When does a quadratic poly factor into linear factors?

Let poly. be $a\lambda^2+b\lambda+c$. Quadratic formula for roots is $\dfrac{-b\pm\sqrt{b^2-4ac}}{2a}$ which is real iff $b^2-4ac \geq 0$.

discriminant of poly is b^2-4ac.

If $b^2-4ac<0$ then get only a pair of Complex roots, not real roots, so poly does not factor over $\mathbb{R}$ or $\mathbb{Q}$.

If $b^2-4ac=0$ get one real root, repeated like $\lambda^2-4\lambda+4=(\lambda-2)^2$

$b^2-4ac=16-4(1)(4)=0$

EX: $A = \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 0 \\ -1 & 0 & 1 \end{bmatrix}$ so $A - \lambda I_3 = \begin{bmatrix} 1-\lambda & 0 & 1 \\ 0 & 1-\lambda & 0 \\ -1 & 0 & 1-\lambda \end{bmatrix}$

$|A - \lambda I_3| = (1-\lambda) \begin{vmatrix} 1-\lambda & 1 \\ -1 & 1-\lambda \end{vmatrix} = -(\lambda-1)\left((1-\lambda)^2 + 1\right)$

(cofactor expansion along row 2)

$= -(\lambda - 1)(\lambda^2 - 2\lambda + 2)$

but $b^2 - 4ac = 4 - 4(1)(2) = -4 < 0$

so $\lambda^2 - 2\lambda + 2$ has no real roots.

only got one real e-value, $\lambda_1 = 1$, $k_1 = 1$, $g_1 = 1$

could not get an e-basis of $\mathbb{R}^3$ for A.

This A is not diag-able over $\mathbb{R}$ or $\mathbb{Q}$.

But over $\mathbb{C}$, $\lambda^2 - 2\lambda + 2 = (\lambda - (1+i))(\lambda - (1-i))$ so

get 3 e-values, $\lambda_1 = 1$, $\lambda_2 = 1+i$, $\lambda_3 = 1-i$ with

alg. & geom. mults. $k_1 = 1 = g_1$, $k_2 = 1 = g_2$, $k_3 = 1 = g_3$. A is diag-able over $\mathbb{C}$

Ex: $A = \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 1 \end{bmatrix}$ so $A - \lambda I_3 = \begin{bmatrix} 1-\lambda & 0 & 1 \\ 0 & 1-\lambda & 0 \\ 1 & 0 & 1-\lambda \end{bmatrix}$

$|A - \lambda I_3| = (1-\lambda)\begin{vmatrix} 1-\lambda & 1 \\ 1 & 1-\lambda \end{vmatrix} = -(\lambda-1)\left((1-\lambda)^2 - 1\right)$

$= -(\lambda-1)(\lambda^2 - 2\lambda) = -(\lambda-1)(\lambda)(\lambda-2)$ has three distinct roots: order by inc. size:

$\lambda_1 = 0$, $\lambda_2 = 1$, $\lambda_3 = 2$, alg. mult.s:

$m_1 = 1$, $m_2 = 1$, $m_3 = 1$, so geom. mults:

$g_1 = 1$, $g_2 = 1$, $g_3 = 1$. Guaranteed to get one e-basis vector for each e-value, indep, e-basis of $\mathbb{R}^3$ for A. Find P s.t.

$P^{-1} A P = D = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 2 \end{bmatrix}$ is diagonal.

$\boxed{\lambda_1 = 0}$: Get A_0: Solve $\left[\begin{array}{ccc|c} 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 1 & 0 & 1 & 0 \end{array}\right] \to \left[\begin{array}{ccc|c} 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right]$

$x_1 = -r$
$x_2 = 0$
$x_3 = r \in \mathbb{R}$

$A_0 = \left\{ \begin{bmatrix} -r \\ 0 \\ r \end{bmatrix} \in \mathbb{R}^3 \,\middle|\, r \in \mathbb{R} \right\}$ has basis $T_1 = \left\{ \begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix} \right\}$

$\boxed{\lambda_2 = 1}$ Get A_1: Solve $\left[\begin{array}{ccc|c} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{array}\right] \to \left[\begin{array}{ccc|c} 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right]$ $x_1 = 0$, $x_2 = r \in \mathbb{R}$, $x_3 = 0$

$A_1 = \left\{ \begin{bmatrix} 0 \\ r \\ 0 \end{bmatrix} \in \mathbb{R}^3 \,\middle|\, r \in \mathbb{R} \right\}$ basis $T_2 = \left\{ \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} \right\}$

$\boxed{\lambda_3 = 2}$: Get A_2: Solve $\left[\begin{array}{ccc|c} -1 & 0 & 1 & 0 \\ 0 & -1 & 0 & 0 \\ 1 & 0 & -1 & 0 \end{array}\right] \to \left[\begin{array}{ccc|c} 1 & 0 & -1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right]$ $x_1 = r$, $x_2 = 0$, $x_3 = r \in \mathbb{R}$

$A_2 = \left\{ \begin{bmatrix} r \\ 0 \\ r \end{bmatrix} \in \mathbb{R}^3 \,\middle|\, r \in \mathbb{R} \right\}$ has basis $T_3 = \left\{ \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix} \right\}$.

So e-basis $T = \left\{ \begin{bmatrix}-1\\0\\1\end{bmatrix}, \begin{bmatrix}0\\1\\0\end{bmatrix}, \begin{bmatrix}1\\0\\1\end{bmatrix} \right\}$ and

$P = {}_S P_T = \begin{bmatrix}-1&0&1\\0&1&0\\1&0&1\end{bmatrix}$ should make $P^{-1}AP = \begin{bmatrix}0&0&0\\0&1&0\\0&0&2\end{bmatrix} = D$

Easier to check that $\boxed{AP = PD}$

$$\underset{A}{\begin{bmatrix}1&0&1\\0&1&0\\1&0&1\end{bmatrix}}\underset{P}{\begin{bmatrix}-1&0&1\\0&1&0\\1&0&1\end{bmatrix}} \stackrel{\checkmark}{=} \begin{bmatrix}0&0&2\\0&1&0\\0&0&2\end{bmatrix} \stackrel{\checkmark}{=} \underset{P}{\begin{bmatrix}-1&0&1\\0&1&0\\1&0&1\end{bmatrix}}\underset{D}{\begin{bmatrix}0&0&0\\0&1&0\\0&0&2\end{bmatrix}}.$$

Pf. Show that $g_i \le k_i$ where $\mathrm{Char}_A(\lambda) = \prod_{i=1}^{r}(\lambda-\lambda_i)^{k_i}$ and $g_i = \dim(A_{\lambda_i})$. We can just do the case $g_1 \le k_1$ since the rest are similar. Let $T_1 = \{w_{11}, \ldots, w_{1g_1}\}$ be a basis of A_{λ_1} and extend T_1 to a basis of F^n, $T = \{w_{11}, \ldots, w_{1g_1}, v_1, \ldots, v_{n-g_1}\}$. Since $Aw_{1j} = \lambda_1 w_{1j}$ for $1 \le j \le g_1$, $L_A : F^n \to F^n$ has values $L_A(w_{1j}) = Aw_{1j} = \lambda_1 w_{1j}$ so $[L_A(w_{1j})]_T = \lambda_1 e_j$ means the matrix $B = {}_T[L_A]_T = P^{-1}AP$ for $P = {}_S P_T$ has a block form

$$B = \left[\begin{array}{c|c} \lambda_1 I_{g_1} & C \\ \hline O & D \end{array}\right].$$

Then $\det(\lambda I_n - B) = \det \left[\begin{array}{c|c} (\lambda-\lambda_1)I_{g_1} & -C \\ \hline 0 & (\lambda I_{n-g_1} - D) \end{array}\right]$

$= \det((\lambda-\lambda_1) I_{g_1}) \cdot \det(\lambda I_{n-g_1} - D)$

$= (\lambda-\lambda_1)^{g_1} \det(\lambda I_{n-g_1} - D)$ because of the block (upper) triangular form. $\det(\lambda I_{n-g_1} - D) = \mathrm{Char}_D(\lambda)$ is some poly. of degree $n-g_1$ which may or may not contain more factors of $(\lambda-\lambda_1)$, but we certainly have at least $(\lambda-\lambda_1)^{g_1}$ as a factor in $\det(\lambda I_n - B) = \det(\lambda I_n - A) = \mathrm{Char}_A(\lambda)$, so $g_1 \le k_1$.