Def. If A is in RREF then the number of leading 1's, which equals the number of non-zero rows, is called the rank of A, $\text{rank}(A)$.

Obvious Fact. For $A \in F_n^m$ (in RREF), $\text{rank}(A) \leq m$ and $\text{rank}(A) \leq n$ so $\text{rank}(A) \leq \text{Min}(m, n)$.
This is clear since each leading 1 occupies a different row, and they are also in different columns.

Def. For $A, B \in F_n^m$ we say A is row equivalat to B, written $A \underset{\text{row}}{\sim} B$, when B can be obtained from A by a finite sequence of elem. row operations.

Ex. Let $A = \begin{bmatrix} 1 & 2 \\ 1 & 3 \end{bmatrix}$. Then

$$\begin{bmatrix} 1 & 2 \\ 1 & 3 \end{bmatrix} \underset{\text{row}}{\sim} \begin{bmatrix} 1 & 2 \\ 0 & 1 \end{bmatrix} \underset{\text{row}}{\sim} \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \underset{\text{row}}{\sim} \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \underset{\text{row}}{\sim} \begin{bmatrix} 0 & 7 \\ 1 & 0 \end{bmatrix} \cdots$$

$-1 \quad -2$ $\qquad\qquad$ $0 \quad -2$ $\qquad\qquad$ RREF ← has rank 2

Ex. $\begin{bmatrix} 1 & 1 & 1 \\ 2 & 2 & 2 \\ 3 & 3 & 3 \end{bmatrix} \underset{\text{row}}{\sim} \begin{bmatrix} 1 & 1 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$ has rank 1.

$+$ $\quad$ $+$

$-2 \ -2 \ -2$

$-3 \ -3 \ -3$ $\qquad$ RREF

Th: The relation $\underset{\text{row}}{\sim}$ on F_n^m is <u>reflexive</u>, <u>symmetric</u> and <u>transitive</u>, that is,

① $A \underset{\text{row}}{\sim} A$, ② $A \underset{\text{row}}{\sim} B$ implies $B \underset{\text{row}}{\sim} A$

③ ($A \underset{\text{row}}{\sim} B$ and $B \underset{\text{row}}{\sim} C$) implies $A \underset{\text{row}}{\sim} C$.

Note the important use of logical concepts ("and", "implies") which are discussed in the Appendix of Heffron. Read about them and we can discuss them in detail later.

<u>Note</u>: A relation on a set with those three properties, reflexive, symmetric, transitive, is called an <u>equivalence</u> <u>relation</u>. Such relations are very important in math.

<u>Def</u>. If $A \underset{\text{row}}{\sim} B$ in F_n^m and B is in RREF with $\text{rank}(B) = r$ then we define $\text{rank}(A) = r$, and the pivot columns of B are also called the pivot columns of A. These are col. numbers, not the entries of the matrices.

What are the ranks and the pivot col's for each of the examples on pages 25 and 28?

<u>Th</u>. For each $A \in F^m_n$ there is a unique $C \in F^m_n$ such that $A \underset{\text{row}}{\sim} C$ and C is in RREF.

<u>Goal</u>: Be able to efficiently and accurately find that RREF C when given A.

<u>Purpose</u>: To solve linear systems $[A|B]$, <u>row reduce</u> to $[C|D]$ so that C is in RREF, then interpret to get solution set.

Many other problems in Lin. Alg. are solved by the process of <u>row reduction</u> to RREF.

Types of problems we have seen so far: 31

① Find all $X \in F^n$ such that $AX = 0_1^m$ for a given $A \in F_n^m$.

② Find all $B \in F^m$ such that $AX = B$ is consistent for a given $A \in F_n^m$.

③ Given $A \in F_n^m$ and $B \in F^m$ find all $X \in F^n$ (if any) such that $AX = B$.

④ Given $A \in F_n^m$ find $C \in F_n^m$ such that $A \underset{\text{row}}{\sim} C$ and C is in RREF. Use C to get $\text{rank}(A) = \text{rank}(C)$ and to find the pivot columns.

Row Reduction Process: Given $A \in F^m_n$.

This is a general description of the process for finding the unique C in RREF such that $A \underset{row}{\sim} C$.

① Find the leftmost non-zero column of A.

② In that column (j) find a row (i) whose entry a_{ij} is non-zero and "convenient" ($a_{ij}=1$ is good)

③ Use a multiplier elem. row op. to change that $a_{ij} \neq 0$ to a leading 1.

④ Use adder elem. row op.s to change all other entries in that column to 0.

⑤ Use a switcher elem. row op. to put that row in row 1.

(6) Look at the part of the modified matrix which is below and to the right of that leading 1 in row 1. Apply the steps above to that part. That should create another leading 1 in row 2 and in a column to the right of that leading 1 in row 1. All entries above and below those leading 1's should be 0.

(7) Continue the process on the part of the modified matrix below and to the right of the second leading 1. That may create a third leading 1, and so on, until nothing more can be done. The final result is the C in RREF that you seek.

Ex. Row reduce the following matrix to RREF.

$$\begin{bmatrix}1&2&3&4\\2&3&4&5\\3&4&5&6\end{bmatrix}\to\begin{bmatrix}1&2&3&4\\0&-1&-2&-3\\0&-2&-4&-6\end{bmatrix}\to\begin{bmatrix}1&0&-1&-2\\0&-1&-2&-3\\0&0&0&0\end{bmatrix}\to\begin{bmatrix}1&0&-1&-2\\0&1&2&3\\0&0&0&0\end{bmatrix}$$

$-2\ -4\ -6\ -8$

$-3\ -6\ -9\ -12$

$0\ -2\ -4\ -6$

$0\ 2\ 4\ 6$

RREF

EX.

$$\begin{bmatrix}3&1\\4&2\end{bmatrix}\to\begin{bmatrix}3&1\\2&1\end{bmatrix}\to\begin{bmatrix}1&0\\2&1\end{bmatrix}\to\begin{bmatrix}1&0\\0&1\end{bmatrix}$$ is in RREF

Mult. Row 2 by $\frac{1}{2}$

$-2\ -1$

$-2\ 0$

or

$$\begin{bmatrix}3&1\\4&2\end{bmatrix}\to\begin{bmatrix}4&2\\3&1\end{bmatrix}\to\begin{bmatrix}1&1\\3&1\end{bmatrix}\to\begin{bmatrix}1&1\\0&-2\end{bmatrix}\to\begin{bmatrix}1&1\\0&-1\end{bmatrix}\to\begin{bmatrix}1&0\\0&1\end{bmatrix}$$

switch

$-3\ -1$

$-3\ -3$

Mult. Row 2 by $\frac{1}{2}$

then mult. Row 2 by -1

Sometimes adders help avoid fractions.

Th. Let $A \in F^m_n$ and $A \underset{row}{\sim} C$ in RREF with rank$(C) = r$. To solve lin. sys. $AX = B$ row reduce $[A|B]$ to $[C|D]$ and interpret $[C|D]$. Then

① If $r < m$ then C has $m-r$ zero rows, so if any of those has a non-zero entry in D then $AX = B$ is <u>inconsistent</u> (no solutions). If B is general, get $m-r$ consistency conditions so if $r = m$ then $AX = B$ is consistent for all B.

② If $AX = B$ is consistent then the solution set has $n-r$ free variables corresponding to the <u>non-pivot</u> columns. When $r = n$ get a unique solution.

Functional Viewpoint:

For $A = [a_{ij}] \in F^m_n$ define the associated map

$L_A : \underset{\text{domain}}{F^n} \longrightarrow \underset{\text{codomain}}{F^m}$ by $L_A(X) = AX$, $\forall X \in F^n$.

Th: The map L_A satisfies, $\forall X, Y \in F^n$, $\forall \alpha \in F$,

① $L_A(X+Y) = L_A(X) + L_A(Y)$

② $L_A(\alpha \cdot X) = \alpha \cdot L_A(X)$, ③ $L_A(0^n_1) = 0^m_1$

Pf. ① $L_A(X+Y) = A(X+Y) = AX + AY = L_A(X) + L_A(Y)$

② $L_A(\alpha \cdot X) = A(\alpha \cdot X) = \alpha \cdot (AX) = \alpha \cdot L_A(X)$.

These are just restatements of the p. 12 Theorem.

③ $L_A(0^n_1) = A 0^n_1 = 0^m_1$. $\square$