<u>Claim</u>: $A_{\lambda_1}^{\perp}$ is an A-invariant subspace, L296

that is, $\forall Y \in A_{\lambda_1}^{\perp}$, $AY \in A_{\lambda_1}^{\perp}$.

<u>Pf</u>. Let $Y \in A_{\lambda_1}^{\perp}$ and $X \in A_{\lambda_1}$. Then

$$X \cdot (AY) = (A^T X) \cdot Y = (AX) \cdot Y = (\lambda_1 X) \cdot Y = \lambda_1 (X \cdot Y) = 0. \square$$

Let $T_1 = \{v_{11}, \dots, v_{1g_1}\}$ be one-basis of A_{λ_1} and $T_1^{\perp}$ any basis of $A_{\lambda_1}^{\perp}$ so $T_1 \cup T_1^{\perp} = S_1$ is a basis of $\mathbb{R}^n$ s.t. $L = L_A : \mathbb{R}^n \to \mathbb{R}^n$ has ${}_{S_1}[L]_{S_1}$ in block diag. form

$$\left[\begin{array}{c|c} \lambda_1 I_{g_1} & 0 \\ \hline 0 & B \end{array}\right] \text{ with } B = {}_{T_1^{\perp}}\left[L|_{A_{\lambda_1}^{\perp}}\right]_{T_1^{\perp}}.$$

Also, $\mathrm{Char}_A(t) = \mathrm{Char}_{\lambda_1 I_{g_1}}(t) \cdot \mathrm{Char}_B(t) = (t-\lambda_1)^{g_1} \mathrm{Char}_B(t)$.

If $g_1 < k_1$ then $(t-\lambda_1)$ divides $Char_B(t)$ so $A_{\lambda_1}^{\perp}$ would have to contain a λ_1-e-vector for A, contradicting $A_{\lambda_1} \cap A_{\lambda_1}^{\perp} = \{\theta\}$. So $g_1 = k_1$ and we are reduced to studing the restriction of $L =$ L_A to $A_{\lambda_1}^{\perp}$ where the $Char_B(t) = \prod_{i=2}^{r} (t-\lambda_i)^{k_i}$.

The problem is that we cannot just replace A by B because B need not be symmetric.

Use Gram-Schmidt process to change T_1 and $T_1^{\perp}$ into orthonormal bases of A_{λ_1} and $A_{\lambda_1}^{\perp}$ respectively. Then S_1 would be an o.n. basis of $\mathbb{R}^n$ and the transition matrix ${}_S P_{S_1} = P$ would be orthogonal, so $P^{-1} = P^T$. It means the block diag. form similar to A is P^TAP so $(P^TAP)^T = P^TA^TP = P^TAP$ says it is symmetric. Then $B = B^T$ in the corner, and we can apply these arguments to B. Inductively, we get B is orthogonally diag-able, each $g_i = k_i$. □

<u>Def</u>. Say $A \in \mathbb{C}^n_n$ is <u>normal</u> when $AA^H = A^HA$, that is, A commutes with $\bar{A}^T$.

<u>Note</u>. A unitary implies A is normal, but not conversely.

See Section 13.9 of (Schaums Outline ~~our textbook~~) for more results about when $A \in \mathbb{C}^n_n$ is guaranteed to be diag-able.