\ 0 \\ 1 \end{bmatrix} \in \mathbb{R}^3 \;\middle|\; y, z \in \mathbb{R} \right\}$$

$$= \left\langle \begin{bmatrix} -2 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} -3 \\ 0 \\ 1 \end{bmatrix} \right\rangle \quad \text{so } B = \left\{ \underbrace{\begin{bmatrix} -2 \\ 1 \\ 0 \end{bmatrix}}_{w_1}, \underbrace{\begin{bmatrix} -3 \\ 0 \\ 1 \end{bmatrix}}_{w_2} \right\} \text{ is a basis of } W.$$

$\dim(W) = 2$.

Can we extend B to a basis of $\mathbb{R}^3$?

When is $w_3 = \begin{bmatrix} x \\ y \\ z \end{bmatrix} \notin \langle B \rangle$? Iff $x + 2y + 3z \neq 0$.

There are infinitely many such choices, but a few are very simple and convenient, e.g., $w_3 = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$

Application of det to finding e-values:

<u>Th</u>: λ is an e-value of $A \in \mathbb{F}^n_n$ iff $A - \lambda I_n$ is not invertible iff $\det(A - \lambda I_n) = 0$.

<u>Ex</u>: $\begin{vmatrix} 1-\lambda & 1 & 1 \\ 1 & 1-\lambda & 1 \\ 1 & 1 & 1-\lambda \end{vmatrix} = \begin{vmatrix} -\lambda & 0 & \lambda \\ 0 & -\lambda & \lambda \\ 1 & 1 & (1-\lambda) \end{vmatrix} = \lambda^2 \begin{vmatrix} -1 & 0 & 1 \\ 0 & -1 & 1 \\ 1 & 1 & (1-\lambda) \end{vmatrix}$

$-1 \quad -1 \quad \lambda-1$

$= \lambda^2 \begin{vmatrix} -1 & 0 & 1 \\ 0 & -1 & 1 \\ 0 & 0 & (3-\lambda) \end{vmatrix} = \lambda^2(3-\lambda)$ is a polynomial of degree $n=3$ whose roots are the e-values of $A = \begin{bmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{bmatrix}$.

Note: $-\lambda^2(\lambda-3)$ is <u>factored</u> into 3 linear factors: $-(\lambda-0)(\lambda-0)(\lambda-3) = -(\lambda-\lambda_1)^2(\lambda-\lambda_2)^1$

<u>Th</u>: For any $A, B \in F^n_n$, $\det(AB) = (\det A)(\det B)$ 21 8

<u>Pf</u>: If A is invertible, $A = E_r \cdots E_2 E_1$ is a product of elem. matrices, so

$$\det(AB) = \det(E_r \cdots E_1 B) = (\det E_r) \cdots (\det E_1)(\det B)$$
$$= \det(E_r \cdots E_1)(\det B) = (\det A)(\det B).$$

Suppose A is not invertible so $C = E_r \cdots E_1 A$ where C has a zero row, so $CB = E_r \cdots E_1 AB$ has a zero row so $\det(CB) = 0 = \det(E_r \cdots E_1 AB)$ $= (\det E_r) \cdots (\det E_1) \det(AB)$ so $\det(AB) = 0$ $= (\det A)(\det B)$ since $\det(A) = 0$. $\square$