Complex vector spaces: Definition

Say $(V, +, \cdot, \theta)$ is a complex vector space (or V is a vector space over $\mathbb{C}$) when V obeys all the usual vector space axioms where scalars are in $\mathbb{C}$ (instead of in $\mathbb{R}$).

Ex 1: $\mathbb{C}^n = \left\{ \begin{bmatrix} z_1 \\ \vdots \\ z_n \end{bmatrix} \middle| z_j \in \mathbb{C},\ 1 \le j \le n \right\}$ with the usual $+$ and $\cdot$

Ex 2: $\mathbb{C}^m_n = \{ A = [a_{ij}] \mid a_{ij} \in \mathbb{C},\ 1 \le i \le m,\ 1 \le j \le n \}$

$=$ $m \times n$ complex matrices. As before, For $A \in \mathbb{C}^m_n$, $L_A : \mathbb{C}^n \to \mathbb{C}^m$ is $L_A(X) = AX$.

Can do <u>any</u> linear algebra problem for complex vector spaces:

Solve linear system $AX=B$,

Find $Ker(L)$, $Range(L)$ for any linear map $L: V \to W$ for complex v. spaces V and W,

Find a basis for a subspace $U \leq V$,

For bases S in V, T in W, $L: V \to W$, find ${}_T[L]_S \in \mathbb{C}^m_n$ if $\dim(V)=n$, $\dim(W)=m$.

Have <u>std</u> basis $S=\left\{e_1=\begin{bmatrix}1\\0\\\vdots\\0\end{bmatrix}, \dots, e_n=\begin{bmatrix}0\\\vdots\\0\\1\end{bmatrix}\right\}$ of $\mathbb{C}^n$

since $\mathbb{C}^n=\left\{Z=\begin{bmatrix}z_1\\\vdots\\z_n\end{bmatrix}=\sum_{j=1}^{n} z_j e_j \;\middle|\; z_j \in \mathbb{C},\ 1 \leq j \leq n\right\}$.

Ex: For $A = \begin{bmatrix} i & 1+i & 1-i \\ 1+i & 1-i & 1 \end{bmatrix}$ Solve $AX = 0^2_1$ $X \in \mathbb{C}^3$

$$\left[\begin{array}{ccc|c} i & 1+i & 1-i & 0 \\ 1+i & 1-i & 1 & 0 \end{array}\right] \quad \begin{array}{l} -iR_1 \to R_1 \\ (1-i)R_2 \to R_2 \end{array}$$

using $(1-i)(1-i) = 1-1-2i = -2i$

$$\to \left[\begin{array}{ccc|c} 1 & 1-i & -1-i & 0 \\ 2 & -2i & 1-i & 0 \end{array}\right] \quad -2R_1 + R_2 \to R_2$$

$$\begin{array}{ccc} -2 & -2+2i & 2+2i \end{array} \qquad + \qquad \begin{array}{ccc} 0 & -1+i & 2-i \end{array}$$

$$\boxed{\begin{array}{l} -\frac{1}{2}(3+i)(-1+i) = \\ -\frac{1}{2}(-4+2i) = 2-i \end{array}}$$

$$\to \left[\begin{array}{ccc|c} 1 & 1-i & -1-i & 0 \\ 0 & -2 & 3+i & 0 \end{array}\right] \to \left[\begin{array}{ccc|c} 1 & 1-i & -1-i & 0 \\ 0 & 1 & -\frac{1}{2}(3+i) & 0 \end{array}\right] \quad (-1+i)R_2 + R_1 \to R_1$$

$$\to \left[\begin{array}{ccc|c} 1 & 0 & 1-2i & 0 \\ 0 & 1 & -\frac{1}{2}(3+i) & 0 \end{array}\right] \quad \begin{array}{l} x_1 = (-1+2i)z \\ x_2 = \frac{1}{2}(3+i)z \\ x_3 = z \in \mathbb{C} \text{ free} \end{array}$$

$\dim(Nul(A)) = 1$

$$Nul(A) = \left\{ z \begin{bmatrix} -1+2i \\ \frac{1}{2}(3+i) \\ 1 \end{bmatrix} \in \mathbb{C}^3 \;\middle|\; z \in \mathbb{C} \right\} = \left\langle \begin{bmatrix} -1+2i \\ \frac{3}{2} + \frac{1}{2}i \\ 1 \end{bmatrix} \right\rangle$$

Std. dot product in $\mathbb{C}^n$:

For $Z = \begin{bmatrix} z_1 \\ \vdots \\ z_n \end{bmatrix}$, $W = \begin{bmatrix} w_1 \\ \vdots \\ w_n \end{bmatrix} \in \mathbb{C}^n$ define

$$Z \cdot W = \sum_{j=1}^{n} z_j \overline{w_j}$$ (note complex conjugate on W coordinates)

$$= Z^T \overline{W}$$ where $\overline{W} = \begin{bmatrix} \overline{w_1} \\ \vdots \\ \overline{w_n} \end{bmatrix}$. $\forall a, b \in \mathbb{C}$,

<u>Then</u>: $(aZ + bZ') \cdot W = a(Z \cdot W) + b(Z' \cdot W)$ but

$Z \cdot (aW + bW') = \overline{a}(Z \cdot W) + \overline{b}(Z \cdot W')$

called "sesquilinear", linear in first input, conjugate linear in second input.

$$\cdot : \mathbb{C}^n \times \mathbb{C}^n \longrightarrow \mathbb{C}$$

Also; $Z \cdot W = \overline{W \cdot Z}$ (conjugate symm.)

and $Z \cdot Z = \sum_{j=1}^{n} z_j \overline{z_j} = \sum_{j=1}^{n} (a_j^2 + b_j^2) \geq 0$ (real)

where $z_j = a_j + b_j i$, and $Z \cdot Z = 0$ iff $Z = 0^n$,

called "positive definite".

This dot product gives geometry on $\mathbb{C}^n$:

$\|Z\| = \sqrt{Z \cdot Z} \geq 0$ length, etc.

Important advantage working over $\mathbb{C}$ is all polynomials factor into linear factors.

Ex: $X^2 + 1 = (X + i)(X - i)$

$ax^2+bx+c=0$ has roots $\dfrac{-b\pm\sqrt{b^2-4ac}}{2a}$

in $\mathbb{R}$ when $b^2-4ac \geq 0$

in $\mathbb{C}$ when $b^2-4ac<0$.

Application to diagonalization:

__Ex.__ $A=\begin{bmatrix}0 & 1\\ -1 & 0\end{bmatrix}$ has $\mathrm{Char}_A(\lambda)=\begin{vmatrix}\lambda & -1\\ 1 & \lambda\end{vmatrix}=\lambda^2+1$

$=(\lambda+i)(\lambda-i)$

has two distinct complex e-values, $\lambda_1=-i$, $\lambda_2=i$.

Espaces: A_{λ_1} : $\left[\begin{array}{cc|c}-i & -1 & 0\\ 1 & -i & 0\end{array}\right]\to\left[\begin{array}{cc|c}1 & -i & 0\\ 0 & 0 & 0\end{array}\right]$ $x_1=iz$, $x_2=z\in\mathbb{C}$ free

$A_{\lambda_1}=\left\langle\begin{bmatrix}i\\1\end{bmatrix}\right\rangle$. A_{λ_2}: $\left[\begin{array}{cc|c}i & -1 & 0\\ 1 & i & 0\end{array}\right]\to\left[\begin{array}{cc|c}1 & i & 0\\ 0 & 0 & 0\end{array}\right]$ $x_1=-iz$, $x_2=z\in\mathbb{C}$ free

$A_{\lambda_2}=\left\langle\begin{bmatrix}-i\\1\end{bmatrix}\right\rangle$ Get e-basis $T=\left\{\begin{bmatrix}i\\1\end{bmatrix},\begin{bmatrix}-i\\1\end{bmatrix}\right\}$ for $\mathbb{C}^2$

If $S=\left\{\begin{bmatrix}1\\0\end{bmatrix},\begin{bmatrix}0\\1\end{bmatrix}\right\}$ is std basis of $\mathbb{C}^2$,

transition matrix $P = {}_S P_T = \begin{bmatrix} i & -i \\ 1 & 1 \end{bmatrix}$

has inverse $P^{-1} = {}_T P_S = \dfrac{1}{2i}\begin{bmatrix} 1 & i \\ -1 & i \end{bmatrix} = \dfrac{1}{2}\begin{bmatrix} -i & 1 \\ i & 1 \end{bmatrix}$

and $P^{-1}AP = \dfrac{1}{2}\begin{bmatrix} -i & 1 \\ i & 1 \end{bmatrix}\begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}\begin{bmatrix} i & -i \\ 1 & 1 \end{bmatrix}$

$$= \frac{1}{2}\begin{bmatrix} -i & 1 \\ i & 1 \end{bmatrix}\begin{bmatrix} 1 & 1 \\ -i & i \end{bmatrix} = \frac{1}{2}\begin{bmatrix} -2i & 0 \\ 0 & 2i \end{bmatrix} = \begin{bmatrix} -i & 0 \\ 0 & i \end{bmatrix}$$

$$= \begin{bmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{bmatrix} = D \text{ is diagonal.}$$

So working over $\mathbb{C}$ allows more matrices to be diagonalizable, but still not all.

Ex: $A = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}$ $\quad \mathrm{Char}_A(\lambda) = \begin{vmatrix} (\lambda-1) & -1 \\ 0 & (\lambda-1) \end{vmatrix} = (\lambda-1)^2$

has only e-value $\lambda_1 = 1$, $k_1 = 2$

$A_{\lambda_1}: \left[\begin{array}{cc|c} 0 & -1 & 0 \\ 0 & 0 & 0 \end{array}\right]$ $\quad x_1 = z \in \mathbb{C}$ free, $x_2 = 0$

$A_{\lambda_1} = \left\{ \begin{bmatrix} z \\ 0 \end{bmatrix} = z\begin{bmatrix} 1 \\ 0 \end{bmatrix} \;\middle|\; z \in \mathbb{C} \right\} = \left\langle \begin{bmatrix} 1 \\ 0 \end{bmatrix} \right\rangle$

$g_1 = 1 < 2 = k_1$

Cannot find a basis of $\mathbb{C}^2$ consisting of e-vectors for A.

A is not diag-able over $\mathbb{C}$.

Th. In $\mathbb{C}^n$ with its standard dot product, $z \cdot w = z^T \overline{w}$, for any $A \in \mathbb{C}^n_n$ we have

$$(Az) \cdot w = z \cdot (\overline{A}^T w).$$

Pf.

$$(Az) \cdot w = (Az)^T \overline{w} = (z^T A^T)\overline{w} = z^T (A^T \overline{w})$$

$$= z^T \overline{(\overline{A}^T w)} = z \cdot (\overline{A}^T w). \quad \square$$

Def. For $A \in \mathbb{C}^n_n$, define the <u>Hermitian conjugate</u>, $A^H = \overline{A}^T = A^* = \overline{(A^T)}$.

Def. Say $A \in \mathbb{C}^n_n$ is <u>Hermitian</u> when $A^H = A$, <u>skew-Hermitian</u> (or anti-Hermitian) when $A^H = -A$, <u>unitary</u> when $A^H = A^{-1}$.

<u>Th</u>. If $A \in \mathbb{C}^n_n$ is unitary then $\forall Z, W \in \mathbb{C}^n$, $(AZ)\cdot(AW) = Z \cdot W$.

<u>Pf</u>. $(AZ)\cdot(AW) = Z\cdot(A^H A W) = Z\cdot(A^{-1}AW) = Z\cdot W$. $\square$

<u>Th</u>. $A \in \mathbb{C}^n_n$ is unitary iff $\{Col_j(A) \mid 1 \le j \le n\}$ is an orthonormal set in $\mathbb{C}^n$ w.r.t. the std. dot product.

<u>Pf</u>. $Col_i(A)\cdot Col_j(A) = Col_i(A)^T \overline{Col_j(A)}$

$= Row_i(A^T)\, Col_j(\bar{A}) = \delta_{ij}$ iff $A^T \bar{A} = I_n$

iff $\bar{A}^T A = I_n$ iff $A^H = A^{-1}$. $\square$

<u>Th</u>. If $A = A^T \in \mathbb{R}^n_n$ then all eigenvalues of A are real.

<u>Pf</u>. Considering $A \in \mathbb{C}^n_n$, $\mathrm{Char}_A(t) = \prod_{i=1}^{r} (t-\lambda_i)^{m_i}$ for $\lambda_1, \dots, \lambda_r \in \mathbb{C}$ the distinct e-values of A. Let $\theta \neq X \in \mathbb{C}^n$ be an e-vector for A with e-value $\lambda_i \in \mathbb{C}$, so $AX = \lambda_i X$. Then

$\lambda_i (X \cdot X) = (\lambda_i X) \cdot X = (AX) \cdot X = X \cdot (A^H X) = X \cdot (AX)$
$= X \cdot (\lambda_i X) = \overline{\lambda_i} (X \cdot X)$ since $A^H = \overline{A}^T = A^T = A$,
so $(\lambda_i - \overline{\lambda_i})(X \cdot X) = 0$. But $X \cdot X > 0$ so $\lambda_i = \overline{\lambda_i}$ means $\lambda_i \in \mathbb{R}$. ∎