Example: For $W = \langle T \rangle = \langle T' \rangle = \langle T'' \rangle$ as before, find $W^{\perp} = \{X \in \mathbb{R}^4 \mid X \perp W\}$ and use its basis vector to extend T' to an orthogonal basis of $\mathbb{R}^4$, extend T'' to an orthonormal basis of $\mathbb{R}^4$, and use that answer to give an orthogonal matrix $A \in \mathbb{R}^4_4$.

Solution: Find $W^{\perp} = \{X \in \mathbb{R}^4 \mid X \cdot w_i' = 0,\ 1 \le i \le 3\}$ by solving

$$\left[\begin{array}{cccc|c} 1 & 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & 1 & 0 \\ -1 & -1 & 1 & 1 & 0 \end{array}\right] \to \left[\begin{array}{cccc|c} 1 & 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & 1 & 0 \\ 0 & 0 & 2 & 2 & 0 \end{array}\right] \to \left[\begin{array}{cccc|c} 1 & 0 & 0 & -1 & 0 \\ 0 & 1 & 0 & 1 & 0 \\ 0 & 0 & 1 & 1 & 0 \end{array}\right] \quad \begin{array}{l} x_1 = r \\ x_2 = -r \\ x_3 = -r \\ x_4 = r \in \mathbb{R} \end{array}$$

$$W^{\perp} = \left\{ \begin{bmatrix} r \\ -r \\ -r \\ r \end{bmatrix} \in \mathbb{R}^4 \,\middle|\, r \in \mathbb{R} \right\} = \left\langle \begin{bmatrix} 1 \\ -1 \\ -1 \\ 1 \end{bmatrix} \right\rangle \text{ so}$$

$$S' = \left\{ \begin{bmatrix} 1 \\ 0 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 0 \\ 1 \end{bmatrix}, \begin{bmatrix} -1 \\ -1 \\ 1 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ -1 \\ -1 \\ 1 \end{bmatrix} \right\}$$ is an orthogonal basis of $\mathbb{R}^4$ extending T'.

Normalizing each vector in S' gives o.n. basis

$$S'' = \left\{ \frac{1}{\sqrt{2}}\begin{bmatrix}1\\0\\1\\0\end{bmatrix}, \frac{1}{\sqrt{2}}\begin{bmatrix}0\\1\\0\\1\end{bmatrix}, \frac{1}{2}\begin{bmatrix}-1\\-1\\1\\1\end{bmatrix}, \frac{1}{2}\begin{bmatrix}1\\-1\\-1\\1\end{bmatrix} \right\}$$ of $\mathbb{R}^4$, and

$$A = \begin{bmatrix} 1/\sqrt{2} & 0 & -1/2 & 1/2 \\ 0 & 1/\sqrt{2} & -1/2 & -1/2 \\ 1/\sqrt{2} & 0 & 1/2 & -1/2 \\ 0 & 1/\sqrt{2} & 1/2 & 1/2 \end{bmatrix}$$

is an orthogonal matrix in $\mathbb{R}^4_4$ whose columns are the vectors in o.n. set S''.

Check that $AA^T = I_4 = A^T A$

Important results about orthog. matrices:

<u>Th</u>: For any $X, Y \in \mathbb{R}^n$ and for any $A \in \mathbb{R}^n_n$, we have $(AX)\cdot Y = X\cdot(A^T Y)$.

<u>Pf</u>. $(AX)\cdot Y = (AX)^T Y = (X^T A^T) Y = X^T(A^T Y) = X\cdot(A^T Y)$

<u>Th</u>: For any $X, Y \in \mathbb{R}^n$ if $A \in \mathbb{R}^n_n$ is orthogonal then $(AX)\cdot(AY) = X\cdot Y$.

<u>Pf</u>. Since A orthog. means $A^T = A^{-1}$, we have
$(AX)\cdot(AY) = X\cdot(A^T A Y) = X\cdot(A^{-1}AY) = X\cdot(I_n Y) = X\cdot Y$.

<u>Cor</u>: If $A^T = A^{-1}$ then for any $X, Y \in \mathbb{R}^n$ we have $\|AX\| = \|X\|$ and $\theta_{X,Y} = \theta_{AX,AY}$ so $L_A : \mathbb{R}^n \to \mathbb{R}^n$ preserves lengths and angles.

Important result about symm. matrices: $\boxed{267}$

<u>Th</u>: Let $A = A^T \in \mathbb{R}^n_n$ be symmetric and let $\lambda \neq \mu$ in $\mathbb{R}$ be e-values of A. Then the e-spaces A_λ and A_μ are perpendicular, $A_\lambda \perp A_\mu$.

<u>pf</u>. Need to show that for any $X \in A_\lambda$, $Y \in A_\mu$ that $X \cdot Y = 0$. It is clear if $X = 0^n_1$ or $Y = 0^n_1$ so suppose $AX = \lambda X$ and $AY = \mu Y$ for $X, Y \in \mathbb{R}^n$ nonzero e-vectors. Then we have

$$\lambda(X\cdot Y) = (\lambda X)\cdot Y = (AX)\cdot Y = X\cdot(A^TY) = X\cdot(AY) = X\cdot(\mu Y) = \mu(X\cdot Y)$$

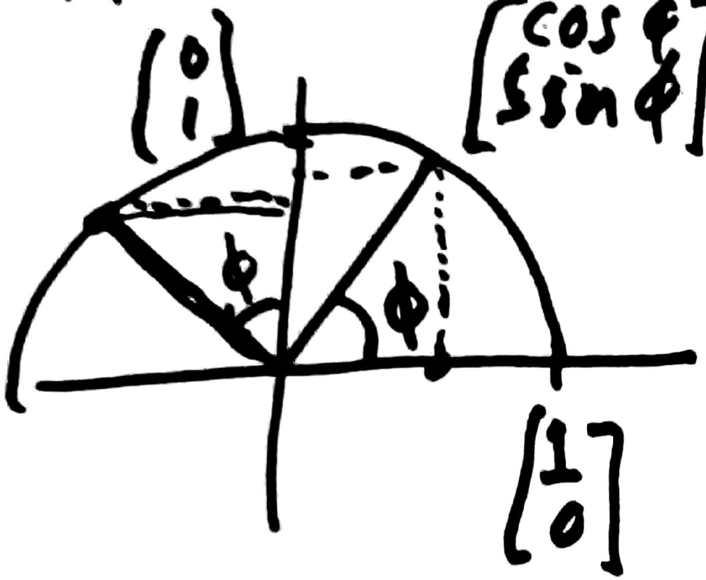
So $\lambda(X\cdot Y) = \mu(X\cdot Y)$ so $(\lambda-\mu)(X\cdot Y) = 0$. But $\lambda - \mu \neq 0$ so $\overline{X\cdot Y = 0}$ $\square$

<u>Th</u>: Let $A = A^T \in \mathbb{R}^n_n$ and suppose $\lambda_1, \dots, \lambda_r \in \mathbb{R}$ are the distinct e-values of A. Let T_i' be an orthonormal basis of e-space A_{λ_i}, $1 \le i \le r$, obtained by Gram-Schmidt process from any basis T_i of A_{λ_i}. Then $T' = T_1' \cup T_2' \cup \dots \cup T_r'$ is an orthonormal basis of $\mathbb{R}^n$ and $P = {}_S P_{T'}$ is an orthogonal matrix (whose columns are the vectors in T') such that $P^{-1} A P = D$ is diagonal with blocks $\lambda_i I_{g_i}$ on the diagonal, $g_i = \dim(A_{\lambda_i}) = k_i$ (geom. = alg. mult.).

Since P is orthog. $P^{-1} = P^T$ so $D = P^T A P$ and we say A can be "orthogonally diag-ized."

Pf. In Advanced Lin. Alg. it is shown that all e-values of symm. $A \in \mathbb{R}^n_n$ are real, and that $g_i = k_i$ so A is diag-able. Since G.-S. gives orthonormal bases T_i' for each A_{λ_i}, and $A_{\lambda_i} \perp A_{\lambda_j}$ for $1 \le i \ne j \le r$ by last Theorem, we get that T' is an orthonormal set of n e-vectors in $\mathbb{R}^n$ so $P = {}_S P_{T'}$ is orthog., $P^{-1} = P^T$ and $D = P^T A P$ is diag. with the e-values λ_i on the diag. repeated $g_i = k_i$ times in blocks corresponding to the order of e-vectors in T'. $\square$

$L_A : \mathbb{R}^2 \to \mathbb{R}^2$ for $A = \begin{bmatrix} \cos\phi & -\sin\phi \\ \sin\phi & \cos\phi \end{bmatrix}$



$$L_A\begin{bmatrix}1\\0\end{bmatrix} = \mathrm{Col}_1(A) = \begin{bmatrix}\cos\phi\\ \sin\phi\end{bmatrix}$$

$$L_A\begin{bmatrix}0\\1\end{bmatrix} = \mathrm{Col}_2(A) = \begin{bmatrix}-\sin\phi\\ \cos\phi\end{bmatrix} = \begin{bmatrix}\cos(\phi+\pi/2)\\ \sin(\phi+\pi/2)\end{bmatrix}$$

$\{L_A(e_1), L_A(e_2)\}$ is another o.n. basis of $\mathbb{R}^2$, just $S = \{e_1, e_2\}$ rotated (c.c.w) by angle ϕ. This L_A preserves lengths and angles

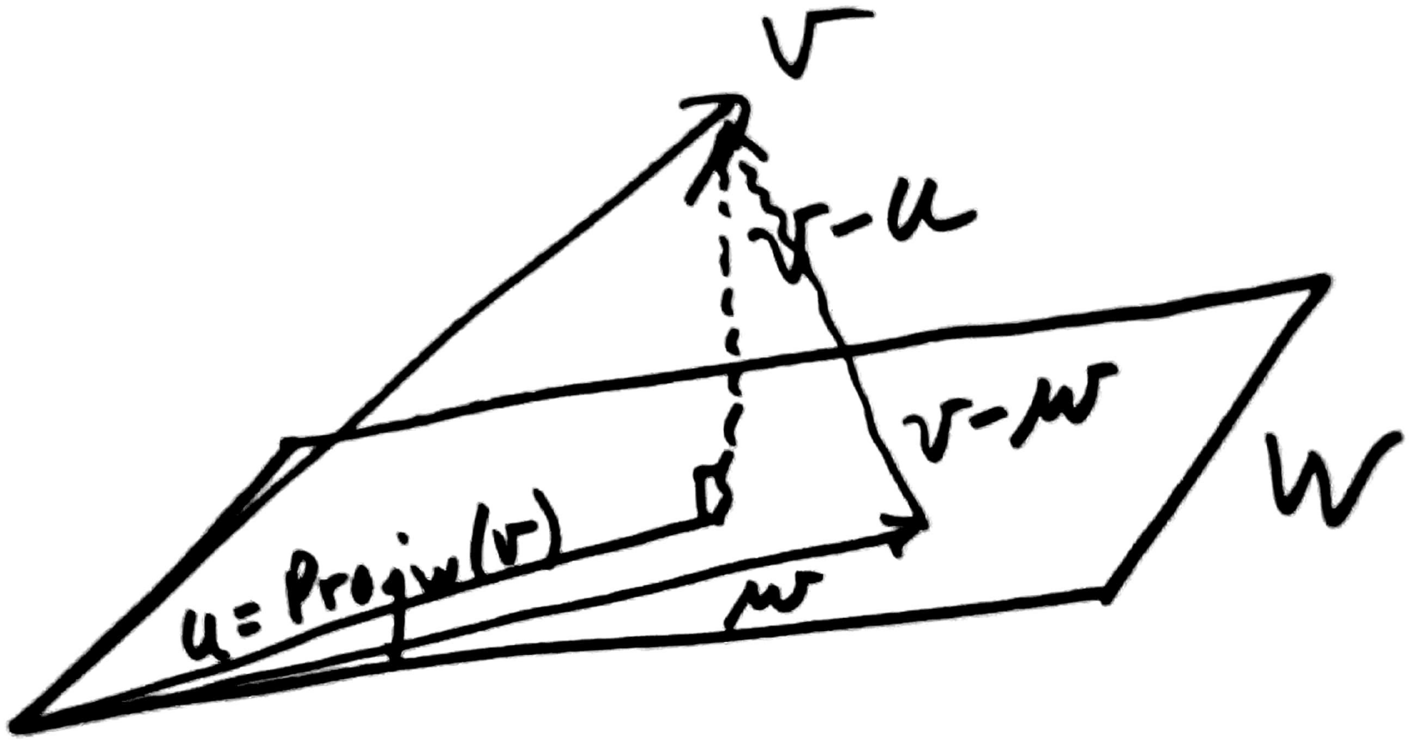
Ex: Reflections: $L_A: \mathbb{R}^2 \longrightarrow \mathbb{R}^2$ is refl. w.r.t. $y=x$, so $L_A\begin{bmatrix}x\\y\end{bmatrix}=\begin{bmatrix}y\\x\end{bmatrix}=\begin{bmatrix}0&1\\1&0\end{bmatrix}\begin{bmatrix}x\\y\end{bmatrix}$

$A=\begin{bmatrix}0&1\\1&0\end{bmatrix}$ has columns $\left\{\begin{bmatrix}0\\1\end{bmatrix},\begin{bmatrix}1\\0\end{bmatrix}\right\}$ an. o.n. basis of $\mathbb{R}^2$

$A^T=A^{-1}$ so A is orthog. matrix.

L_A preserved lengths & angles.

Meaning of $\mathrm{Proj}_W(v)$ as "best approximation to v in W"



$$\|v-u\| < \|v-w\| \quad \text{if } w \neq u \quad \forall w \in W$$

$\leq$

"Best approx. Thm"

Th (Pythagorean Thm in $\mathbb{R}^n$).
For $X, Y \in \mathbb{R}^n$, if $X \cdot Y = 0$ (so $X \perp Y$) then $\|X+Y\|^2 = \|X\|^2 + \|Y\|^2$. For $v_1, v_2, \dots, v_m \in \mathbb{R}^n$, if $\{v_1, \dots, v_m\}$ is orthogonal (so $v_i \cdot v_j = 0$ for $i \neq j$) then $\|\sum_{i=1}^m v_i\|^2 = \sum_{i=1}^m \|v_i\|^2$.

Pf. $\|X+Y\|^2 = (X+Y)\cdot(X+Y) = X\cdot X + X\cdot Y + Y\cdot X + Y\cdot Y$ $= X\cdot X + Y\cdot Y = \|X\|^2 + \|Y\|^2$. The general case of m orthogonal vectors follows by induction, using $X = v_1 + \cdots + v_{m-1}$, $Y = v_m$. $\square$

Th (Triangle Inequality in $\mathbb{R}^n$)

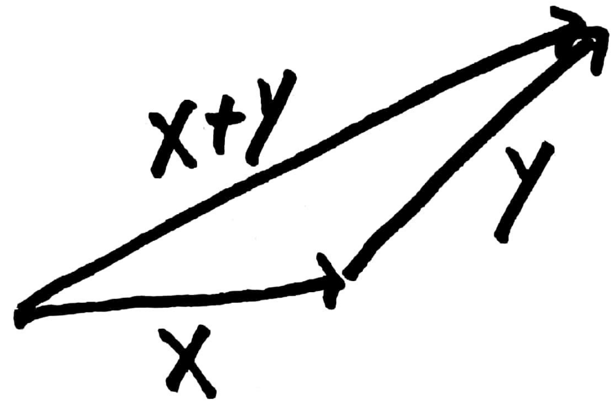
For any $X, Y \in \mathbb{R}^n$, $\|X+Y\| \le \|X\| + \|Y\|$.

Pf.
$$\|X+Y\|^2 = \|X\|^2 + 2(X \cdot Y) + \|Y\|^2$$
$$\le \|X\|^2 + 2|X \cdot Y| + \|Y\|^2 \quad \text{so by}$$

Cauch-Schw. Ineq.
$$\le \|X\|^2 + 2(\|X\|)(\|Y\|) + \|Y\|^2$$
$$= (\|X\| + \|Y\|)^2. \quad \text{This gives}$$

$$0 \le \|X+Y\| \le \|X\| + \|Y\|. \qquad \square$$

Geometrical Picture:



Complex Numbers: $\mathbb{C} = \{a+bi \mid a,b \in \mathbb{R},\ i^2=-1\}$

The "imaginary" number $i \in \mathbb{C}$, $i \notin \mathbb{R}$, is a special number such that $i^2 = -1$.

$\mathbb{C}$ is a "field", like $\mathbb{R}$ = real numbers and $\mathbb{Q}$ = rational numbers, where we can do arithmetic, use for scalars in Lin. Alg.

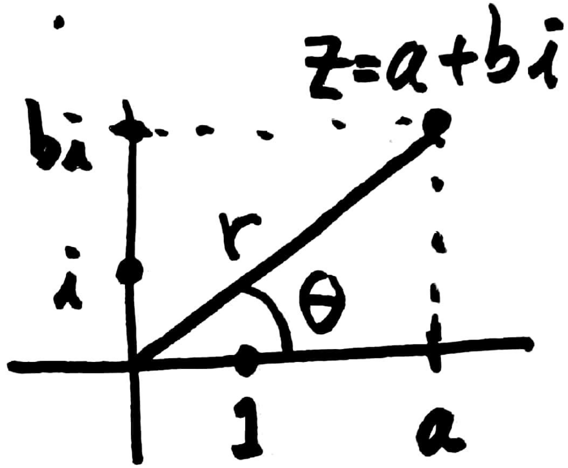
Addition: $(a+bi)+(c+di) = (a+c)+(b+d)i$

Mult: $(a+bi)\cdot(c+di) = ac+adi+bci+bdi^2$
(commutative) $\qquad = (ac-bd)+(ad+bc)i$

<u>Def.</u> For $z=a+bi$ let "complex conjugate" of z be $\bar{z} = a-bi$, so $z\bar{z} = a^2+b^2 \geq 0$ and $z\bar{z}=0$ iff $z = 0+0i = 0$.

<u>Note</u>: For $0 \neq z = a+bi \in \mathbb{C}$, $z\bar{z} > 0$
and $z^{-1} = \dfrac{\bar{z}}{a^2+b^2} \in \mathbb{C}$ is mult. inverse of z.

<u>Ex</u>: If $z = 3+4i$ then $z\bar{z} = 3^2+4^2 = 25$
so $z\left(\dfrac{\bar{z}}{25}\right) = 1$, $z^{-1} = \dfrac{3-4i}{25}$.



<u>Graphical Picture of $\mathbb{C}$:</u>
looks like $\mathbb{R}^2$ with $1 \leftrightarrow \begin{bmatrix}1\\0\end{bmatrix}$
$i \leftrightarrow \begin{bmatrix}0\\1\end{bmatrix}$

but $\mathbb{C}$ has a <u>mult</u>
while $\mathbb{R}^2$ does <u>not</u>.

Related to "polar coordinates"

$$z = (r\cos\theta) + (r\sin\theta)\,i$$
$$= a + b\,i$$