<u>Def</u>. For $1 \leq m, n \in \mathbb{Z}$ we say that a map (function) $L : F^n \to F^m$ is <u>linear</u> when

① $L(X+Y) = L(X) + L(Y)$, $\forall$ (<u>for all</u>) $X, Y \in F^n$, and

② $L(\alpha \cdot X) = \alpha \cdot L(X)$, $\forall X \in F^n$, $\forall \alpha \in F$.

So L_A defined on p. 36 is <u>linear</u>.

Important questions about (lin.) maps are:

When is L <u>injective</u>? <u>surjective</u>? <u>bijective</u>? <u>invertible</u>?

These concepts come from the general Theory of functions between sets, $f : S \to T$, so we will review them. But first have definitions:

<u>Def.</u> For $L : F^n \to F^m$ linear, define
$\ker(L) = \{X \in F^n \mid L(X) = 0_1^m\}$ and
$\mathrm{Range}(L) = \mathrm{Image}(L) = \{L(X) \in F^m \mid X \in F^n\}$

<u>Note:</u> When $L = L_A$ for some $A \in F_n^m$ then
$\ker(L_A) = \{X \in F^n \mid AX = 0_1^m\}$ is just the solution set of the homog. lin. sys. $AX = 0_1^m$ we know how to find by row reducing $[A \mid 0_1^m]$.

$\mathrm{Range}(L_A) = \{B \in F^m \mid AX = B \text{ is consistent}\}$
which we know how to find through consistency conditions coming from $[A|B] \xrightarrow[\text{red.}]{\text{row}} [\underset{\substack{\uparrow \\ \text{RREF}}}{C} \mid D]$.

<u>Ex</u>. Let $L : F^3 \to F^3$ be $L\begin{bmatrix}x_1\\x_2\\x_3\end{bmatrix} = \begin{bmatrix}x_1+2x_2+3x_3\\2x_1+3x_2+4x_3\\3x_1+4x_2+5x_3\end{bmatrix}$

$= \begin{bmatrix}1&2&3\\2&3&4\\3&4&5\end{bmatrix}\begin{bmatrix}x_1\\x_2\\x_3\end{bmatrix} = AX$ so $L = L_A$ for that $A \in F^3_3$.

To find $Ker(L) = Ker(L_A)$, row reduce $[A|0^3_1]$

$$\left[\begin{array}{ccc|c}1&2&3&0\\2&3&4&0\\3&4&5&0\end{array}\right] \to \left[\begin{array}{ccc|c}1&2&3&0\\0&-1&-2&0\\0&-2&-4&0\end{array}\right] \to \left[\begin{array}{ccc|c}1&0&-1&0\\0&1&2&0\\0&0&0&0\end{array}\right]$$

$-2\ -4\ -6\ \ 0$

$-3\ -6\ -9\ \ 0$

<u>Interp</u>

$x_1 = x_3 = r$

$x_2 = -2x_3 = -2r$

$x_3 = r \in F$ (free)

$$Ker(L) = \left\{X = r\begin{bmatrix}1\\-2\\1\end{bmatrix} \in F^3 \,\middle|\, r \in F\right\}$$

To find Range(L) find when $AX = B$ is consistent.

$$\left[\begin{array}{ccc|c}1&2&3&b_1\\2&3&4&b_2\\3&4&5&b_3\end{array}\right] \to \left[\begin{array}{ccc|c}1&2&3&b_1\\0&-1&-2&b_2-2b_1\\0&-2&-4&b_3-3b_1\end{array}\right] \to \left[\begin{array}{ccc|c}1&2&3&b_1\\0&1&2&2b_1-b_2\\0&0&0&b_1-2b_2+b_3\end{array}\right]$$

$-2\ -4\ -6\ -2b_1$

$-3\ -6\ -9\ -3b_1$

$0\ \ 2\ \ 4\ \ 4b_1-2b_2$

so need

$0 = b_1 - 2b_2 + b_3$

$$\text{Range}(L) = \left\{ B = \begin{bmatrix} b_1 \\ b_2 \\ b_3 \end{bmatrix} \in F^3 \,\middle|\, b_1 - 2b_2 + b_3 = 0 \right\}$$

$$= \left\{ \begin{bmatrix} 2b_2 - b_3 \\ b_2 \\ b_3 \end{bmatrix} \in F^3 \,\middle|\, b_2, b_3 \in F \right\}$$

$$= \left\{ b_2 \begin{bmatrix} 2 \\ 1 \\ 0 \end{bmatrix} + b_3 \begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix} \in F^3 \,\middle|\, b_2, b_3 \in F \right\}$$

$$= \left\{ \begin{bmatrix} b_1 \\ b_2 \\ -b_1 + 2b_2 \end{bmatrix} = b_1 \begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix} + b_2 \begin{bmatrix} 0 \\ 1 \\ 2 \end{bmatrix} \in F^3 \,\middle|\, b_1, b_2 \in F \right\}$$

expresses the answer as all linear combinations of two particular col. vectors. in two different ways. There can be many correct ways to express an answer.

Math 507 | Review of theory of functions [4]
between sets:

Let S and T be any sets. Say $f: S \to T$ is a function from __domain__ S to __codomain__ T when $\forall s \in S$, $\exists t \in T$ such that $f(s) = t$.

__Def.__ $\text{Range}(f) = \text{Image}(f) = \{f(s) \in T \mid s \in S\}$
$= \{t \in T \mid \exists s \in S \text{ s.t. } f(s) = t\} \subseteq T$.

Def. Say $f: S \to T$ is __surjective__ (onto) if $\text{Range}(f) = T$.

__Def.__ Say $f: S \to T$ is __injective__ (one-to-one) when
$\forall s_1, s_2 \in S$, $f(s_1) = f(s_2)$ implies $s_1 = s_2$.

This is logically equivalent to its __contrapositive__:
$\forall s_1, s_2 \in S$, $s_1 \neq s_2$ implies $f(s_1) \neq f(s_2)$.

<u>Def.</u> Say $f : S \to T$ is <u>bijective</u> (one-to-one correspondence) when f is both injective and surjective.

<u>Def.</u> Say $f : S \to T$ is <u>invertible</u> when $\exists g : T \to S$ such that ① $\forall s \in S$, $g(f(s)) = s$, and
② $\forall t \in T$, $f(g(t)) = t$.

<u>Th.</u> If $f : S \to T$ is invertible then there is only one $g : T \to S$ satisfying ① and ② above, so we can denote that <u>unique</u> inverse of f by f^{-1}.

<u>Pf:</u> Given $f : S \to T$ suppose $\exists g_1, g_2 : T \to S$ such that ① $\forall s \in S$, $g_i(f(s)) = s$ for $i = 1, 2$, and
② $\forall t \in T$, $f(g_i(t)) = t$ for $i = 1, 2$.

By ① with $i=1$ and $s = g_2(t)$ we have, $\forall t \in T$,
$g_1(f(g_2(t))) = g_2(t)$, and by ② with $i=2$,
$g_1(f(g_2(t))) = g_1(t)$, so $g_1(t) = g_2(t)$ so $g_1 = g_2$ $\square$

<u>Def</u>. Let $f: S \to T$ and $g: R \to S$ for sets R, S and T. Define the <u>composition</u> function $(f \circ g): R \to T$ by $(f \circ g)(r) = f(g(r))$, $\forall r \in R$.

This is summarized by the <u>diagram</u>:

$$\begin{tikzcd} R \arrow[r, "g"] \arrow[dr, "f \circ g"'] & S \arrow[d, "f"] \\ & T \end{tikzcd}$$

$$\underset{\in R}{\xrightarrow{r}} \boxed{g} \underset{\in S}{\xrightarrow{g(r)=s}} \boxed{f} \underset{\in T}{\xrightarrow{f(g(r))}}$$

Def: For any set S, the identity function (map) is $I_S : S \to S$ where $\forall s \in S$, $I_S(s) = s$.

Note: $f : S \to T$ is invertible when $\exists g : T \to S$ s.t. ① $g \circ f = I_S$ and ② $f \circ g = I_T$.

Also, $I_T \circ f = f$ for **any** $f : S \to T$ and

$g \circ I_T = g$ for **any** $g : T \to S$.

Th: $f : S \to T$ is invertible iff f is bijective.

Th: Composition of functions is associative.

Pf. let

$$\begin{tikzcd}
R \arrow[r, "g"] \arrow[dr, "(f\circ g)"'] \arrow[drr, bend left=60, "(h\circ f)\circ g"] \arrow[drr, bend right=60, "h\circ(f\circ g)"'] & S \arrow[d, "f"] \arrow[dr, "(h\circ f)"] & \\
 & T \arrow[r, "h"] & U
\end{tikzcd}$$

Then $\forall r \in R$,

$(h\circ(f\circ g))(r) = h(f(g(r)))$

$= ((h\circ f)\circ g)(r)$, so $\square$

$h\circ(f\circ g) = (h\circ f)\circ g$.

Brief review of logic and set theory:

In basic logic we talk about "statements" or "propositions", that is, assertions which are "true" or "false". We use single letters to stand for assertions but we can combine them with logical connectives like "and", "or", "implies", "not", "if and only if" to get more complicated assertions. Those have "truth value" depending on the truth value (T or F) of the simpler pieces in the expression. In some books symbols are used for the connectives, for example, "and" = "$\wedge$", "or" = "$\vee$", "implies" = "$\Rightarrow$", "not" = "$\sim$"

and "if and only if" = "iff" = "$\Leftrightarrow$".

These connectives are best defined by "truth tables" as follows:

P and Q is defined by:

P\Q	T	F
T	T	F
F	F	F

so "P and Q" is T only when P is T <u>and</u> Q is T, false otherwise.

P or Q is defined by:

P\Q	T	F
T	T	T
F	T	F

so "P or Q" is T when either P = T or Q = T but is F only when both P = F and Q = F.

not(P) is defined by table to have the opposite value of P.

P	T	F
not(P)	F	T

"If P then Q" = "P implies Q" = "$P \Rightarrow Q$"

is defined by the table

$P \backslash Q$	T	F
T	T	F
F	T	T

so $P \Rightarrow Q$ is only false when $P = T$ and $Q = F$, otherwise it is T.

"$P \Leftrightarrow Q$" = "P iff Q" has table

$P \backslash Q$	T	F
T	T	F
F	F	T

so it is T when P an Q have the same truth value, F if not the same.

More advanced logic includes "quantifiers" "for all" = "$\forall$" and "there exists" = "$\exists$". These are used a lot in math with set theory.