

Def. For  $1 \leq m, n \in \mathbb{Z}$  we say that a map (function)  $L: F^n \rightarrow F^m$  is linear when

①  $L(X+Y) = L(X) + L(Y)$ ,  $\forall$  (for all)  $X, Y \in F^n$ , and

②  $L(\alpha \cdot X) = \alpha \cdot L(X)$ ,  $\forall X \in F^n$ ,  $\forall \alpha \in F$ .

So  $L_A$  defined on p. 36 is linear.

Important questions about (lin.) maps are:

When is  $L$  injective? surjective?

bijjective? invertible?

These concepts come from the general theory of functions between sets,  $f: S \rightarrow T$ , so we will review them. But first here definitions:

Def. For  $L: F^n \rightarrow F^m$  linear, define 38  
 $\ker(L) = \{X \in F^n \mid L(X) = 0^m\}$  and  
 $\text{Range}(L) = \text{Image}(L) = \{L(X) \in F^m \mid X \in F^n\}$   
Note: When  $L = L_A$  for some  $A \in F_n^m$  then  
 $\ker(L_A) = \{X \in F^n \mid AX = 0^m\}$  is just the solution  
set of the homog. lin. sys.  $AX = 0^m$ , we know  
how to find by row reducing  $[A \mid 0^m]$ .  
 $\text{Range}(L_A) = \{B \in F^m \mid AX = B \text{ is consistent}\}$   
which we know how to find through consistency  
conditions coming from  $[A \mid B] \xrightarrow[\text{red.}]{\text{row}} \begin{matrix} [C \mid D] \\ \uparrow \\ \text{RREF} \end{matrix}$ .

Ex. Let  $L: F^3 \rightarrow F^3$  be  $L \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} x_1 + 2x_2 + 3x_3 \\ 2x_1 + 3x_2 + 4x_3 \\ 3x_1 + 4x_2 + 5x_3 \end{bmatrix}$  39

$= \begin{bmatrix} 1 & 2 & 3 \\ 2 & 3 & 4 \\ 3 & 4 & 5 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = AX$  so  $L = L_A$  for that  $A \in F_3^3$ .

To find  $\text{Ker}(L) = \text{Ker}(L_A)$ , row reduce  $[A|0^3]$

$$\begin{bmatrix} 1 & 2 & 3 & | & 0 \\ 2 & 3 & 4 & | & 0 \\ 3 & 4 & 5 & | & 0 \end{bmatrix} \rightarrow \begin{bmatrix} 1 & 2 & 3 & | & 0 \\ 0 & -1 & -2 & | & 0 \\ 0 & -2 & -4 & | & 0 \end{bmatrix} \rightarrow \begin{bmatrix} 1 & 0 & -1 & | & 0 \\ 0 & 1 & 2 & | & 0 \\ 0 & 0 & 0 & | & 0 \end{bmatrix}$$

Interp  
 $x_1 = x_3 = r$   
 $x_2 = -2x_3 = -2r$   
 $x_3 = r \in F$  (free)

$$\text{Ker}(L) = \left\{ X = r \begin{bmatrix} 1 \\ -2 \\ 1 \end{bmatrix} \in F^3 \mid r \in F \right\}$$

To find  $\text{Range}(L)$  find when  $AX=B$  is consistent.

$$\begin{bmatrix} 1 & 2 & 3 & | & b_1 \\ 2 & 3 & 4 & | & b_2 \\ 3 & 4 & 5 & | & b_3 \end{bmatrix} \rightarrow \begin{bmatrix} 1 & 2 & 3 & | & b_1 \\ 0 & -1 & -2 & | & b_2 - 2b_1 \\ 0 & -2 & -4 & | & b_3 - 3b_1 \end{bmatrix} \rightarrow \begin{bmatrix} 1 & 2 & 3 & | & b_1 \\ 0 & 1 & 2 & | & 2b_1 - b_2 \\ 0 & 0 & 0 & | & b_1 - 2b_2 + b_3 \end{bmatrix}$$

so need

$$\begin{matrix} -2 & -4 & -6 & -2b_1 \\ -3 & -6 & -9 & -3b_1 \end{matrix} \quad \begin{matrix} 0 & 2 & 4 & 4b_1 - 2b_2 \\ 0 & 0 & 0 & 0 = b_1 - 2b_2 + b_3 \end{matrix}$$

$$\text{Range}(L) = \left\{ B = \begin{bmatrix} b_1 \\ b_2 \\ b_3 \end{bmatrix} \in F^3 \mid b_1 - 2b_2 + b_3 = 0 \right\} \quad \underline{40}$$

$$= \left\{ \begin{bmatrix} 2b_2 - b_3 \\ b_2 \\ b_3 \end{bmatrix} \in F^3 \mid b_2, b_3 \in F \right\}$$

$$= \left\{ b_2 \begin{bmatrix} 2 \\ 1 \\ 0 \end{bmatrix} + b_3 \begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix} \in F^3 \mid b_2, b_3 \in F \right\}$$

$$= \left\{ \begin{bmatrix} b_1 \\ b_2 \\ -b_1 + 2b_2 \end{bmatrix} = b_1 \begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix} + b_2 \begin{bmatrix} 0 \\ 1 \\ 2 \end{bmatrix} \in F^3 \mid b_1, b_2 \in F \right\}$$

expresses the answer as all linear combinations of two particular col. vectors. in two different ways. There can be many correct ways to express an answer.

# Math 507 | Review of theory of functions [4]

between sets:

Let  $S$  and  $T$  be any sets. Say  $f: S \rightarrow T$  is a function from domain  $S$  to codomain  $T$  when

$\forall a \in S, \exists t \in T$  such that  $f(a) = t$ .

Def.  $\text{Range}(f) = \text{Image}(f) = \{f(a) \in T \mid a \in S\}$

$= \{t \in T \mid \exists a \in S \text{ s.t. } f(a) = t\} \subseteq T$ .

Def. Say  $f: S \rightarrow T$  is surjective (onto) if  $\text{Range}(f) = T$ .

Def. Say  $f: S \rightarrow T$  is injective (one-to-one) when

$\forall a_1, a_2 \in S, f(a_1) = f(a_2)$  implies  $a_1 = a_2$ .

This is logically equivalent to its contrapositive:

$\forall a_1, a_2 \in S, a_1 \neq a_2$  implies  $f(a_1) \neq f(a_2)$ .

Def. Say  $f: S \rightarrow T$  is bijjective (one-to-one correspondence) when  $f$  is both injective and surjective. 142

Def. Say  $f: S \rightarrow T$  is invertible when  $\exists g: T \rightarrow S$  such that

- ①  $\forall a \in S, g(f(a)) = a$ , and
- ②  $\forall t \in T, f(g(t)) = t$ .

Th. If  $f: S \rightarrow T$  is invertible then there is only one  $g: T \rightarrow S$  satisfying ① and ② above, so we can denote that unique inverse of  $f$  by  $f^{-1}$ .

Pf. Given  $f: S \rightarrow T$  suppose  $\exists g_1, g_2: T \rightarrow S$  such that

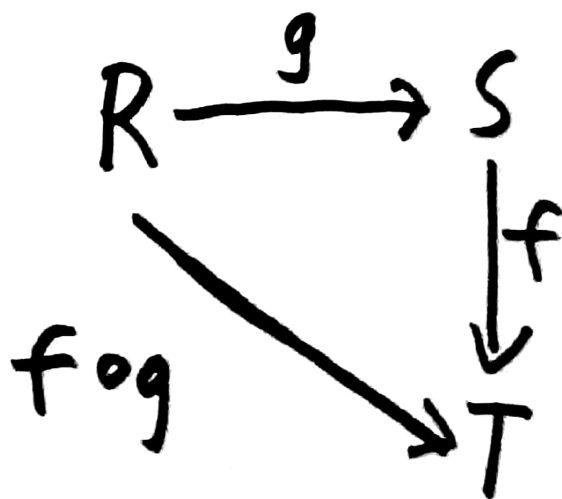
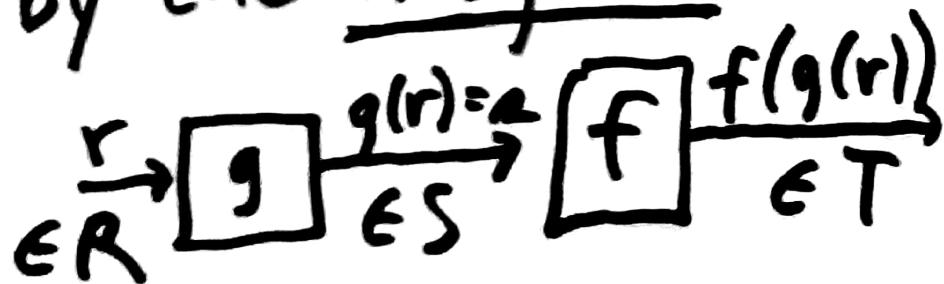
- ①  $\forall a \in S, g_i(f(a)) = a$  for  $i=1, 2$ , and
- ②  $\forall t \in T, f(g_i(t)) = t$  for  $i=1, 2$ .

By ① with  $i=1$  and  $a = g_2(t)$  we have,  $\forall t \in T$ , 43  
 $g_1(f(g_2(t))) = g_2(t)$ , and by ② with  $i=2$ ,  
 $g_1(f(g_2(t))) = g_1(t)$ , so  $g_1(t) = g_2(t)$  so  $g_1 = g_2$   $\square$

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Def. Let  $f: S \rightarrow T$  and  $g: R \rightarrow S$  for sets  $R, S$  and  $T$ . Define the composition function  $(f \circ g): R \rightarrow T$  by  $(f \circ g)(r) = f(g(r))$ ,  $\forall r \in R$ .

This is summarized by the diagram:



Def. For any set  $S$ , the identity function (map) is  $I_S: S \rightarrow S$  where  $\forall a \in S, I_S(a) = a$ . [44]

Note:  $f: S \rightarrow T$  is invertible when  $\exists g: T \rightarrow S$  s.t. ①  $g \circ f = I_S$  and ②  $f \circ g = I_T$ .

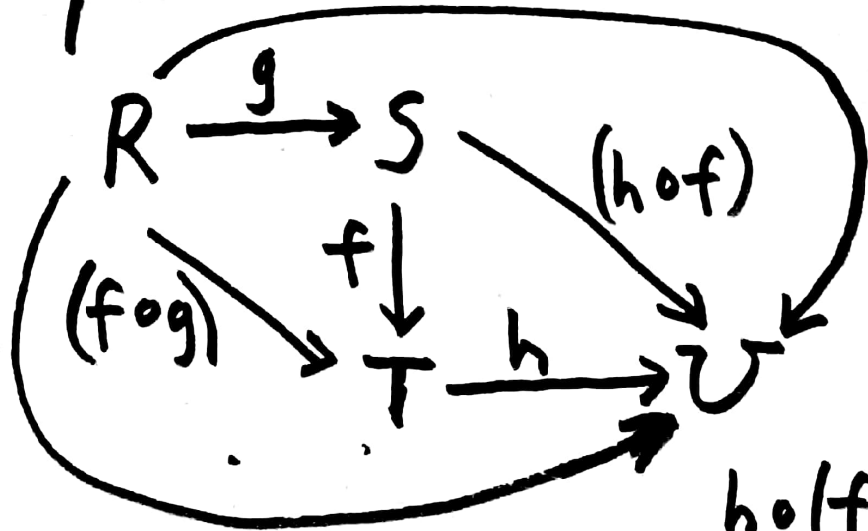
Also,  $I_T \circ f = f$  for any  $f: S \rightarrow T$  and

$g \circ I_T = g$  for any  $g: T \rightarrow S$ .

Th:  $f: S \rightarrow T$  is invertible iff  $f$  is bijective.

Th: Composition of functions is associative. Then  $\forall r \in R$ ,

Pf. let



$(h \circ f) \circ g$

$$(h \circ (f \circ g))(r) = h(f(g(r)))$$

$$= ((h \circ f) \circ g)(r), \text{ so } \square$$

$$h \circ (f \circ g) = (h \circ f) \circ g.$$



## Brief review of logic and set theory: 145

In basic logic we talk about "statements" or "propositions", that is, assertions which are "true" or "false". We use single letters to stand for assertions but we can combine them with logical connectives like "and", "or", "implies", "not", "if and only if" to get more complicated assertions. Those have "truth value" depending on the truth value (T or F) of the simpler pieces in the expression. In some books symbols are used for the connectives, for example, "and" = " $\wedge$ ", "or" = " $\vee$ ", "implies" = " $\Rightarrow$ ", "not" = " $\sim$ "

and "if and only if" = "iff" = " $\Leftrightarrow$ ". [46]  
 These connectives are best defined by  
 "truth tables" as follows:

P and Q is defined by:

P \ Q	T	F
T	T	F
F	F	F

so "P and Q" is T  
 only when P is T and  
 Q is T, false otherwise.

P or Q is defined by:

P \ Q	T	F
T	T	T
F	T	F

so "P or Q" is T when  
 either  $P=T$  or  $Q=T$   
 but is F only when both  $P=F$  and  $Q=F$ .

not(P) is defined by table

P	T	F
not(P)	F	T

to have the opposite value of P.

"If P then Q" = "P implies Q" = " $P \Rightarrow Q$ " 147

is defined by the table  
so  $P \Rightarrow Q$  is only false  
when  $P=T$  and  $Q=F$ ,  
otherwise it is T.

$P \backslash Q$	T	F
T	T	F
F	T	T

" $P \Leftrightarrow Q$ " = "P iff Q" has table  
so it is T when P and Q have  
the same truth value,  
F if not the same.

$P \backslash Q$	T	F
T	T	F
F	F	T

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More advanced logic includes "quantifiers"  
"for all" = " $\forall$ " and "there exists" = " $\exists$ ".  
These are used a lot in math with set theory.