Dot Product in $\mathbb{R}^2$: Def:

For $u = \begin{bmatrix} a_1 \\ b_1 \end{bmatrix}$, $v = \begin{bmatrix} a_2 \\ b_2 \end{bmatrix} \in \mathbb{R}^2$ define the dot product

$u \cdot v = a_1 a_2 + b_1 b_2 \in \mathbb{R}$. $\cdot : \mathbb{R}^2 \times \mathbb{R}^2 \to \mathbb{R}$

Note: ① $u \cdot v = v \cdot u$ since $a_1 a_2 + b_1 b_2 = a_2 a_1 + b_2 b_1$

so the dot product is symmetric.

② $\forall r \in \mathbb{R}$, $(ru) \cdot v = (ra_1)a_2 + (rb_1)b_2 = r(u \cdot v)$

and $\forall u, v, w \in \mathbb{R}^2$, $(u+v) \cdot w = \begin{bmatrix} a_1 + a_2 \\ b_1 + b_2 \end{bmatrix} \cdot \begin{bmatrix} a_3 \\ b_3 \end{bmatrix} =$

$(a_1 + a_2)a_3 + (b_1 + b_2)b_3 = a_1 a_3 + a_2 a_3 + b_1 b_3 + b_2 b_3$

$= (a_1 a_3 + b_1 b_3) + (a_2 a_3 + b_2 b_3) = (u \cdot w) + (v \cdot w)$

So $\forall r, s \in \mathbb{R}$, $\forall u, v, w \in \mathbb{R}^2$ we have

$(ru + sv) \cdot w = r(u \cdot w) + s(v \cdot w)$ linearity.

③ By symmetry ① we also have linearity
in the second factor:

$$w\cdot(ru+sv) = r(w\cdot u)+s(w\cdot v).$$

② and ③ together are called "**bilinearity**".

④ $\forall u=\begin{bmatrix}a_1\\b_1\end{bmatrix}\in\mathbb{R}^2$, $u\cdot u = a_1^2+b_1^2 \geq 0$ and

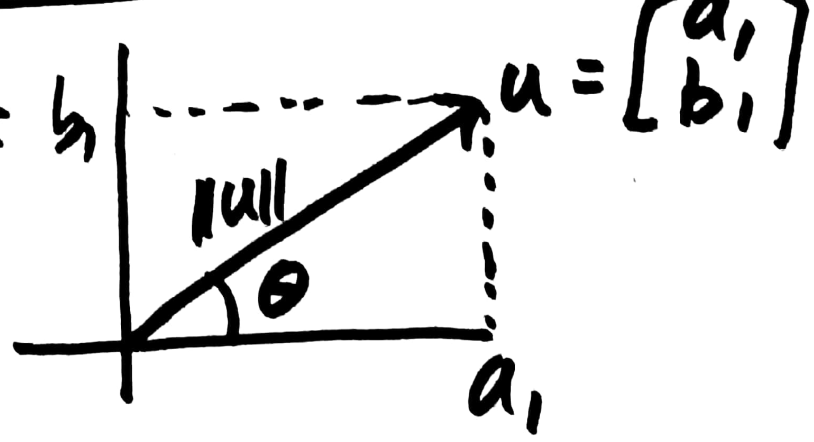
$$u\cdot u = 0 \iff a_1=0=b_1 \iff u=\theta_{\mathbb{R}^2}=\begin{bmatrix}0\\0\end{bmatrix}.$$

This property of the dot product in $\mathbb{R}^2$ is called "positive definite". Def: Length of u is

$$\|u\| = \sqrt{u\cdot u} = \sqrt{a_1^2+b_1^2} \geq 0$$

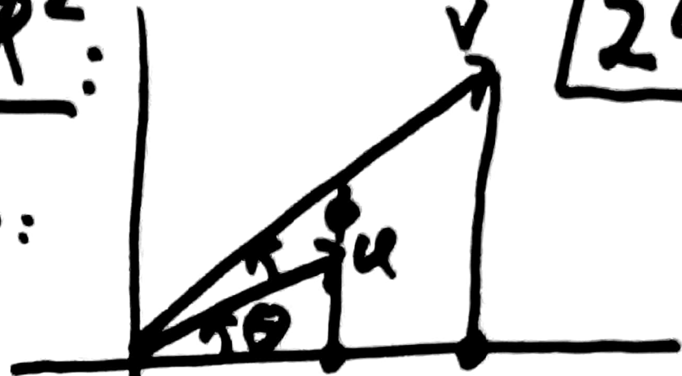
Picture:

$\mathbb{R}^2$



Angle between vectors in $\mathbb{R}^2$:

Let $u = \begin{bmatrix} a_1 \\ b_1 \end{bmatrix}$, $v = \begin{bmatrix} a_2 \\ b_2 \end{bmatrix}$. Picture:



using trig. can write

$$\begin{bmatrix} a_1 \\ b_1 \end{bmatrix} = \|u\| \begin{bmatrix} \cos\theta \\ \sin\theta \end{bmatrix} \text{ and } \begin{bmatrix} a_2 \\ b_2 \end{bmatrix} = \|v\| \begin{bmatrix} \cos(\theta+\phi) \\ \sin(\theta+\phi) \end{bmatrix}$$

Also have trig. formulas:

$$\cos(\theta+\phi) = \cos\theta\cos\phi - \sin\theta\sin\phi$$
$$\sin(\theta+\phi) = \sin\theta\cos\phi + \cos\theta\sin\phi$$

We can use these to write a formula for $u \cdot v$ in terms of $\|u\|$, $\|v\|$ and $\cos(\phi)$ where ϕ is the angle between u and v.

Th. For any $u, v \in \mathbb{R}^2$ with ϕ the angle from u to v, we have $u \cdot v = \|u\| \|v\| \cos(\phi)$.

Pf. $u \cdot v = \left(\|u\| \begin{bmatrix} \cos\theta \\ \sin\theta \end{bmatrix}\right) \cdot \left(\|v\| \begin{bmatrix} \cos(\theta+\phi) \\ \sin(\theta+\phi) \end{bmatrix}\right)$

$= \|u\| \|v\| \left((\cos\theta)\cos(\theta+\phi) + (\sin\theta)\sin(\theta+\phi)\right)$

$= \|u\| \|v\| \left((\cos\theta)(\cos\theta\cos\phi - \sin\theta\sin\phi)\right.$
$\left. + (\sin\theta)(\sin\theta\cos\phi + \cos\theta\sin\phi)\right)$

$= \|u\| \|v\| \left(\cos^2\theta \cos\phi - \cos\theta\sin\theta\sin\phi\right.$
$\left. + \sin^2\theta\cos\phi + \sin\theta\cos\theta\sin\phi\right)$

$= \|u\| \|v\| (\cos^2\theta + \sin^2\theta)\cos\phi$

$= \|u\| \|v\| \cos\phi.$ $\square$

<u>Cor.</u> For any $u, v \in \mathbb{R}^2$, $|u \cdot v| \le \|u\| \, \|v\|$

<u>Pf.</u> Since $\|u\| \ge 0$ and $\|v\| \ge 0$ we have
$|u \cdot v| = \big| \|u\| \, \|v\| \cos\phi \big| = \|u\| \, \|v\| \, |\cos\phi| \le \|u\| \, \|v\|$
since $|\cos\phi| \le 1$. $\square$

The inequality $|u \cdot v| \le \|u\| \, \|v\|$ is called the <u>Cauchy-Schwarz inequality</u>, and it is true for $u, v \in \mathbb{R}^3$ and for $u, v \in \mathbb{R}^n$ if we generalize the definition of dot product in the obvious way.

Inner Product Spaces and Orthogonality:

Recall the standard **dot product on $\mathbb{R}^n$**:

For $v = \begin{bmatrix} a_1 \\ \vdots \\ a_n \end{bmatrix}$, $w = \begin{bmatrix} b_1 \\ \vdots \\ b_n \end{bmatrix} \in \mathbb{R}^n$ define $v \cdot w = \sum_{i=1}^{n} a_i b_i \in \mathbb{R}$.
$= v^T w$

Properties: **Symmetric**: $v \cdot w = w \cdot v$

Bilinear: $(c_1 v_1 + c_2 v_2) \cdot w = c_1 (v_1 \cdot w) + c_2 (v_2 \cdot w)$

and $v \cdot (c_1 w_1 + c_2 w_2) = c_1 (v \cdot w_1) + c_2 (v \cdot w_2)$

Positive Definite: $v \cdot v = \sum_{i=1}^{n} a_i^2 \geq 0$ and

$v \cdot v = 0$ implies $v = 0_1^n$.

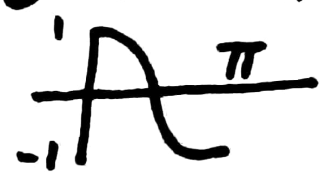
Def. Length of $v \in \mathbb{R}^n$ is $\|v\| = \sqrt{v \cdot v}$

Distance between $v, w \in \mathbb{R}^n$ is $\|v - w\|$.

Say v is a **unit vector** when $\|v\| = 1$.

Th: (Cauchy-Schwarz Inequality)
For any $v, w \in \mathbb{R}^n$, $|v \cdot w| \leq \|v\| \cdot \|w\|$ so
$-1 \leq \dfrac{v \cdot w}{\|v\| \cdot \|w\|} \leq 1$ for $\|v\| \neq 0 \neq \|w\|$.

Def: For $v, w \in \mathbb{R}^n$ the angle between v and w, $\theta_{v,w}$ is the unique angle between 0 and π such that $\cos(\theta_{v,w}) = \dfrac{v \cdot w}{\|v\| \, \|w\|}$.



Def. Say $v \perp w$ (perpendicular, orthogonal) when $\theta_{v,w} = \pi/2$, Same as $\cos(\theta_{v,w}) = 0$, iff $v \cdot w = 0$.

Def: Say $S = \{v_1, \ldots, v_m\} \subseteq \mathbb{R}^n$ is <u>orthogonal</u> when $v_i \perp v_j$ for all $1 \leq i \neq j \leq m$.

Def. Say $S=\{v_1,\dots,v_m\}\subseteq\mathbb{R}^n$ is **orthonormal** when $v_i\cdot v_j=\delta_{ij}=\begin{cases}1 & \text{if } i=j\\ 0 & \text{if } i\neq j.\end{cases}$
So S consists of unit vectors which are mutually perpendicular.

Th: If $S=\{v_1,\dots,v_m\}\subseteq\mathbb{R}^n$ is an orthogonal set of non-zero vectors then S is indep.

Pf: Suppose $\sum_{i=1}^m c_i v_i=\theta$ then for any $1\le j\le m$
$\left(\sum_{i=1}^m c_i v_i\right)\cdot v_j=\theta\cdot v_j=0$. By bilinearity, get
$\sum_{i=1}^m c_i(v_i\cdot v_j)=0$ but for $i\neq j$, $v_i\cdot v_j=0$ so
$c_j(v_j\cdot v_j)=0$. Since $v_j\neq\theta$, $v_j\cdot v_j>0$ (Pos. Definite property) so $c_j=0$, true for all $1\le j\le m$. □

Th. Suppose $S=\{v_1,\dots,v_n\}$ is an orthogonal basis of $\mathbb{R}^n$. For any $v\in\mathbb{R}^n$, $v=\sum_{i=1}^n c_i v_i$ and $c_j=\dfrac{v\cdot v_j}{v_j\cdot v_j}$ for each $1\le j\le n$ gives the coordinates of v with respect to S, $[v]_S$.

Pf. For each $1\le j\le n$, $v\cdot v_j=\sum_{i=1}^n c_i(v_i\cdot v_j)$ $=c_j(v_j\cdot v_j)$ since $v_i\cdot v_j=0$ for $i\neq j$. Since $v_j\cdot v_j\neq 0$ (pos. def.) get $c_j=\dfrac{v\cdot v_j}{v_j\cdot v_j}$. $\square$

Cor: If $S=\{v_1,\dots,v_n\}$ is an orthonormal basis of $\mathbb{R}^n$ then $\forall v\in\mathbb{R}^n$, $v=\sum_{i=1}^n (v\cdot v_i)\, v_i$.

Ex: Std. basis $S=\{e_1,\dots,e_n\}$ of $\mathbb{R}^n$ is an orthonormal basis of $\mathbb{R}^n$.

<u>Ex</u>: For any angle ϕ let $S = \left\{ \begin{bmatrix} \cos\phi \\ \sin\phi \end{bmatrix}, \begin{bmatrix} -\sin\phi \\ \cos\phi \end{bmatrix} \right\}$

$\phantom{Ex: For any angle \phi let S = \{} v_1 \qquad\qquad v_2$

Then $v_1 \cdot v_1 = \cos^2\phi + \sin^2\phi = 1$

$v_2 \cdot v_2 = (-\sin\phi)^2 + (\cos^2\phi) = 1$

$v_1 \cdot v_2 = -\cos\phi\sin\phi + \sin\phi\cos\phi = 0$

so S is an orthonormal set in $\mathbb{R}^2$, indep, basis of $\mathbb{R}^2$. $\forall v = \begin{bmatrix} a \\ b \end{bmatrix} \in \mathbb{R}^2$

$$\begin{bmatrix} v \cdot v_1 \\ v \cdot v_2 \end{bmatrix} = \begin{bmatrix} a\cos\phi + b\sin\phi \\ -a\sin\phi + b\cos\phi \end{bmatrix} = [v]_S \text{ since}$$

$$(a\cos\phi + b\sin\phi)\begin{bmatrix} \cos\phi \\ \sin\phi \end{bmatrix} + (-a\sin\phi + b\cos\phi)\begin{bmatrix} -\sin\phi \\ \cos\phi \end{bmatrix}$$

$$= \begin{bmatrix} a\cos^2\phi + b\sin\phi\cos\phi + a\sin^2\phi - b\cos\phi\sin\phi \\ a\cos\phi\sin\phi + b\sin^2\phi - a\sin\phi\cos\phi + b\cos^2\phi \end{bmatrix} = \begin{bmatrix} a \\ b \end{bmatrix}$$

Let $S=\{v_1,\dots,v_n\}$ be an orthonormal (o.n.) basis of $\mathbb{R}^n$ and let $A\in R^n_n$ be the matrix whose columns are the vectors from S, so $\mathrm{Col}_j(A)=v_j$ for $1\le j\le n$. Then

$$\underset{n\times 1}{\mathrm{Col}_i(A)}\cdot\underset{n\times 1}{\mathrm{Col}_j(A)}=\underset{1\times n}{\left(\mathrm{Col}_i(A)\right)^T}\,\underset{n\times 1}{\mathrm{Col}_j(A)}=\delta_{ij}$$

But $\left(\mathrm{Col}_i(A)\right)^T=\mathrm{Row}_i\left(A^T\right)$ so

$$\underset{1\times n}{\mathrm{Row}_i(A^T)}\,\underset{n\times 1}{\mathrm{Col}_j(A)}=\delta_{ij}$$

is the (i,j)-entry of $\underset{(n\times n)(n\times n)}{A^TA}=[\delta_{ij}]=I_n$ = the identity matr.

so $A^T=A^{-1}$.

Def. We say $A \in \mathbb{R}_n^n$ is an <u>orthogonal</u> matrix when $A^T = A^{-1}$.

<u>Th</u>: $A \in \mathbb{R}_n^n$ is orthogonal iff $S = \{Col_1(A), \ldots, Col_n(A)\}$ is an <u>orthonormal</u> set in $\mathbb{R}^n$ iff $T = \{Row_1(A), \ldots, Row_n(A)\}$ is an o.n. set in $\mathbb{R}_n$ (with respect to the std dot product in $\mathbb{R}_n$).

<u>EX</u>: For each $\phi \in \mathbb{R}$, $A = \begin{bmatrix} \cos\phi & -\sin\phi \\ \sin\phi & \cos\phi \end{bmatrix}$ is orthogonal.

$$A^TA = \begin{bmatrix} \cos\phi & \sin\phi \\ -\sin\phi & \cos\phi \end{bmatrix}\begin{bmatrix} \cos\phi & -\sin\phi \\ \sin\phi & \cos\phi \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

and $AA^T = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$ so $A^T = A^{-1}$.