Def. Let $S \subseteq \mathbb{R}^n$. Define "S perp" to be
$S^\perp = \{X \in \mathbb{R}^n \mid X \cdot Y = 0, \forall Y \in S\}$.

Th. For any nonempty $S \subseteq \mathbb{R}^n$, $S^\perp = \langle S \rangle^\perp \leq \mathbb{R}^n$.

Pf. $\theta = 0_1^n \in S^\perp$ since $0_1^n \cdot Y = 0$, $\forall Y \in S$.
$\forall X_1, X_2 \in S^\perp$, $(X_1 + X_2) \cdot Y = (X_1 \cdot Y) + (X_2 \cdot Y) = 0 + 0 = 0$
$\forall Y \in S$ so $X_1 + X_2 \in S^\perp$ (closed under $+$).
$\forall X \in S^\perp$, $\forall a \in \mathbb{R}$, $(aX) \cdot Y = a(X \cdot Y) = a(0) = 0$
$\forall Y \in S$ so $aX \in S^\perp$ (closed under scalar prod. $\cdot$)

$\forall Y_1, \ldots, Y_m \in S$, $\forall a_1, \ldots, a_m \in \mathbb{R}$, $\forall X \in S^\perp$, $X \cdot Y_i = 0$
for $1 \leq i \leq m$, so $X \cdot \sum_{i=1}^m a_i Y_i = \sum_{i=1}^m a_i (X \cdot Y_i) = 0$ so $X \in \langle S \rangle^\perp$,
so $S^\perp \leq \langle S \rangle^\perp$.

Also, $S \subseteq \langle S \rangle$, so $X \in \langle S \rangle^\perp$ implies $X \in S^\perp$ so $\langle S \rangle^\perp \subseteq S^\perp$ giving $S^\perp = \langle S \rangle^\perp$. □

<u>Note</u>. If $S = \{v_1, \dots, v_m\} \subseteq \mathbb{R}^n$ then $X \in S^\perp$ iff $v_i \cdot X = 0$ for $1 \le i \le m$ iff $v_i^T X = 0$, $1 \le i \le m$ is the homog. lin. sys. $AX = 0$ where $Row_i(A) = v_i^T$, $A \in \mathbb{R}^m_n$.

<u>Ex</u>: If $S = \left\{ \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix} \right\}$ then find $S^\perp$ by solving

$$\left[\begin{array}{ccc|c} 1 & 1 & 1 & 0 \\ 1 & 2 & 3 & 0 \end{array}\right] \to \left[\begin{array}{ccc|c} 1 & 0 & -1 & 0 \\ 0 & 1 & 2 & 0 \end{array}\right]. \quad \begin{array}{l} x_1 = r \\ x_2 = -2r \\ x_3 = r \in \mathbb{R} \end{array}$$

so $S^\perp = \left\{ \begin{bmatrix} r \\ -2r \\ r \end{bmatrix} \in \mathbb{R}^3 \,\middle|\, r \in \mathbb{R} \right\} = \left\langle \begin{bmatrix} 1 \\ -2 \\ 1 \end{bmatrix} \right\rangle$

and $\mathbb{R}^3 = \langle S \rangle \oplus S^\perp$. $\oplus$ means "direct sum"

Def. Let V be a vector space over field F | 290
and let $W_1, \dots, W_m \le V$ be subspaces of V.
Then $W = W_1 + \cdots + W_m = \{w_1 + \cdots + w_m \in V \mid w_i \in W_i\}$
is the set of all sums of vectors from those
subspaces.

Th: $W_1 + \cdots + W_m \le V$ is a subspace called the
sum of $W_1, \dots, W_m$.

Pf. Exercise. Show $W = W_1 + \cdots + W_m$ is closed
under $+$ and $\cdot$ and contains θ_V. $\square$

Def. Say that sum $W = W_1 + \cdots + W_m$ is a **direct sum** when each $w \in W$ has a **unique** expression
as $w = w_1 + \cdots + w_m$ for $w_i \in W_i$. Notation is:

$$W = W_1 \hat{\oplus} \cdots \hat{\oplus} W_m .$$

<u>Ex</u>. Let $V = F^2$, $W_1 = \left\{ \begin{bmatrix} x \\ 0 \end{bmatrix} \in F^2 \middle| x \in F \right\}$, $W_2 = \left\{ \begin{bmatrix} 0 \\ y \end{bmatrix} \in F^2 \middle| y \in F \right\}$, $W_3 = \left\{ \begin{bmatrix} z \\ z \end{bmatrix} \in F^2 \middle| z \in F \right\}$.

Then $V = W_1 + W_2$ since $\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} x \\ 0 \end{bmatrix} + \begin{bmatrix} 0 \\ y \end{bmatrix} = w_1 + w_2$ for $w_1 = \begin{bmatrix} x \\ 0 \end{bmatrix} \in W_1$, $w_2 = \begin{bmatrix} 0 \\ y \end{bmatrix} \in W_2$. This expression is unique since $\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} a \\ 0 \end{bmatrix} + \begin{bmatrix} 0 \\ b \end{bmatrix} = \begin{bmatrix} a \\ b \end{bmatrix}$ is only true for $a = x$ and $b = y$. So $V = W_1 \oplus W_2$ is a dir. sum.

Also, $W_1 + W_3 = \left\{ \begin{bmatrix} x \\ 0 \end{bmatrix} + \begin{bmatrix} z \\ z \end{bmatrix} = \begin{bmatrix} x+z \\ z \end{bmatrix} \in F^2 \middle| x, z \in F \right\} = V$ since $\forall \begin{bmatrix} a \\ b \end{bmatrix} \in F^2$, $\begin{bmatrix} a \\ b \end{bmatrix} = \begin{bmatrix} x+z \\ z \end{bmatrix}$ for $x = a-b$, $z = b$ and that is unique so $V = W_1 \oplus W_3$. Show that $V = W_2 \oplus W_3$.

Questions: ① Is $V = W_1 + W_2 + W_3$? Yes 292

② Is $W_1 + W_2 + W_3$ a direct sum? No

Ex: Let $V = F^3$, $W_1 = \left\{ \begin{bmatrix} x \\ y \\ 0 \end{bmatrix} \in F^3 \,\middle|\, x, y \in F \right\}$,

$W_2 = \left\{ \begin{bmatrix} 0 \\ y \\ z \end{bmatrix} \in F^3 \,\middle|\, y, z \in F \right\}$, $W_3 = \left\{ \begin{bmatrix} x \\ 0 \\ z \end{bmatrix} \in F^3 \,\middle|\, x, z \in F \right\}$

Exercises: ① Find: $W_1 + W_2$. Is it a direct sum?

② Find $W_1 + W_3$. Is it a direct sum?

③ Find $W_2 + W_3$. Is it a direct sum?

④ For $1 \leq i \leq 3$ find $W_i^{\perp}$ and show $V = W_i \oplus W_i^{\perp}$.

⑤ Find the intersections $W_1 \cap W_2$, $W_1 \cap W_3$ and $W_2 \cap W_3$. Find $W_1 \cap W_2 \cap W_3$.

⑥ Is $W_1 + W_2 + W_3$ a direct sum?

Th. For subspaces $W_1, \dots, W_m \leq V$ the sum $W = W_1 + \cdots + W_m$ is a direct sum iff $W_i \cap \left(\sum_{j \neq i} W_j\right) = \{\theta_V\}$ for each $1 \leq i \leq m$.

For $m=2$ this says $W = W_1 + W_2$ is a direct sum iff $W_1 \cap W_2 = \{\theta_V\}$.

For $m=3$ this says $W = W_1 + W_2 + W_3$ is a dir. sum iff $W_1 \cap (W_2 + W_3) = \{\theta_V\}$ and $W_2 \cap (W_1 + W_3) = \{\theta_V\}$ and $W_3 \cap (W_1 + W_2) = \{\theta_V\}$.

Cor. Let B_i be a basis for $W_i \leq V$, $1 \leq i \leq m$. Then $W = W_1 + \cdots + W_m$ is a direct sum iff $B = B_1 \cup \cdots \cup B_m$ is a basis of W. Here, each B_i and B is understood to be a list.

Def. Let $W_1, \dots, W_t \leq \mathbb{R}^n$, and $W = W_1 \oplus \cdots \oplus W_t \leq \mathbb{R}^n$. If $W_i \perp W_j$ for $1 \leq i \neq j \leq t$ then say $W = W_1 \oplus \cdots \oplus W_t$ is an **orthogonal direct sum.**

Note: If $W_1 \perp W_2$ and $W_1 \perp W_3$ then $W_1 \perp (W_2 + W_3)$ and similarly for larger sums. So for mutually orthog. subspaces any sum is a direct sum.

Th: For any subspace $W \leq \mathbb{R}^n$, $W + W^{\perp} = \mathbb{R}^n$ is an orthog. direct sum decomposition of $\mathbb{R}^n$.

Pf. This is clear if $W = \{\theta\}$ is trivial.
Suppose $\{\theta\} \subsetneq W \subsetneq \mathbb{R}^n$.

$\forall X \in \mathbb{R}^n$, $\mathrm{Proj}_W(X) \in W$ is defined so that $X - \mathrm{Proj}_W(X) \in W^\perp$ so $X = \mathrm{Proj}_W(X) + (X - \mathrm{Proj}_W(X)) \in W + W^\perp$ shows $\mathbb{R}^n = W + W^\perp$. Clearly, $W \perp W^\perp$ by definition and $v \in W \cap W^\perp$ implies $v \cdot v = 0$ so $v = \theta$ so $W \cap W^\perp = \{\theta\}$. $\mathbb{R}^n = W \oplus W^\perp$ is an orthog. direct sum. $\square$

Let's try to apply these concepts to the eigenspaces $A_{\lambda_1}, \dots, A_{\lambda_r}$ for $A = A^T \in \mathbb{R}^n_n$ where we know $\lambda_1, \dots, \lambda_r \in \mathbb{R}$ and $\mathrm{Char}_A(t) = \prod_{i=1}^{r} (t - \lambda_i)^{m_i}$. $\mathbb{R}^n = A_{\lambda_1} \oplus A_{\lambda_1}^\perp$ is an orthog. direct sum.