Review for Exam 2 (continued):

Cutting a spanning set down to a basis:

<u>EX</u>: $S = \left\{ \underset{v_1}{\begin{bmatrix} 1 \\ 1 \end{bmatrix}}, \underset{v_2}{\begin{bmatrix} 2 \\ 2 \end{bmatrix}}, \underset{v_3}{\begin{bmatrix} 3 \\ 3 \end{bmatrix}} \right\} \subseteq F^2$, $W = \langle S \rangle$

But S is dep. $x_1 v_1 + x_2 v_2 + x_3 v_3 = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$ has solutions: $\left[\begin{array}{ccc|c} 1 & 2 & 3 & 0 \\ 1 & 2 & 3 & 0 \end{array}\right] \to \left[\begin{array}{ccc|c} 1 & 2 & 3 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right]$ $\begin{array}{l} x_1 = -2r-3s \\ x_2 = r \in F \\ x_3 = s \in F \end{array}$ free

$\left\{ \begin{bmatrix} -2r-3s \\ r \\ s \end{bmatrix} \in F^3 \middle| r, s \in F \right\}$ is solution set so

$(-2r-3s)\begin{bmatrix} 1 \\ 1 \end{bmatrix} + r\begin{bmatrix} 2 \\ 2 \end{bmatrix} + s\begin{bmatrix} 3 \\ 3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$ $\forall r, s \in F$ says

$(r=1, s=0)$: $-2\begin{bmatrix} 1 \\ 1 \end{bmatrix} + 1\begin{bmatrix} 2 \\ 2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$ so $\begin{bmatrix} 2 \\ 2 \end{bmatrix} = 2\begin{bmatrix} 1 \\ 1 \end{bmatrix}$ $v_2 = 2v_1$

$(r=0, s=1)$: $-3\begin{bmatrix} 1 \\ 1 \end{bmatrix} + 1\begin{bmatrix} 3 \\ 3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$ so $\begin{bmatrix} 3 \\ 3 \end{bmatrix} = 3\begin{bmatrix} 1 \\ 1 \end{bmatrix}$, $v_3 = 3v_1$

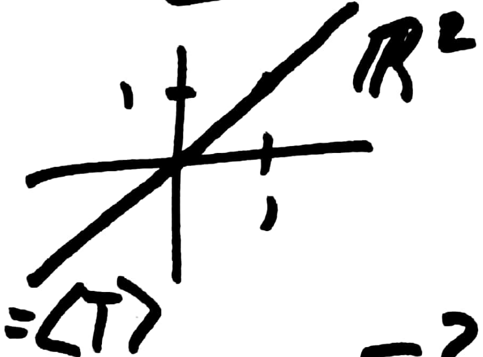
So $v_2, v_3 \in \langle v_1 \rangle$ are redundant so
$W = \langle S \rangle = \langle v_1 \rangle$ and $\{v_1\}$ is indep so
$T = \{v_1\}$ is a basis of W, we cut down S to
T (basis of W).

Generally, If $W = \langle S \rangle$ and S is dep. $S = \{w_1, \ldots, w_n\}$
we can find and remove redundant vectors
from S to get a basis $T \subseteq S$ of W.

$$\underset{\text{as columns}}{[S \mid 0_1^m]} \xrightarrow{\text{r.r.}} \underset{\text{RREF}}{[C \mid 0_1^m]} \quad \left.\begin{matrix} x_1 = \\ \vdots \\ x_n = \end{matrix}\right\} \begin{matrix} \text{some free} \\ \text{var's } (n-r) \\ \text{some not } (r) \end{matrix}$$

$\operatorname{rank}(C) = r$ For each free var.
get a dep. rel. on S. Keep pivot col's of S
to get basis T.

How can we extend $T=\left\{\begin{bmatrix}1\\1\end{bmatrix}\right\}$ to a basis of F^2? 218.5



Want $v_2 \in \begin{bmatrix}a\\b\end{bmatrix} \in F^2$ s.t.

$S=\left\{v_1=\begin{bmatrix}1\\1\end{bmatrix}, v_2=\begin{bmatrix}a\\b\end{bmatrix}\right\}$ is indep and spans F^2.

When is S indep? Answer: when $v_2 \notin \langle v_1\rangle$

$v_2 \neq \begin{bmatrix}r\\r\end{bmatrix}$ for any $r\in F$. Infinitely many such choices. Easy choices: $v_2=\begin{bmatrix}1\\0\end{bmatrix}$ or $\begin{bmatrix}0\\1\end{bmatrix}$.

<u>Tool</u>: If $S=\{w_1,\dots,w_n\}$ then $\forall w\in V$, and $T=S\cup\{w\}$, we have $\langle T\rangle \supsetneq \langle S\rangle$ iff $w\notin\langle S\rangle$. Also, if S indep then T indep iff $w\notin\langle S\rangle$

Recall that for $A \in F_n^m$, have

$Nul(A) = \{X \in F^n \mid AX = 0_1^m\}$ and

$L_A : F^n \to F^m$ by $L_A(X) = AX$ has

$Ker(L_A) = \{X \in F^n \mid L_A(X) = 0_1^m\}$ but

$L_A(X) = AX$ so $X \in Nul(A)$ iff $X \in Ker(L_A)$

$Nul(A) = Ker(L_A)$. Also,

$Col(A) = \langle Col_1(A), \ldots, Col_n(A) \rangle =$

$\{\sum_{j=1}^{n} x_j Col_j(A) \in F^m \mid x_j \in F\} = \{AX \in F^m \mid X \in F^n\}$

$= Range(L_A)$

In Pr. Ex2, #1 (c), can use "row-space" method to get a "simpler" basis of Col(A): 218.7

Transpose $Col_1(A)$, $Col_3(A)$, $Col_5(A)$ to get

$$\begin{bmatrix} 1&1&1&1 \\ 1&2&3&4 \\ 1&0&1&0 \end{bmatrix} \xrightarrow{r.r.} \begin{bmatrix} 1&1&1&1 \\ 0&1&2&3 \\ 1&0&1&0 \end{bmatrix} \to \begin{bmatrix} 1&0&1&0 \\ 0&1&2&3 \\ 0&1&0&1 \end{bmatrix} \to \begin{bmatrix} 1&0&1&0 \\ 0&1&0&1 \\ 0&0&2&2 \end{bmatrix}$$

$-1\ -1\ -1\ -1$ $\quad$ $-1\ 0\ -1\ 0$ $\quad$ $0\ -1\ 0\ -1$

$$\downarrow$$

$$\begin{bmatrix} 1&0&1&0 \\ 0&1&0&1 \\ 0&0&1&1 \end{bmatrix} \text{RREF}$$

Transpose back the nonzero rows to get "nice" basis

$$\left\{ \begin{bmatrix} 1\\0\\1\\0 \end{bmatrix}, \begin{bmatrix} 0\\1\\0\\1 \end{bmatrix}, \begin{bmatrix} 0\\0\\1\\1 \end{bmatrix} \right\}$$

Prob. 2 ([illegible]) $S=\{v_1,\dots,v_m\}$ dep. so $\quad$ 218.8

$\sum_{i=1}^{m} \underset{\text{not all } 0}{x_i} v_i = \theta_V$ so $L: V \to W$ gives

$$L\left(\sum_{i=1}^{m} x_i v_i\right) = L(\theta_V) = \theta_W$$

$= \sum_{i=1}^{m} x_i L(v_i) = \theta_W$ so $L(S) = \{L(v_1),\dots, L(v_m)\}$ in W is <u>dep</u>

not all 0

A invertible $\Rightarrow \exists A^{-1} \in F^n_n$ so if $AX = 0^n_1$ then

$A^{-1}(AX) = A^{-1}0^n_1 = 0^n_1$ $\quad$ so $Nul(A) = \{0^n_1\}$

$= (A^{-1}A)X = I_n X = X$ $\qquad$ $= Ker(L_A)$

A invertible iff $A \overset{row}{\sim} I_n$ iff $[A \mid 0^n_1] \to [I_n \mid 0^n_1]$

$\dim(V) = \dim \text{Ker}(L) + \dim \text{Range}(L)$ 218.9

3(a) $L : \mathbb{R}^4_4 \to \mathbb{R}^{16}$

$$\dim(V) = \dim(\mathbb{R}^4_4) = 16 = \dim \text{Ker}(L) + \underbrace{\dim(\text{Range}(L))}_{=16}$$

L <u>surj</u> means $\text{Range}(L) = \mathbb{R}^{16}$ so 16

so $\dim \text{Ker}(L) = 0$ so $\text{Ker}(L) = \{0^4_4\}$

is trivial so L is <u>inj</u>, so L is <u>bij</u>, <u>invertible</u>

and an <u>isomorphism</u>.