Another method to compute $\det(A)$:

Cofactor Expansion:

Def: For $A = [a_{ij}] \in \mathbb{F}_n^n$, $1 \le r, s \le n$, let $M_{rs} \in \mathbb{F}_{n-1}^{n-1}$ be the matrix obtained from A by deleting row r and column s.

Ex: For $A = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix}$, $M_{11} = [a_{22}]$, $M_{12} = [a_{21}]$

$M_{21} = [a_{12}]$, $M_{22} = [a_{11}]$

For $A = \begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{bmatrix}$, $M_{11} = \begin{bmatrix} 5 & 6 \\ 8 & 9 \end{bmatrix}$, $M_{12} = \begin{bmatrix} 4 & 6 \\ 7 & 9 \end{bmatrix}$

$M_{13} = \begin{bmatrix} 4 & 5 \\ 7 & 8 \end{bmatrix}$, $M_{21} = \begin{bmatrix} 2 & 3 \\ 8 & 9 \end{bmatrix}$, $M_{22} = \begin{bmatrix} 1 & 3 \\ 7 & 9 \end{bmatrix}$, etc.

Def: With notation as above, let
$A_{rs} = (-1)^{r+s}|M_{rs}| = (-1)^{r+s}\det(M_{rs})$.

Th (Cofactor Expansion) For each $1 \le r, s \le n$, wehare

(a) $\det(A) = \sum_{j=1}^{n} a_{rj} A_{rj}$ (expansion along row r)

(b) $\det(A) = \sum_{i=1}^{n} a_{is} A_{is}$ (expansion along column s)

Ex: $n=2$: $|A| = a_{11}A_{11} + a_{12}A_{12}$
$= a_{11}(-1)^{1+1}|M_{11}| + a_{12}(-1)^{1+2}|M_{12}|$
$= a_{11}a_{22} - a_{12}a_{21}$

row 1 cofactor expansion.

$$|A| = a_{21}A_{21} + a_{22}A_{22} \quad \text{(row 2 expansion)}$$
$$= a_{21}(-1)^{2+1}|M_{21}| + a_{22}(-1)^{2+2}|M_{22}|$$
$$= -a_{21}a_{12} + a_{22}a_{11}$$

$$|A| = a_{11}A_{11} + a_{21}A_{21} \quad \text{(column 1 expansion)}$$
$$= a_{11}(-1)^{1+1}|M_{11}| + a_{21}(-1)^{2+1}|M_{21}|$$
$$= a_{11}a_{22} - a_{21}a_{12}$$

<u>Ex</u>: $n=3$: $|A| = a_{11}A_{11} + a_{12}A_{12} + a_{13}A_{13}$

$$= a_{11}(-1)^{1+1}|M_{11}| + a_{12}(-1)^{1+2}|M_{12}| + a_{13}(-1)^{1+3}|M_{13}|$$

$$= a_{11}\begin{vmatrix} a_{22} & a_{23} \\ a_{32} & a_{33} \end{vmatrix} - a_{12}\begin{vmatrix} a_{21} & a_{23} \\ a_{31} & a_{33} \end{vmatrix} + a_{13}\begin{vmatrix} a_{21} & a_{22} \\ a_{31} & a_{32} \end{vmatrix}$$

Value of cofactor expansion is best pqz when used on a row (or column) with most number of 0 entries.

Ex: $\begin{vmatrix} 1 & 2 & 3 \\ 0 & 5 & 0 \\ 7 & 8 & 9 \end{vmatrix} = 5(-1)^{2+2}\begin{vmatrix} 1 & 3 \\ 7 & 9 \end{vmatrix} = 5(9-21) = -60$

(using cofactor expansion along row 2)

$$\begin{vmatrix} 1 & 0 & -1 & 1 \\ 2 & 0 & 3 & 4 \\ 3 & 2 & 1 & 0 \\ 1 & 0 & -1 & 2 \end{vmatrix} = 2(-1)^{3+2}\begin{vmatrix} 1 & -1 & 1 \\ 2 & 3 & 4 \\ 1 & -1 & 2 \end{vmatrix} = -2\begin{vmatrix} 1 & -1 & 1 \\ 0 & 5 & 2 \\ 0 & 0 & 1 \end{vmatrix} = -10$$

(cof. exp along col. 2) (row ops.)

shows efficient use of a combination of methods.

Recall: $\det(AB) = (\det A)(\det B)$.

Cor: If A is invertible then
$$\det(A^{-1}) = \frac{1}{\det(A)} = (\det A)^{-1}.$$

Pf: $I_n = AA^{-1}$ so $1 = \det(I_n) = (\det A)(\det(A^{-1}))$
so $\det(A^{-1}) = \dfrac{1}{(\det A)}$. $\square$

Application: If $|A| = 5$, $|B| = 6$, $|C| = 7$
then $$|A^2 B^T C^{-1}| = \frac{|A|^2 |B|}{|C|} = \frac{(25)(6)}{7} = \frac{150}{7}$$
since $|A^2| = |A \cdot A| = |A| \cdot |A| = |A|^2$ and $|B^T| = |B|$.
(transpose)

<u>Th</u>: For $A \in \mathbb{F}^n_n$, $|A - \lambda I_n| = (-1)^n |\lambda I_n - A|$ is a polynomial in λ of degree n whose highest term is $(-1)^n \lambda^n$ and whose lowest (constant) term is $|A|$.

<u>Pf</u>: By definition of $|A - \lambda I_n|$, it is a sum of products of the entries of $A - \lambda I_n$ $= [a_{ij} - \lambda \delta_{ij}]$ so each factor in each term is either a constant a_{ij} or a linear polynomial $a_{ii} - \lambda$. The sum of such products is a polynomial in λ with top term coming from $(a_{11} - \lambda)(a_{22} - \lambda) \cdots (a_{nn} - \lambda)$.

$\square$

Def: The *characteristic polynomial* of A is $|\lambda I_n - A| = \lambda^n + \cdots + (-1)^n |A|$.

Def: If char. poly. of A factors as

$$|\lambda I_n - A| = \prod_{i=1}^{r} (\lambda - \lambda_i)^{k_i} = (\lambda - \lambda_1)^{k_1} \cdots (\lambda - \lambda_r)^{k_r}$$

for *distinct* roots $\lambda_1, \ldots, \lambda_r \in \mathbb{F}$, the powers $k_1, \ldots, k_r$ are called the *algebraic multiplicities* of those roots, which are the *e-values* of A, say

k_i = algeb. mult. of λ_i for A.

Note: $n = k_1 + k_2 + \cdots + k_r$

Ex: We did $A = \begin{bmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{bmatrix}$ and found

$|A - \lambda I_3| = -\lambda^2(\lambda - 3) = (-1)^3(\lambda - 0)^2(\lambda - 3)$ so

char. poly. of A is $|\lambda I_3 - A| = \lambda^2(\lambda - 3)$

$\lambda_1 = 0$, $k_1 = 2$ and $\lambda_2 = 3$, $k_2 = 1$.

Note: $g_1 = 2$ and $g_2 = 1$ were found.

Ex: For $A = \begin{bmatrix} a_{11} & \cdots & a_{1n} \\ & a_{22} & \vdots \\ 0 & & a_{nn} \end{bmatrix}$ upper triangular, char poly is

$$|\lambda I_n - A| = \begin{vmatrix} \lambda - a_{11} & \cdots & -a_{1n} \\ & \lambda - a_{22} & \vdots \\ 0 & & \lambda - a_{nn} \end{vmatrix} = \prod_{i=1}^{n} (\lambda - a_{ii})$$

So, for example;

$$\begin{vmatrix} \lambda-2 & & & & * \\ & \lambda-2 & & & \\ & & \lambda+5 & & \\ & & & \lambda+5 & \\ 0 & & & & \lambda-6 \end{vmatrix} = (\lambda-2)^2(\lambda+5)^2(\lambda-6)$$

has degree $n=5$ but only 3 distinct roots; $\lambda_1=2$, $\lambda_2=-5$, $\lambda_3=6$ with alg. mults. $k_1=2$, $k_2=2$, $k_3=1$.

To find e-values of A, must factor char. poly. to get its roots. Some polys. don't factor over $\mathbb{R}$ into all linear factors, but every poly. <u>does</u> factor over $\mathbb{C}$ into all linear factors.

<u>Th</u>: If A is similar to B, so $B = P^{-1}AP$ for some invertible P, then $|B| = |A|$.

<u>Pf</u>: $|P^{-1}AP| = |P^{-1}| \cdot |A| \cdot |P| = \frac{1}{|P|} |A| \cdot |P| = |A|$
since this is a product of scalars in field F. $\square$

<u>Th</u>: If A is similar to B, then they have the same char. poly., $|\lambda I_n - A| = |\lambda I_n - B|$.

<u>Pf</u>: $|\lambda I_n - B| = |\lambda I_n - P^{-1}AP| = |P^{-1}\lambda I_n P - P^{-1}AP|$
$= |P^{-1}(\lambda I_n - A)P| = |P^{-1}| \cdot |\lambda I_n - A| \cdot |P| = |\lambda I_n - A|$,
since this is a product of two scalar numbers with a polynomial in λ. $\square$

Note: Suppose A is diagonalizable and [illegible]

$$P^{-1}AP = D = \begin{bmatrix} \boxed{\begin{matrix} \lambda_1 & & \\ & \ddots & \\ & & \lambda_1 \end{matrix}} & & & \\ & \boxed{\begin{matrix} \lambda_2 & & \\ & \ddots & \\ & & \lambda_2 \end{matrix}} & & \\ & & \ddots & \\ & & & \boxed{\begin{matrix} \lambda_r & & \\ & \ddots & \\ & & \lambda_r \end{matrix}} \end{bmatrix}$$

for distinct e-values $\lambda_1, \ldots, \lambda_r$ with λ_i repeated g_i times, $1 \le i \le r$. Then

$$|\lambda I_n - A| = |\lambda I_n - D| = \prod_{i=1}^{r} (\lambda - \lambda_i)^{g_i}$$

so $g_i = k_i$ for all $1 \le i \le r$,

geom. mult. = alg. mult. for all e-values, when A is diag.-able.

<u>Th.</u> Suppose $|\lambda I_n - A| = \prod_{i=1}^{r} (\lambda - \lambda_i)^{k_i}$ for distinct $\lambda_1, \dots, \lambda_r \in \mathbb{F}$, and recall that $g_i = \dim(A_{\lambda_i}) =$ geom. mult. of λ_i for A.
Then $1 \le g_i \le k_i$ for each $1 \le i \le r$.

<u>Cor</u>: A is diag-able iff each $g_i = k_i$.

<u>Practical Application</u>: If $k_i = 1$ then $g_i = 1$, but if $1 < k_i$ there is a chance that $1 \le g_i < k_i$. Check <u>largest</u> k_i first. If any $g_i < k_i$ then A <u>not</u> diag-able, can stop process. Don't waste time on finding other e-vectors if A not diag-able.

Def: For $L: V \longrightarrow V$, S any basis of V, $A = {}_S[L]_S$, let char. poly. of L be $|\lambda I_n - A|$.
If T is any other basis of V, let $B = {}_T[L]_T$ so $B = P^{-1}AP$ for $P = {}_S P_T$ (transition matrix). Then we know $|\lambda I_n - B| = |\lambda I_n - A|$ is the same char. poly. giving a consistent definition of char. poly. for L.

Possible Notations: $\mathrm{Char}_A(\lambda) = |\lambda I_n - A|$
$= \mathrm{Char}_L(\lambda)$

Some books use $\Delta_A(\lambda)$ or $p_A(\lambda)$ for char. poly of A.